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SOME SPECIAL CLASSES OF n -ABELIAN GROUPS

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ABSTRACT. Given an integer n , we denote by \mathfrak{B}_n and \mathfrak{C}_n the classes of all groups G for which the map $\phi_n : g \mapsto g^n$ is a monomorphism and an epimorphism of G , respectively. In this paper we give a characterization for groups in \mathfrak{B}_n and for groups in \mathfrak{C}_n . We also obtain an arithmetic description of the set of all integers n such that a group G is in $\mathfrak{B}_n \cap \mathfrak{C}_n$.

1. Introduction

Let n be an integer. A group G is said to be n -abelian if the map $\phi_n : g \mapsto g^n$ is an endomorphism of G . Then $(xy)^n = x^n y^n$ for all $x, y \in G$, from which it follows $[x^n, y] = [x, y]^n = [x, y^n]$. It is also easy to see that a group G is n -abelian if and only if it is $(1 - n)$ -abelian. The structure of n -abelian groups has been described in [2] and [1]. If $n \neq 0, 1$ and G is an n -abelian group, then the quotient group $G/Z(G)$ has finite exponent dividing $n(n - 1)$. This implies that every torsion-free n -abelian group is abelian.

In this paper we denote by \mathfrak{B}_n and \mathfrak{C}_n the classes of all groups G for which ϕ_n is a monomorphism and an epimorphism of G , respectively. Then $\mathfrak{B}_0 = \mathfrak{C}_0$ contains only the trivial group, $\mathfrak{B}_1 = \mathfrak{C}_1$ is the class of all groups, and $\mathfrak{B}_{-1} = \mathfrak{C}_{-1}$ is the class of all abelian groups. Furthermore, with $|n| > 1$, $G \in \mathfrak{B}_n$ if and only if G is an n -abelian group having no elements of order dividing $|n|$. Similarly, $G \in \mathfrak{C}_n$ if and only if G is n -abelian and for every $g \in G$ there exists an element $x \in G$ such that $g = x^n$. We also set $\mathfrak{A}_n = \mathfrak{B}_n \cap \mathfrak{C}_n$. This class has been studied in [6], [7] and [5]. Of course, groups of exponent dividing $|n - 1|$ are in \mathfrak{A}_n .

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For all integers $n \neq 0$, every divisible abelian group is in \mathfrak{C}_n . In particular, the additive group \mathbb{Q} of rational numbers is in \mathfrak{A}_n , as well as every Prüfer group $\mathbb{Z}(p^\infty)$, with $\gcd(p, n) = 1$. The class \mathfrak{B}_n is subgroup closed, but the class \mathfrak{C}_n is not: in fact the group \mathbb{Z} of all integer numbers is not in \mathfrak{C}_n . The class \mathfrak{C}_n is quotient closed, but the class \mathfrak{B}_n is not: for example \mathbb{Q}/\mathbb{Z} is not in \mathfrak{B}_n . Each of these classes is closed under forming direct products of its members. However, they are both not closed under extensions. For, let G be the wreath product of a cyclic group of order p by $\mathbb{Z}(p^\infty)$. Then G is an extension of groups in \mathfrak{A}_n , for all integers n with $\gcd(p, n) = 1$. But $Z(G)$ is trivial and so G is not n -abelian when $n \neq 0, 1$.

We describe the structure of groups in \mathfrak{B}_n and \mathfrak{C}_n in Section 2. In Section 3, we provide an arithmetic description of the set of all integers n such that a group G is in \mathfrak{A}_n .

2. Groups in $\mathfrak{B}_n \cup \mathfrak{C}_n$

Given an integer n , a group G is said to be n -central if $[x^n, y] = 1$ for all $x, y \in G$. If $n \neq 0$ then n -central is equivalent to $G/Z(G)$ having finite exponent dividing $|n|$.

Lemma 2.1. *Let G be an n -abelian group. Then G is $(n-1)$ -central if and only if it is $(n-1)$ -abelian.*

Proof. Since G is n -abelian, for all $x, y \in G$, we get

$$x^n y^{n-1} x^{-1} = x^n y^n y^{-1} x^{-1} = (xy)^n (xy)^{-1} = (xy)^{n-1}.$$

Thus, the group G is $(n-1)$ -abelian if and only if we have $x^n y^{n-1} x^{-1} = x^{n-1} y^{n-1}$, which is equivalent to $[y^{n-1}, x] = 1$. \square

In the sequel, we always assume $|n| > 1$. Moreover, we denote by \mathbb{P} the set of all (positive) primes and, with n integer, by π_n the set of all primes dividing n . Finally, if π is a set of primes, we set $\pi' = \mathbb{P} \setminus \pi$.

Proposition 2.2. *If $G \in \mathfrak{B}_n \cup \mathfrak{C}_n$, then G is $(n-1)$ -central and $(n-1)$ -abelian. In particular, G' has finite exponent dividing $|n-1|$.*

Proof. The group G is n -abelian, so $1 = [x^{n(n-1)}, y] = [x^{n-1}, y]^n$ for all $x, y \in G$. If $G \in \mathfrak{B}_n$, then G has no elements of order dividing $|n|$ and therefore $[x^{n-1}, y] = 1$. If $G \in \mathfrak{C}_n$, there exists $z \in G$ such that $x = z^n$. Hence $[x^{n-1}, y] = [z^{n(n-1)}, y] = 1$. In both cases G is $(n-1)$ -abelian by Lemma 2.1. \square

It is well-known that, for every n -abelian group G , the elements of finite order form a subgroup T such that G/T is abelian and T is the direct product of a π_{n-1} -group, a π_n -group and an abelian $\pi'_{n(n-1)}$ -group (see [2]). This allows us to characterize torsion groups in $\mathfrak{B}_n \cup \mathfrak{C}_n$.

Theorem 2.3. *Let G be a torsion group. Then:*

- (i) $G \in \mathfrak{B}_n$ if and only if $G = A \times B$ where A is an n -abelian π_{n-1} -group and B is an abelian $\pi'_{n(n-1)}$ -group;
- (ii) $G \in \mathfrak{C}_n$ if and only if $G = A \times B$ where A is an n -abelian π_{n-1} -group and $B = B^n$ is an abelian π'_{n-1} -group.

Proof. (i) Suppose $G \in \mathfrak{B}_n$. Since G has no elements of order dividing $|n|$, by Theorem A of [2], we get $G = A \times B$ where A is an n -abelian π_{n-1} -group and B is an abelian $\pi'_{n(n-1)}$ -group. The converse is clear.

(ii) Let $G \in \mathfrak{C}_n$. Then $G = A \times C \times D$ where A is a π_{n-1} -group, C is π_n -group and D is an abelian $\pi'_{n(n-1)}$ -group (see [2], Theorem A). In particular C is abelian by Proposition 2.2. Thus, $B = C \times D$ is an abelian π'_{n-1} -group. Since $G/A \in \mathfrak{C}_n$, we also have $B = B^n$.

Conversely, for all $g \in A$, there exist integers α and β , depending on g , such that $1 = \alpha|g| + \beta n$. Then $g = g^{\beta n}$ and $A = A^n$. This implies $G \in \mathfrak{C}_n$. \square

The following is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Let G be a torsion group in \mathfrak{B}_n . Then $G \in \mathfrak{A}_n$.*

Hence, if \mathfrak{T} denotes the class of all torsion groups, we have

$$\mathfrak{A}_n \cap \mathfrak{T} = \mathfrak{B}_n \cap \mathfrak{T} \subseteq \mathfrak{C}_n \cap \mathfrak{T}$$

for all integers n , where the inclusion can be proper: for each prime p , the group $\mathbb{Z}(p^\infty)$ is in $\mathfrak{C}_p \setminus \mathfrak{B}_p$.

However, for groups of finite exponent we have:

Proposition 2.5. *Let $G \in \mathfrak{C}_n$ be a group of finite exponent. Then $G \in \mathfrak{A}_n$.*

Proof. Let $\exp(G) = k$ and suppose $\gcd(k, n) = d \neq 1$. Then $k = \alpha d$ for some $\alpha < k$ and $n = \beta d$. Now, for all $g \in G$, there exists $x \in G$ such that $g = x^n = (x^\beta)^d$, so $g^\alpha = 1$. Therefore $\exp(G)$ divides α , which is a contradiction. \square

Now we use Theorem 2.3 to obtain characterizations of groups in \mathfrak{B}_n and in \mathfrak{C}_n .

Theorem 2.6. *Let G be a group. Then $G \in \mathfrak{B}_n$ if and only if G is isomorphic to a subgroup of the direct product of an n -abelian π_{n-1} -group by an abelian group without elements of order dividing $|n|$.*

Proof. Assume $G \in \mathfrak{B}_n$. Let V be a maximal torsion-free subgroup of $Z(G)$ and let W/V be the subgroup consisting of all π_n -elements of $Z(G)/V$. Notice that G/W is torsion because so are $G/Z(G)$ and $Z(G)/V$. Furthermore, if $x \in W \cap G'$, then there exists a π_n -number m such that $x^m \in V$. As $\gcd(m, n-1) = 1$ and $x^{n-1} = 1$, we get $x = 1$. Hence, W has trivial intersection with G' .

Let $x \in G$. First, suppose $(xW)^n = W$. Then $x^n V \in W/V$ and $x^{ns} V = V$ for some π_n -number s . By Proposition 2.2 we have $x^{n-1} \in Z(G)$, so that $x \in Z(G)$. Thus $xV \in Z(G)/V$ is a π_n -element. Therefore $x \in W$ and G/W has no elements of order dividing $|n|$, that is $G/W \in \mathfrak{B}_n$. Suppose now $(xG')^n = G'$. Then $x^n \in G'$, so $x^{n(n-1)} = 1$ by Proposition 2.2. This implies $x^{n-1} = 1$. Hence $x \in G'$ and $G/G' \in \mathfrak{B}_n$.

Finally, since G is isomorphic to a subgroup of $G/W \times G/G'$, the claim follows by (i) of Theorem 2.3.

The converse is clear. \square

Theorem 2.7. *Let G be an n -abelian group and denote by T its torsion group. Then $G \in \mathfrak{C}_n$ if and only if $T = A \times B$, where A is a π_{n-1} -group, $B = B^n$ is an abelian π'_{n-1} -group, and G/T is a p -divisible abelian group for every prime p dividing n .*

Proof. Assume $G \in \mathfrak{C}_n$ and let p be a prime dividing n . Then $n = \alpha p$ for some integer α and, for all $g \in G$, there exists $x \in G$ such that $g = x^n = (x^\alpha)^p \in G^p$. This means that G/T is in \mathfrak{C}_p . The rest follows by (ii) of Theorem 2.3.

Conversely, given $g \in G$, we have $gT = x^n T$ for some $x \in G$: in fact, G/T is n -divisible. Thus $g^{-1}x^n \in T$ and, by Corollary 2.4, there exists $y \in T$ such that $g^{-1}x^n = y^n$.

Therefore $g = x^n y^{-n} = (xy^{-1})^n$, that is $G = G^n$. \square

Notice that in Theorem 2.7 one cannot replace the hypothesis that G is n -abelian by the weaker hypothesis that A is n -abelian. For example, consider the wreath product G of the cyclic group of order 2 by \mathbb{Q} . The torsion part T of G is the base group, that is an infinite group of exponent 2. Moreover, $G/T = \mathbb{Q}$. Finally $Z(G) = 1$ and so G is not n -abelian for all $n \neq 0, 1$. In particular, $G \notin \mathfrak{C}_n$.

3. The semigroup $\mathbb{A}(G)$

Let G be a group. In [4] F. W. Levi introduced the set

$$\mathbb{E}(G) = \{n \in \mathbb{Z} : (xy)^n = x^n y^n \text{ for all } x, y \in G\},$$

the so-called *exponent semigroup* of G . It is a multiplicative subsemigroup of \mathbb{Z} containing 0 and 1; moreover, $n \in \mathbb{E}(G)$ if and only if $1 - n \in \mathbb{E}(G)$. It has been shown (see [4] and [3]) that $\mathbb{E}(G)$ is either $\{0, 1\}$, or \mathbb{Z} , or the union of certain residue classes modulo some integers depending on G .

Similarly, starting from the map $\phi_n : g \in G \mapsto g^n \in G$, we introduce the set

$$\mathbb{A}(G) = \{n \in \mathbb{Z} : \phi_n \in \text{Aut}(G)\}.$$

This is a subsemigroup of $\mathbb{E}(G)$ containing 1. Obviously $0 \in \mathbb{A}(G)$ if and only if $G = \{1\}$; in that case $\mathbb{A}(G) = \mathbb{Z}$. From now on, we assume $G \neq \{1\}$.

Lemma 3.1. *With H and K groups, we have $\mathbb{A}(H \times K) = \mathbb{A}(H) \cap \mathbb{A}(K)$.*

Proof. A direct product is in \mathfrak{A}_n if and only if each of its direct factors is. \square

Lemma 3.2. *Let G be a group and suppose that $\mathbb{A}(G)$ satisfies one of the following conditions:*

- (i) $2 \in \mathbb{A}(G)$;
- (ii) $3 \in \mathbb{A}(G)$;
- (iii) $n \in \mathbb{A}(G)$ and $-n \in \mathbb{A}(G)$ for some $n \neq 0$;
- (iv) $n \in \mathbb{A}(G)$ and $m \in \mathbb{A}(G)$ with $\gcd(n - 1, m - 1) \leq 2$.

Then G is abelian.

Proof. (i) is trivial, (ii) follows from Proposition 2.2.

(iii) The group G is n and $(n + 1)$ -abelian and so, by Lemma 2.1, it is n -central. It is also $(n - 1)$ -central by Proposition 2.2. This means that G is abelian.

(iv) Let $\gcd(n - 1, m - 1) = 1$. By Proposition 2.2 the quotient group $G/Z(G)$ has exponent dividing $n - 1$ and $m - 1$. Hence G is abelian. Assume $\gcd(n - 1, m - 1) = 2$. Then $G/Z(G)$ has exponent 2

by Proposition 2.2 and so G is nilpotent of class ≤ 2 . In particular, we have $[x^2, y] = [x, y]^2 = 1$ for all $x, y \in G$. This implies that, for every integer k with $k \equiv 0 \pmod{4}$ and for all $x, y \in G$,

$$(xy)^k = x^k y^k [y, x]^{k(k-1)/2} = x^k y^k.$$

Without loss of generality we can suppose $n \equiv -1 \pmod{4}$. Thus G is $(n+1)$ -abelian and, as in (iii), we conclude that G is abelian. \square

Notice that $\mathbb{E}(G) = \mathbb{Z}$ if and only if G is abelian. On the other hand, it is easy to show that $\mathbb{A}(G) = \mathbb{Z} \setminus \{0\}$ if and only if G is isomorphic to a direct sum of copies of \mathbb{Q} . However, G is abelian if and only if $-1 \in \mathbb{A}(G)$. So, in that case, $-n \in \mathbb{A}(G)$ for all $n \in \mathbb{A}(G)$. Furthermore, if G is abelian, $n \in \mathbb{A}(G)$ if and only if $p \in \mathbb{A}(G)$ for all primes p dividing n . The semigroup $\mathbb{A}(G)$ is thus generated by -1 and all primes in $\mathbb{A}(G)$. Therefore, if $\pi(G)$ denotes the set of all primes involved in the decomposition of orders of elements of G , and $\delta(G)$ denotes the set of all primes p such that G is p -divisible, one can easily realize that:

Theorem 3.3. *Let G be an abelian group. Then $\mathbb{A}(G)$ is the multiplicative subsemigroup of \mathbb{Z} generated by $(\delta(G) \setminus \pi(G)) \cup \{-1\}$. Moreover:*

- (i) *if G is torsion then $\mathbb{A}(G)$ is generated by $(\mathbb{P} \setminus \pi(G)) \cup \{-1\}$;*
- (ii) *if G is torsion-free then $\mathbb{A}(G)$ is generated by $\delta(G) \cup \{-1\}$.*

By Theorem 3.3, $\mathbb{A}(\mathbb{Z}) = \{-1, 1\} = \mathbb{A}(\mathbb{Q}/\mathbb{Z})$ and, if G is a torsion abelian group, then

$$\mathbb{A}(G) = \mathbb{Z} \setminus \cup_{p \in \pi(G)} p\mathbb{Z}.$$

Concerning (ii) of Theorem 3.3 we also point out that, given a set π of primes, there always exists a torsion-free abelian group G such that $\mathbb{A}(G)$ is the multiplicative subsemigroup of \mathbb{Z} generated by $\pi \cup \{-1\}$. For example, the additive group of all rational numbers with a π -number as denominator has the above properties.

Now let G be a non-abelian group and suppose that G is n -abelian for some integer $n \neq 0, 1$, so $\{0, 1\} \subset \mathbb{E}(G) \subset \mathbb{Z}$. It has been shown in [4] (see also [3]) that the set $\mathbb{E}_0(G)$ of all integers n such that G is both n -abelian and n -central is an ideal of \mathbb{Z} . Let $\mathbb{E}_0(G) = w\mathbb{Z}$. Then $w > 2$, G is both w -abelian and w -central, and w is the least positive integer with such properties. Let $w = q_1 q_2 \dots q_t$ be a factorization of w , with $t \geq 1$, $q_i > 2$ for all $i = 1, 2, \dots, t$ and $\gcd(q_i, q_j) = 1$ for $i \neq j$. Then $\mathbb{E}(G)$ is the union of the 2^t residue classes modulo w which are the solutions of the 2^t systems of congruential equations $x \equiv \delta_i \pmod{q_i}$, where $i = 1, 2, \dots, t$ and $\delta_i \in \{0, 1\}$. In [4] and [3] it has also been shown that, if $n, n+1 \in \mathbb{E}(G)$, then $n \equiv 0 \pmod{w}$. This, together with Proposition 2.2, gives $n-1 \equiv 0 \pmod{w}$ for all $n \in \mathbb{A}(G)$. Therefore we have:

Theorem 3.4. *For each group G , $\mathbb{A}(G) \subseteq [1]_w$.*

In general, the equality does not hold in the theorem above. For example, consider a non-abelian group H of exponent 3, and the direct product G of H by the cyclic group of order 4. Clearly, $\mathbb{E}_0(G) = 3\mathbb{Z}$. However, $4 \notin \mathbb{A}(G)$ since G has an element of order 4. Thus $\mathbb{A}(G) \neq [1]_3$.

REFERENCES

- [1] J. L. Alperin, *A classification of n -abelian groups*, *Canad. J. Math.* **21** (1969), 1238–1244.
- [2] R. Baer, *Factorization of n -soluble and n -nilpotent groups*, *Proc. Amer. Math. Soc.* **4** (1953), 15–26.
- [3] L.-C. Kappe, *On n -Levi groups*, *Arch. Math.* **47** (1986), 198–210.
- [4] F. W. Levi, *Notes on group theory VII*, *J. Indian Math. Soc.* **9** (1945), 37–42.
- [5] D. MacHale, *Power mappings and group morphisms*, *Proc. Roy. Irish. Acad. Sect. A* **74** (1974), 91–93.
- [6] E. Schenkman and L. I. Wade, *The mapping which takes each element of a group onto its n th power*, *Amer. Math. Monthly* **65** (1958), 33–34.
- [7] H. F. Trotter, *Groups in which raising to a power is an automorphism*, *Canad. Math. Bull.* **8** (1965), 825–827.

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