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ON SOLUBLE GROUPS WHOSE SUBNORMAL SUBGROUPS ARE INERT

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Abstract. A subgroup H of a group G is called inert if, for each $g \in G$, the index of $H \cap H^g$ in H is finite. We give a classification of soluble-by-finite groups G in which subnormal subgroups are inert in the cases where G has no nontrivial torsion normal subgroups or G is finitely generated.

1. Introduction and Main Results

According to [1] and [11], a subgroup H of a group G is said to be *inert* if, for each $g \in G$, the index of $H \cap H^g$ in H is finite. This is equivalent to saying that H is commensurable with each of its conjugates. Recall that two subgroups H and K of a group are said *commensurable* if and only if both indices $|H : (H \cap K)|$ and $|K : (H \cap K)|$ are finite.

A group all of whose subgroups are inert is called *inertial* in [11] where, in the context of generalized soluble groups (with some finiteness conditions), a characterization of inertial groups was given. Inertial groups received attention also in the context of locally finite groups (see [1] and [7] for example).

Recall that Theorem A of [11] states that *a hyper-(abelian or finite) group G without non-trivial torsion normal subgroups is inertial if and only if G is abelian or dihedral.*

In Theorem B of [11] it is shown that *a finitely generated hyper-(abelian or finite) group G is inertial if and only if it has a torsion-free abelian normal subgroup of finite index in which elements of G induce power automorphisms.* Moreover, this is equivalent to the fact that G has a finite normal subgroup F

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such that G/F is torsion-free abelian or dihedral on a torsion-free abelian group (see Proposition 6.1 of [11]).

For terminology, notation and basic facts we refer to [9] and [10]. In particular, a *dihedral group* G on an abelian group A is a group $G = \langle x \rangle \ltimes A$ where x acts faithfully on A as the inversion map. Also, an automorphism γ of a group A is said a *power automorphism* if and only if γ maps each subgroup into itself. Thus $a^\gamma = a^n$ for all $a \in A$ where $n = n(a) \in \mathbb{Z}$. If A is non-torsion, then $n = \pm 1$ is independent of a.

The class T of groups in which subnormal subgroups are normal and its generalizations received much attention in the literature. In the present work we consider the class $\tilde{T}$ of groups whose *subnormal subgroups are inert.*

In Proposition 2.1 we show that *an inert subnormal subgroup H of a group G is strongly inert* (according to the terminology of [6]), that is H has the property that $|\langle H, H^g \rangle : H|$ is finite for all $g \in G$. Clearly, strongly inert subgroups are inert. Note that in a dihedral group D each subgroup H is inert, since $|H : H_D| \leq 2$. However, if $D = \langle x \rangle \ltimes \langle a \rangle$ is infinite, then non-normal subgroups of order 2 are not strongly inert. Recall that if H is a subgroup of a group G, then H^G (resp. H_G) denotes the smallest (resp. largest) normal subgroup of G containing H (resp. contained in H). Groups all of whose subgroups are strongly inert have been studied in [6].

Before stating our main results, we introduce some further terminology. An automorphism of a (possibly non-abelian) group A is said to be an *inertial automorphism* if and only if it maps each subgroup of A to a commensurable one. Clearly a $\tilde{T}$-group acts on its normal abelian sections by means of inertial automorphisms. In [2] we noticed that inertial automorphisms of a group A form a group $\mathrm{IAut}(A)$ and that, *if A is torsion-free abelian, then* $\mathrm{IAut}(A)$ *consists of maps γ such that there are coprime integers m, n (with $n > 0$)* such that

$$(a^\gamma)^n = a^m \ \forall a \in A$$

where clearly m, n are uniquely defined and $mn = \pm 1$ *if A has infinite rank* (see also Proposition 2.3(1) below). Recall that a group is said to have finite (Prüfer) rank if there is an integer r such that every finitely generated subgroup can be generated by at most r elements.

For a complete description of inertial automorphisms of any abelian group A see [3] and [4].

Definition *A group G is said to be* semidihedral *on a torsion-free abelian subgroup A if G acts on A by means of inertial automorphims and $C_G(A) = A$.*

The reader is warned that the word *semidihedral* has been also used with a different meaning in other areas of group theory. Note that, by Proposition 2.3(1) below, it follows that in the above conditions we have that G/A is abelian. Moreover, $A = \mathrm{Fit}(G)$ is uniquely determined, since the elements of $G \setminus A$ act on A as fixed-point-free automorphisms. Furthermore, if elements of G induce power automorphisms in A (which are only the identity and the inversion map, since A is torsion-free), then G is abelian or dihedral. This is the case when A has infinite rank.

We are now in a position to state a theorem corresponding to Theorem A of [11]. Notice that clearly *a semidihedral group has no non-trivial torsion normal subgroups.*

Theorem $\tilde{\mathbf{A}}$ *A hyper-(abelian or finite) group G without non-trivial torsion normal subgroups is a $\tilde{T}$-group if and only if G is semidihedral on a torsion-free abelian subgroup.*

Then we have a statement that corresponds to Theorem B in [11] and answers Question C in [8].

Theorem $\tilde{\mathbf{B}}$ *For a finitely generated soluble-by-finite group G, the following are equivalent:*
i) *G is a $\tilde{T}$-group;*
ii) *G has a finite normal subgroup F such that G/F is a semidihedral group.*

We remark that, for a finitely generated group G, condition (*ii*) is equivalent to the fact that G has a semidihedral normal subgroup G_0 with finite index such that G acts by means of inerial automorphisms on $\mathrm{Fit}(G_0)$ and trivially on G/G_0 (see Proposition 3.1 for details). Moreover, above F is uniquely determined, since $F = \tau(G)$, the *maximum normal torsion subgroup* of G.

2. Preliminary results and proof of Theorem $\tilde{\mathbf{A}}$

Our first result seems to be missing in the literature.

Proposition 2.1. *Inert subnormal subgroups are strongly inert.*

Above statement follows from the next lemma.

Lemma 2.2. *Let H be an inert subnormal subgroup of a group G and $K \leq G$. If $|K : (H \cap K)|$ is finite, then $|\langle H, K\rangle : H|$ is finite.*

Proof. We may assume $G = \langle H, K\rangle = H^G K$ and argue by induction on the subnormal defect d of H in G. The result is trivial if $d \leq 1$. Thus, since $|G : H| = |G : H^G| \cdot |H^G : H|$ and $|G : H^G| \leq |K : (H \cap K)| =: n$ is finite, we only have to show that $|H^G : H|$ is finite. To this aim note that H^G is the join of at most n conjugates of H. Then the statement follows from the following claim: *for any positive integer r, any subgroup H^+ generated by at most s conjugates of H is commensurable with H.* To show this fact argue by induction on s. The claim is trivial if $s = 1$ since H is inert by hypothesis. Assume then that the claim is true for s and consider a subgroup $H^+ = \langle H_1, \ldots, H_{s+1}\rangle$ where each H_i is conjugate to H. Denote commensurability by $\sim$ and note that it is a transitive relation. Then, by induction on s, we have that $L := \langle H_1, \ldots, H_s\rangle \sim H \sim H_{s+1}$. Further, since H_{s+1} has subnormal defect at most $d-1$ in H^G, we may apply induction (on d) to the group H^G and its subgroups H_{s+1} and L. We have that $|\langle H_1, \ldots, H_{s+1}\rangle : H_{s+1}|$ is finite. Thus $H^+ \sim H_{s+1} \sim H$. □

Recall that if an abelian group A is bounded, the group $\mathrm{PAut}(A)$ of power automorphism of A is well-known to be finite. If A is non-torsion abelian, then the only power automorphisms of A are the identity and the inversion map (see 13.4.3 in [10]).

By the next proposition we recall some facts about the group $\mathrm{IAut}(A)$ of inertial automorphisms of an abelian group A. They follow from Theorem 1 of [2] and Proposition 2.2 and Theorem A of [3]. Here we deal with finitely generated subgroups of $\mathrm{IAut}(A)$, while for the structure of the whole group $\mathrm{IAut}(A)$ see [5].

To handle the inertial automorphisms mentioned in the introduction, we introduce some terminology. If γ is the automorphism of the torsion-free abelian group A such that $(a^\gamma)^n = a^m\ \forall a \in A$ (with m, n coprime integers) we say that γ is a *rational-power automorphism*. By abuse of notation, we write $\gamma = m/n$. Note that in [3] and [4] we call such a γ a "multiplication" since we are regarding A as a $\mathbb{Q}^\pi$-module, where $\mathbb{Q}^\pi$ is the *ring of rational numbers whose denominator is a π-number* and π is the set of primes p such that $A^p = A$. As in [9], we will denote by $\mathbb{Q}_\pi$ the *additive group* of the ring $\mathbb{Q}^\pi$.

Proposition 2.3. *Let be a finitely generated group of inertial automorphisms of an abelian group A. Then,*

1) *if A is torsion-free and $\pm 1 \neq \gamma \in \mathrm{Aut}(A)$, then γ is inertial if and only if $\gamma = m/n \in \mathbb{Q}$ and $A^n = A^m = A$ has finite rank, thus* $\mathrm{IAut}(A)$ *is abelian in this case;*

2) *if A is bounded, then there is a finite -invariant subgroup $F \leq A$ such that acts on A/F by means of power automorphisms;*

3) *if A is torsion, then for each $X \leq A$ there is a -invariant subgroup $X^{\geq X}$ such that $X^{/X}$ is finite;*

4) *if A is any abelian group, then there is a -invariant torsion-free subgroup $V \leq A$ such that A/V is torsion.*

Clearly *the class of $\tilde{T}$-groups is closed with respect to the formation of normal subgroups and homomorphic images.* Let us give instances of $\tilde{T}$-groups.

Lemma 2.4. *Let $G_1 \leq G_0$ be normal subgroups of a group G with G_1 and G/G_0 finite. If any subnormal subgroup of G_0/G_1 is inert in G/G_1, then G is $\tilde{T}$.*

Proof. Let H be a subnormal subgroup of G. Then $H_* := G_1H \cap G_0 = G_1(H \cap G_0) \leq G_0$ is subnormal in G, hence inert in G. Thus, since commensurability is a transitive relation, it is enough to verify that H_* is commensurable with H. In fact we have that $H_* \cap H = H \cap G_0$ has index at most $|G_1|$ in $H_* = G_1(H \cap G_0)$. On the other hand, $H \cap G_0$ has index at most $|G/G_0|$ in H. □

Proof of Theorem $\tilde{A}$. Suppose that G is a $\tilde{T}$-group. By Theorem A of [11], any torsion-free nilpotent normal subgroup of G is abelian. Thus $A := \mathrm{Fit}(G)$ is abelian and by Proposition 2.3(1) it follows that $G/C_G(A)$ is abelian too. Suppose, by the way of contradiction, that $A \neq C := C_G(A)$. Since G is hyper-(abelian or finite), there exists a G-invariant subgroup U of C properly containing A and such that U/A is finite or abelian. In the latter case U is nilpotent and so $U = A$, a contradiction. Then U/A is finite, so $U/Z(U)$ is finite and U' is finite. Then $U' = 1$, a contradiction again. Hence $A = C$ and G is semidihedral on A.

Conversely, let G be a semidihedral group on a torsion-free abelian subgroup A. If A has infinite rank, then G is dihedral and every subgroup is inert. Then assume that A has finite rank. Let H

be a subnormal subgroup of G. If $H \leq A$, then H is inert since G acts on A by means of inertial automorphisms. Otherwise, by Proposition 2.3(1), there is an element $h \in H \setminus A$ acting on A as the rational power automorphism $m/n \neq 1$. If H has subnormality defect i, we have $H \geq [A,_i H] \geq A^{(m-n)^i}$. Since A has finite rank, then $A/A^{(m-n)^i}$ is finite and so $|HA : H|$ is finite. Then $|H^G : H|$ is finite and H is strongly inert, hence inert. Thus G is a $\tilde{T}$-group.

3. Finitely generated groups and proof of Theorem $\tilde{\mathbf{B}}$

Let us investigate the structure of groups under consideration.

Proposition 3.1. *Consider the following properties for a group G:*
i) G has a semidihedral normal subgroup G_0 with finite index such that G acts by means of rational-power automorphisms on $A_0 := \mathrm{Fit}(G_0)$ (therefore G acts trivially on G_0/A_0);
ii) G has a finite normal subgroup F such that G/F is semidihedral.

Then (i) implies (ii). Moreover (i) and (ii) are equivalent, provided G has finite rank.

Proof. Let (i) hold. By Proposition 2.3(1), $G/C_G(A_0)$ is abelian. Thus for any $g \in G$ and $g_0 \in G_0$, we have $[g, g_0] \in C_{G_0}(A_0) = A_0$. Hence g acts trivially on G_0/A_0. Let $C := C_G(A_0)$. Since $C \cap G_0 = A_0$, we have that C/A_0 is finite. It follows that C' and $F/C' := \tau(C/C')$ are finite as well. Thus F is finite.

We claim that $\bar{G} := G/F$ *is semidihedral on* $\bar{C}$ (we use bar notation to denote subgroups and elements modulo F). To show this consider two cases. If A_0 has finite rank, then G acts on A_0 by means of inertial automorphisms. Otherwise, if the rank of A_0 is infinite, then A_0 has finite index in G_0, hence G acts on A_0 by means of periodic rational-power automorphisms, that is by ± 1. In both cases G acts on A_0 by means of inertial automorphisms. Since C/A_0 is finite, by the same argument as in Lemma 2.4, we have that G acts by means of inertial automorphisms on the whole C. Moreover, $C_{\bar{G}}(\bar{C}) = \bar{C}$ as if $\bar{x} \in C_{\bar{G}}(\bar{C})$, then $[x, A_0] \leq A_0 \cap F = 1$. Thus the claim is proved.

Assume now that (ii) holds and G has finite rank. Let $n := |F|$, $A_1/F := \mathrm{Fit}(G/F)$ and $C := C_{A_1}(F)$. Then $Z(F) = F \cap C \leq Z(C)$ and $C/(F \cap C)$ is abelian. Thus $[C^n, C^n] \leq (F \cap C)^{n^2} = 1$. Therefore C^n is abelian and $A_0 := C^{n^2}$ is torsion-free abelian. Moreover A_0 has finite index, say s, in A_1, since G has finite rank. By using bar notation in $\bar{G} = G/A_0$, let $\bar{G}_1 := C_{\bar{G}}(\bar{A}_1)$. Then $[\bar{G}_1^s, \bar{G}_1^s] \leq \bar{A}_1^{s^2} = 1$ and so $\bar{G}_1^s$ is abelian. Moreover $\bar{G}_0 := \bar{G}_1^{2s^2}$ is torsion-free abelian and has finite index in $\bar{G}$, since $\bar{G}$ has finite rank.

Since $G_0 \cap F = A_0 \cap F = 1$, then $A_0 \simeq_G A_0F/F$. In particular, every element of $G_0 \setminus A_0$ acts fixed-point-free on A_0. Hence $C_{G_0}(A_0) = A_0$ and G_0 is semidihedral on A_0. □

Note that if G is a group such that G' has prime order, G/G' is free abelian with infinite rank and $G/Z(G)$ is infinite, then G has (ii) but not (i).

Now we consider finitely generated groups.

Lemma 3.2. *Let $G = \langle g_1, \dots, g_r \rangle$ be a finitely generated group with a torsion-free abelian normal subgroup A such that G/A is abelian. If, for each i, the element g_i acts on A by means of the rational-power automorphism $m_i/n_i \in \mathbb{Q}$, then A is a free $\mathbb{Q}^\pi$-module of finite rank where $\pi = \pi(m_1 \cdots m_r n_1 \cdots n_r)$.*

Proof. In the natural embedding of $\bar{G} := G/C_G(A)$ in $\mathrm{IAut}(A)$, each generator $\bar{g}_i$ corresponds to the rational-power automorphisms $m_i/n_i \in \mathbb{Q}$ (see Proposition 2.3(1)). Thus the subring of End(A) generated by the image of $\bar{G}$ is isomorphic to $\mathbb{Q}^\pi$.

Since G/A is finitely presented, we have that A is finitely G-generated and A is a finitely generated $\mathbb{Q}^\pi$-module. Then A is isomorphic to a direct sum of finitely many quotients of the additive group $\mathbb{Q}_\pi$. Thus A is a free $\mathbb{Q}^\pi$-module, since it is torsion-free as an abelian group. □

Notice that a finitely generated semidihedral group may be obtained by a sequence of *finitely many HNN-extensions* starting with a free abelian group of finite rank as a base group. However, generally such extensions are not *ascending* and A is not finitely presented, an easy example being the extension of $\mathbb{Q}_{\{2,3\}}$ by the (inertial) rational-power automorphism $\gamma = 2/3$, see Proposition 11.4.3 of [9]. On the other hand, since finitely generated semidihedral groups have finite rank, for such groups conditions (i) and (ii) of Proposition 3.1 are equivalent.

Lemma 3.3. *Let G be a finitely generated group with a normal subgroup N such that G/N is abelian and G acts on N/N' by means of inertial automorphisms. If N' is finite, then $\tau(G)$ is finite.*

Proof. By arguing mod N' we may assume that N is abelian.

If N is torsion, then it is bounded being G-finitely generated. Thus by Proposition 2.3(2) there is a finite G-subgroup $F \leq N$ such that G acts by means of power automorphisms on N/F. We may assume $F = 1$. Then $G/C_G(N)$ is finite, as a group of power automorphisms of a bounded abelian group. Therefore the nilpotent group $C_G(N)$ is finitely generated. Thus G is polycyclic and $\tau(G)$ is finite.

If N is any abelian group, then, by Proposition 2.3(4), there is a torsion-free G-subgroup $V \leq N$ such that N/V is torsion. By the above $\tau(G/V)$ is finite, whence $\tau(G)$ is finite. □

Lemma 3.4. *Let G be a finitely generated $\tilde{T}$-group. If G' is nilpotent of class 2 and G'' is a p-group, then $\tau(G)$ is finite.*

Proof. By Lemma 3.3, we have that $\tau(G/G'')$ is finite. Then by Theorem $\tilde{A}$ and Proposition 3.1, there is a subgroup G_0 of finite index in G such that $G_0 \geq G''$ and G_0/G'' is torsion-free semidihedral. Then $T := \tau(G_0) \leq G''$ is abelian. Applying again Lemma 3.3 to G_0/G_0'', we have that T/G_0'' is finite. Since it suffices to prove that $\tau(G_0)$ is finite, we may replace G by G_0. Therefore we reduce to the case where G/T is torsion-free semidihedral, $T = \tau(G)$ is abelian and T/G'' is finite.

Then there is a finite subgroup F, such that $T = FG''$. By Proposition 2.3(3) we may assume F to be G-invariant and factor out by F. Thus, denoting $N := G'$, we have the following:

- $G = \langle g_1, \dots, g_r \rangle$ is finitely generated with a nilpotent subgroup N of class 2 such that G/N is abelian, N/N' is torsion-free and $N' = \tau(G)$ is a p-group;

- each g_i acts on N/N' by means of a rational-power automorphism, say $m_i/n_i \in \mathbb{Q}$ (by Proposition 2.3(1)).

By Lemma 3.2, N/N' is a free $\mathbb{Q}^{\pi}$-module of finite rank where $\pi := \pi(m_1 \cdots m_r n_1 \cdots n_r)$. Further, we have that, for each $a, b \in N$ and $g \in G$, if $a^{ng} = a^m z_1$ and $b^{ng} = b^m z_2$ with $z_1, z_2 \in N' \leq Z(N)$ and $m, n \in \mathbb{Z}$, then $[a,b]^{n^2 g} = [a^{ng}, b^{ng}] = [a^m z_1, b^m z_2] = [a^m, b^m] = [a,b]^{m^2}$. Since N' is a p-group, we have $p \notin \pi$.

By a classical argument (see 5.2.5 in [11]), we have that the p-group $\tau(G) = N'$ is finite since it is isomorphic to an epimorphic image of $N/N' \otimes N/N'$ which is a direct sum of finitely many copies of $\mathbb{Q}_{\pi}$ where $p \notin \pi$. □

Proof of Theorem $\tilde{B}$. Clearly, (ii) implies (i) by Lemma 2.4 and Theorem $\tilde{A}$.

Let G be a $\tilde{T}$-group and note that (ii) is equivalent to saying that $\tau(G)$ is finite. Clearly it suffices to prove the statement in the case where G is soluble. By induction on the derived length of G we may assume that G has a normal abelian subgroup A such that $\tau(G/A)$ is finite. By Theorem $\tilde{A}$, Proposition 3.1 and Lemma 3.2, there is a subgroup G_0 of finite index in G such that G_0/A is torsion-free semidihedral of finite rank. We may assume $G := G_0$.

Consider first the case where A is a p-group. If A is unbounded and $B \leq A$ is such that A/B is a Prüfer group, then, by Proposition 2.3(3), such is A/B^G. Using bar notation in $\bar{G} := G/B^G$, we have that $\bar{G}/C_{\bar{G}}(\bar{A})$ is abelian and $\bar{G}'$ is nilpotent of class 2, since $G'' \leq A$. Then we may apply Lemma 3.4 and we have that $\tau(\bar{G}) = \bar{A}$ is finite, a contradiction. Thus A is bounded and, by Proposition 2.3(2), there is a finite G-invariant subgroup $F \leq A$ such that G acts on A/F by means of power automorphisms. Then $G/C_G(A/F)$ is abelian and $A = \tau(G)$ is finite, again by Lemma 3.4.

Let A be any torsion abelian group. By the above, the primary components of A are finite. By the way of contradiction, if A is infinite, then there is $B \leq A$ such that A/B is infinite with cyclic primary components. By Proposition 2.3(3), we may assume $B := B^G$ to be G-invariant. Since A/B has finite Prüfer rank, then $\bar{G} := G/B$ has finite Prüfer rank (and is finitely generated). Thus, by Corollary 10.5.3 of [9], $\bar{G}$ is a minimax group. Therefore its torsion normal subgroups have min (the minimal condition on subgroups) while $\bar{A}$ has not, a contradiction.

In the general case, since G acts by means of inertial automorphisms on A, by Proposition 2.3(4) there is a torsion-free G-invariant subgroup V of A such that A/V is torsion. By the above, $\tau(G/V)$ is finite, whence $\tau(G)$ is finite.

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