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## ON THE NUMBER OF MAXIMUM INDEPENDENT SETS OF GRAPHS

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**ABSTRACT.** Let  $G$  be a simple graph. An independent set is a set of pairwise non-adjacent vertices. The number of vertices in a maximum independent set of  $G$  is denoted by  $\alpha(G)$ . In this paper, we characterize graphs  $G$  with  $n$  vertices and with maximum number of maximum independent sets provided that  $\alpha(G) \leq 2$  or  $\alpha(G) \geq n - 3$ .

### 1. Introduction

Throughout this paper we will consider only simple graphs. Let  $G = (V, E)$  be a simple graph. The *order* of  $G$  is the number of vertices of  $G$ . For every vertex  $v \in V$ , the *closed neighborhood* of  $v$  is the set  $[v] = \{u \in V : uv \in E\} \cup \{v\}$ . For every edge  $e \in E$  with end points  $u$  and  $v$ , the *closed neighborhood* of  $e$  is the set  $[e] = [u] \cup [v]$ . For every vertex  $v \in V(G)$ , the *degree* of  $v$  is the number of edges incident with  $v$  and is denoted by  $d_G(v)$ . By  $\Delta(G)$  we mean the maximum degree of vertices of  $G$ . For two vertex-disjoint graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , the *disjoint union* of  $G_1$  and  $G_2$  is denoted by  $G_1 + G_2$  is the graph with vertex set  $V_1 \cup V_2$  and edge set  $E_1 \cup E_2$ . The graph  $rG$  denotes the disjoint union of  $r$  copies of  $G$ . By  $K_n$ ,  $C_n$  and  $P_n$  we mean the complete graph, the cycle and the path of order  $n$ , respectively. A set  $S \subseteq V(G)$  is an *independent set* if there is no edge between any pair of the vertices of  $S$ . The *independence number* of  $G$ ,  $\alpha(G)$ , is the maximum cardinality of an independent set of  $G$ . A *maximal independent set* is an independent set that is not a proper subset of any other independent set. Note that a maximum independent set is maximal but the converse is not always true.

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An appealing problem in extremal graph theory is to find the graphs which have the maximum or the minimum number of maximal (maximum) independent sets. Erdős and Moser posed the problem of determining the maximum value of the number of maximal independent sets among all graphs of order  $n$  and those graphs achieving this maximum value. In 1965 Moon and Moser showed that every graph of order  $n$  has at most  $3^{\frac{n}{3}}$  maximal independent sets [9]. There are many papers related to this question which studied this problem among trees, forests, unicyclic graphs, bipartite graphs, and triangle-free graphs. For a survey on this subject see [8]. There are some papers related to counting the number of independent sets [1, 2, 13] and the number of maximum independent sets in graphs [5, 6, 14]. In [7] the maximum value of the number of maximum independent sets of some families of graphs such as trees, forests and triangle-free graphs was determined. In this paper, we characterize graphs  $G$  with  $n$  vertices and with maximum number of maximum independent sets provided that  $\alpha(G) \leq 2$  or  $\alpha(G) \geq n - 3$ .

### 2. Balanced Graphs and Balanced Sequence

Let  $G$  be a graph. For every positive integer  $k$ , let  $\mathcal{I}(G, k)$  be the set of all independent sets of  $G$  with cardinality  $k$ . By  $i(G, k)$  we mean the cardinality of  $\mathcal{I}(G, k)$ . Let  $\mathcal{A}_{n,\alpha}$  be the set of graphs of order  $n$  and independence number  $\alpha$ . The purpose of this paper is to find the maximum value of  $i(G, \alpha)$ , where  $G \in \mathcal{A}_{n,\alpha}$ . Let  $n_1 \geq \dots \geq n_t \geq 1$  be a sequence of integers. Suppose that  $n_1 + \dots + n_t = n$ . We say  $(n_1, \dots, n_t)$  is *balanced* if  $n_i \in \{\lfloor \frac{n}{t} \rfloor, \lceil \frac{n}{t} \rceil\}$ , for every  $1 \leq i \leq t$ . For example  $(3, 3, 2)$  is a balanced sequence. It is not hard to see that for every two positive integers  $n$  and  $t$  there exists only one balanced sequence and it is the following sequence

$$\underbrace{\left(\lceil \frac{n}{t} \rceil, \dots, \lceil \frac{n}{t} \rceil\right)}_{n-t\lfloor \frac{n}{t} \rfloor} \underbrace{\left(\lfloor \frac{n}{t} \rfloor, \dots, \lfloor \frac{n}{t} \rfloor\right)}_{t-n+t\lfloor \frac{n}{t} \rfloor}.$$

Let  $(n_1, \dots, n_t)$  be a descending sequence of positive integers. By  $K(n_1, \dots, n_t)$  we mean the connected graph which has a vertex of degree  $n_1 + t - 2$ , say  $v$ , such that  $K(n_1, \dots, n_t) \setminus v = K_{n_1-1} + K_{n_2} + \dots + K_{n_t}$  and  $v$  is adjacent to every vertex of  $K_{n_1-1}$ . For example  $K(4, 4, 3, 2)$  is shown in Figure 1. We note that  $K(n)$  is the complete graph  $K_n$  and  $K(2, 1)$  and  $K(2, 2)$  are the paths  $P_3$  and  $P_4$ , respectively. We say that the graph  $K(n_1, \dots, n_t)$  is *balanced* if the descending sequence  $(n_1, \dots, n_t)$  is balanced. We note that there is a connection between the complement of Turán graphs and the balanced graphs. In fact, by removing  $t - 1$  edges which are adjacent to  $v$  from the balanced graph  $K(n_1, \dots, n_t)$  one obtain the complement of Turán graph  $T(n, t)$ .

We think that the balanced graphs have the maximum number of maximum independent set. More precisely we have the following:

**Conjecture 2.1.** *Let  $G$  be a connected graph of order  $n$  and independence number  $\alpha$ . Then  $i(G, \alpha) \leq i(K, \alpha)$ , where  $K$  is the balanced graph of order  $n$  and independence number  $\alpha$ .*

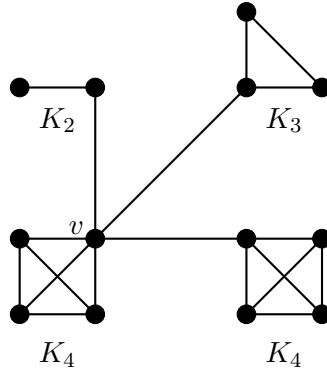


FIGURE 1. The graph  $K(4, 4, 3, 2)$ .

**Remark 2.2.** We recall that among all graphs of order  $n$  and independence number  $\alpha$ , the maximum number of maximum independent sets is attained for the complement of the Turán graph  $T(n, \alpha)$  [11]. See [4, 10, 12] for more details about the other families of graphs.

In sequel we will prove that Conjecture 2.1 holds for  $\alpha \in \{1, 2, n - 3, n - 2, n - 1\}$ .

### 3. Graphs with Independence Number Two

In this section we find those graphs  $G$  which attained the maximum value of  $i(G, \alpha)$  among all graphs in the set  $\mathcal{A}_{n,2}$ , where  $\mathcal{A}_{n,\alpha}$  is the set of all graphs of order  $n$  and independence number  $\alpha$ .

**Theorem 3.1.** Let  $G$  be a disconnected graph of order  $n$  with  $\alpha(G) = 2$ . Then

$$i(G, 2) \leq i(K_{\lceil \frac{n}{2} \rceil} + K_{\lfloor \frac{n}{2} \rfloor}, 2) = \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Moreover the equality holds if and only if  $G \cong K_{\lceil \frac{n}{2} \rceil} + K_{\lfloor \frac{n}{2} \rfloor}$ .

*Proof.* Since  $\alpha(G) = 2$ ,  $G$  has exactly two connected components with independence number one. Thus there exist positive integers  $r$  and  $s$  such that  $G = K_r + K_s$  and  $r + s = n$ . Thus  $i(G, 2) = rs$  and the maximum value of  $rs$  is attained on  $\{r, s\} = \{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor\}$ . □

Now, we investigate the maximum number of independent sets among all connected graphs.

**Theorem 3.2.** Let  $G$  be a connected graph of order  $n$  with  $\alpha(G) = 2$ . Then

$$i(G, 2) \leq i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2) = \begin{cases} \frac{n^2}{4} - 1, & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Moreover if  $n \neq 5$  the equality holds if and only if  $G \cong K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$ . If  $n = 5$ , the equality holds if and only if  $G \cong K(3, 2)$  or  $G \cong C_5$ .

*Proof.* It is easy to see that

$$i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2) = \begin{cases} \frac{n^2}{4} - 1, & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

By induction on  $n$  we prove that if  $H$  is a connected graph of order  $n$  and  $\alpha(H) = 2$  which has the maximum number of independent sets of size 2, then  $H \cong K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$ . Thus we can assume that  $i(H, 2) \geq i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2)$ . We want to show that  $H \cong K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$ . For  $n \leq 5$  it is not hard to see that the result follows. Using [3] one can investigate that for  $n = 6$  the result follows, as well. Now suppose that  $n \geq 7$ . First we claim that  $\Delta(H) < n - 1$ . By contradiction assume that  $H$  has a vertex, say  $v$ , with degree  $n - 1$ . Thus  $\alpha(H \setminus v) = 2$  and  $i(H, 2) = i(H \setminus v, 2)$ . If  $H \setminus v$  is disconnected, then  $H \setminus v = K_r + K_s$ , for some positive integers  $r$  and  $s$  such that  $r + s = n - 1$ . Thus  $i(H, 2) = rs \leq \lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor < i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2)$ , a contradiction. Therefore  $H \setminus v$  is connected. By the induction hypothesis  $i(H \setminus v, 2) \leq i(K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor), 2)$ . On the other hand  $i(H, 2) = i(H \setminus v, 2)$ . Thus  $i(H, 2) \leq i(K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor), 2) < i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2)$ , a contradiction. So the claim is proved.

Consider a vertex  $u \in V(H)$ . Since  $\Delta(H) < n - 1$  and  $\alpha(H) = 2$ ,  $\alpha(H \setminus [u]) = 1$ . In the other words  $H \setminus [u] \cong K_{n-d_H(u)-1}$ . If  $\alpha(H \setminus u) = 1$ , then  $H \setminus u \cong K_{n-1}$ . Thus  $i(H, 2) = n - 1 - \deg_H(u) \leq n - 2 < i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2)$ , a contradiction. Therefore  $\alpha(H \setminus u) = 2$ . Let  $t = n - d_H(u) - 1$ . Then  $i(H, 2) = i(H \setminus u, 2) + t$ . First suppose that  $H \setminus u$  is disconnected. Thus  $H \setminus u = K_r + K_s$ , for some natural numbers  $r$  and  $s$  with  $r + s = n - 1$ . If there are vertices  $a \in V(K_r)$  and  $b \in V(K_s)$  such that  $u$  is not adjacent to  $a$  and  $b$ , then  $\{a, b, u\}$  is an independent set in  $H$ , a contradiction. Thus  $u$  is adjacent to every vertex of  $K_r$  or every vertex of  $K_s$ . Suppose that  $r \leq s$ . Thus  $d_H(u) \geq r + 1$ . Since  $i(H, 2) = rs + n - 1 - d_H(u)$ ,  $i(H, 2) \leq rs + n - 1 - (r + 1) = r(s - 1) - 2$ . On the other hand  $r + (s - 1) = n - 2$ . Thus

$$i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2) \leq i(H, 2) \leq \lceil \frac{n-2}{2} \rceil \lfloor \frac{n-2}{2} \rfloor - 2.$$

This shows that  $\{r, s - 1\} = \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil\}$ . Using the latter fact one can see that  $H \cong K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$ .

As we see, the result follows if there exists a vertex  $u$ , such that  $H \setminus u$  is disconnected. Now suppose for every vertex  $u \in V(H)$ ,  $H \setminus u$  is connected. We claim that  $H$  has a vertex  $v$  such that  $d_H(v) \geq \lfloor \frac{n}{2} \rfloor$ . By contradiction let  $\Delta(H) < \lfloor \frac{n}{2} \rfloor$ . Let  $w \in V(H)$ . Thus  $H \setminus [w] \cong K_{n-d_H(w)-1}$ . Since  $H$  is connected, there exists a vertex  $z \in V(H \setminus w)$ , with  $d_H(z) \geq n - d_H(w) - 1$ . On the other hand  $d_H(w), d_H(z) < \lfloor \frac{n}{2} \rfloor$ , a contradiction. This proves the claim.

Let  $v_0 \in V(H)$  and  $d_H(v_0) \geq \lfloor \frac{n}{2} \rfloor$ . Then  $H \setminus v_0$  is connected and  $\alpha(H \setminus v_0) = 2$ . In addition  $i(H, 2) = i(H \setminus v_0, 2) + n - d_H(v_0) - 1$ . We claim that  $H \setminus v_0 \cong K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor)$ . By contradiction let  $H \setminus v_0 \not\cong K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor)$ . Thus by the induction hypothesis  $i(H \setminus v_0, 2) < i(K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor), 2)$ . This shows that

$$i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2) \leq i(H, 2) < i(K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor), 2) + n - 1 - \lfloor \frac{n}{2} \rfloor.$$

This is a contradiction. Thus  $H \setminus v_0 \cong K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor)$ . In the graph  $H \setminus v_0$ , assume that  $a$  and  $b$  are two vertices from  $K_{\lceil \frac{n-1}{2} \rceil}$  and  $K_{\lfloor \frac{n-1}{2} \rfloor}$ , respectively, such that  $a$  and  $b$  are adjacent. Since  $\alpha(H) = 2$ , one of the following cases holds:

- 1)  $v$  is adjacent to every vertex of  $K_{\lceil \frac{n-1}{2} \rceil}$ . Let  $m$  be the number of edges of  $H$ . Since  $i(H, 2) = \binom{n}{2} - m$  and  $H$  has the maximum number of independent sets of size two,  $m$  is minimum

among all graphs of order  $n$  and independence number 2. Using the latter fact and that  $H \setminus v_0 \cong K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor)$ , one can see that  $H \cong K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$ .

- 2)  $v$  is adjacent to every vertex of  $K_{\lfloor \frac{n-1}{2} \rfloor}$ . Similar to case 1, one can obtain the result.
- 3)  $v$  is adjacent to every vertex of  $K_{\lceil \frac{n-1}{2} \rceil} \setminus a$  and  $K_{\lfloor \frac{n-1}{2} \rfloor} \setminus b$ . Thus  $i(H, 2) = i(H \setminus v_0, 2) + 2$ .  
Therefore

$$i(K(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), 2) \leq i(H, 2) = i(K(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor), 2) + 2.$$

This shows that  $n \leq 5$ , a contradiction. Thus the case 3 does not happen.

The proof is complete. □

### 4. Graphs with Large Independence Number

In this section we investigate the maximum value of the number of maximum independent sets in some families of graphs which have large independence number.

**Remark 4.1.** *Let  $G$  be a graph of order  $n$  with  $\alpha = \alpha(G) = n - 1$ . Then  $G = K_{1,r} + tK_1$ , for some positive integer  $r$  and a non-negative integer  $t$ . Thus  $i(G, \alpha) \leq 2$ . Moreover  $i(G, \alpha) = 2$  if and only if  $r = 1$ , that is  $G = K_2 + (n - 2)K_1$ .*

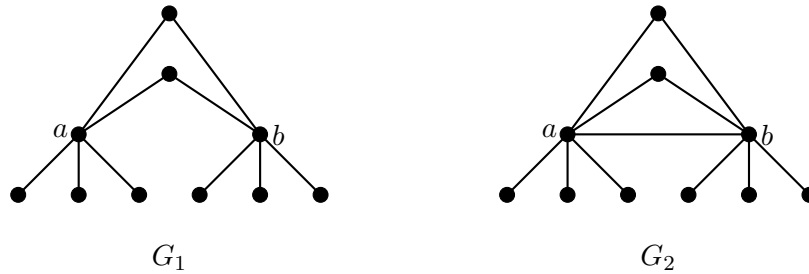


FIGURE 2. Graphs of order  $n$  and  $\alpha = n - 2$ .

**Theorem 4.2.** *Let  $G$  be a graph of order  $n$  with  $\alpha(G) = \alpha = n - 2$ . Then  $i(G, \alpha) \leq 4$ . Moreover the following hold:*

- 1)  $i(G, \alpha) = 4$  if and only if  $G \cong 2K_2 + (n - 4)K_1$  for some  $n \geq 4$ .
- 2)  $i(G, \alpha) = 3$  if and only if  $G \cong P_4 + (n - 4)K_1$  for some  $n \geq 4$  or  $G \cong K_3 + (n - 3)K_1$  for some  $n \geq 3$ .
- 3)  $i(G, \alpha) = 2$  if and only if  $G \cong K_2 + K_{1,r} + tK_1$  or  $G \cong C_4 + (n - 4)K_1$  (for some  $n \geq 4$ ) or  $G \cong K(2, 2, \underbrace{1, \dots, 1}_h) + tK_1$  or  $G \cong K(3, \underbrace{1, \dots, 1}_h) + tK_1$  for some integers  $r \geq 2, h \geq 1$  and  $t \geq 0$ .

*Proof.* Let  $S$  be an independent set of  $G$  with size  $n - 2$ . Let  $V(G) \setminus S = \{a, b\}$ . Assume that  $S_1$  and  $S_2$  are the set of all vertices of  $S$  which are adjacent only to  $a$  and  $b$ , respectively. Let  $S_3$  be the set of all vertices of  $S$  which are adjacent to both vertices  $a$  and  $b$ . Let  $|S_1| = r, |S_2| = s$  and  $|S_3| = k$ . Let

$t = n - 2 - (r + s + k)$ . Then  $G$  has  $t$  isolated vertices and only one of the graphs  $G_1$  or  $G_2$  which are shown in Figure 2 (depending on the case that  $a$  and  $b$  are adjacent or not). We note that if  $k = 0$  then  $G_1$  is a disjoint union of two stars. By examining the maximum independent sets containing  $\{a\}, \{b\}, \{a, b\}$  and discussing about the values of  $r, s$  and  $k$  one can obtain the result. For instance if  $a$  and  $b$  are adjacent and  $r = n - 3, s = 0$  and  $k = 1$ , then  $t = 0$  and  $G \cong K(3, \underbrace{1, \dots, 1}_{n-3})$ . Also if  $a$  and  $b$  are not adjacent and  $r = s = 1, k = 0$ , then  $G \cong 2K_2 + (n - 4)K_1$ .  $\square$

**Remark 4.3.** Let  $G$  be a graph of order  $n \geq 2$ . Then for every vertex  $v$  of  $G$ ,  $\alpha(G) - 1 \leq \alpha(G \setminus v) \leq \alpha(G)$ . Moreover if  $G \not\cong nK_1$ , then there exists  $u \in V(G)$  such that  $\alpha(G \setminus v) = \alpha(G)$ .

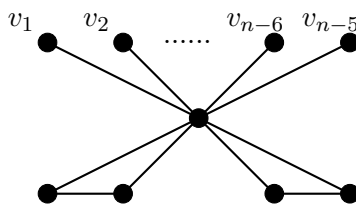


FIGURE 3. The graph  $F_n$ .

Now, we are in position to prove the main result of this section.

**Theorem 4.4.** Let  $G$  be a connected graph of order  $n \geq 4$ . If  $\alpha(G) = \alpha = n - 3$ , then  $i(G, \alpha) \leq 5$ . Moreover the equality holds if and only if  $G \cong C_5$  or  $G \cong K(3, 2)$  or  $G \cong K(2, 2, 2)$ .

*Proof.* We proceed by induction on  $n$ . If  $n = 4$ , then  $G \cong K_4$ . Thus  $i(G, \alpha) = 4$ . Let  $n = 5$ . Thus  $\alpha(G) = 2$ . By Theorem 3.2,  $i(G, \alpha) \leq 5$  and the equality holds if and only if  $G \cong C_5$  or  $G \cong K(3, 2)$ . Thus for  $n \leq 5$  the result follows. Now suppose that  $n \geq 6$ . It is sufficient to show that  $i(G, \alpha) \leq 5$  and the equality holds if and only if  $G \cong K(2, 2, 2)$  ( equivalently  $i(G, \alpha) \geq 5$  if and only if  $G \cong K(2, 2, 2)$ ). By Remark 4.3 we can assume the following:

- 1) Assume that  $G$  has a vertex  $u$  such that  $G \setminus u$  is connected and  $\alpha(G \setminus u) = n - 4$ . Let  $h = i(G, \alpha)$  and  $S_1, \dots, S_h$  be all independent sets of  $G$  with size  $n - 3$ . Since  $\alpha(G \setminus u) = n - 4$ , for every  $i, u \in S_i$ . By removing  $u$  from each  $S_i$ , we conclude that  $G \setminus u$  has at least  $h$  independent sets of size  $n - 4$ . Since  $G \setminus u$  has order  $n - 1$  and  $\alpha(G \setminus u) = n - 4$ , by the induction hypothesis,  $h \leq 5$ . Now, suppose that  $h = 5$ . If  $n = 6$ , then  $G \setminus u \cong K(3, 2)$  or  $G \setminus u \cong C_5$ . For both of these cases, since  $u$  is in every maximum independent set of  $G$  and  $V(G \setminus u) = \bigcup_{i=1}^5 S_i \setminus u$ , we conclude that  $v$  has no neighbor in  $G$ , a contradiction ( because  $G$  is connected). Thus  $h \leq 4$  and  $n \geq 7$ .
- 2) Suppose that  $G$  has a vertex  $v$  such that  $G \setminus v$  is connected and  $\alpha(G \setminus v) = \alpha(G) = \alpha = n - 3$ . Thus by the proof of Theorem 4.2,  $G \setminus v$  is isomorphic to one of the graphs which are shown in Figure 2. Since  $n \geq 6$ , by the first part of Theorem 4.2,  $i(G \setminus v, \alpha) \leq 2$ . First we show that  $i(G, \alpha) \leq 5$ . By contradiction, assume that  $i(G, \alpha) \geq 6$ . Since  $\alpha(G \setminus v) = \alpha(G)$  and

$i(G \setminus v, \alpha) \leq 2$  and  $i(G, \alpha) \geq 6$ , we conclude that  $\alpha(G \setminus [v]) = n - 4$  and  $i(G \setminus [v], \alpha - 1) \geq 4$ . Since  $\alpha(G \setminus [v]) = n - 4$ ,  $d_G(u) \leq 3$ . We can consider the following cases:

- A) Suppose that  $d_G(v) = 1$ . Thus  $G \setminus [v]$  has  $n - 2$  vertices. By the first part of Theorem 4.2,  $G \setminus [v] \cong 2K_2 + (n - 4)K_1$ . Since  $G$  is connected, one can see that  $G \cong K(3, 2, \underbrace{1, \dots, 1}_{n-5})$  or  $G \cong K(2, 2, 2, \underbrace{1, \dots, 1}_{n-6})$  or  $G \cong F_n$ , where  $F_n$  is shown in Figure 3. In any of these cases we have  $\alpha(G) = n - 3$  and  $i(G, \alpha) = 4$  ( if  $n = 6$ ,  $i(K(2, 2, 2), 3) = 5$ ), a contradiction.
- B) Assume that  $d_G(v) = 2$ . That is  $G \setminus [v]$  has order  $n - 3$ . Using Remark 4.1, we have  $i(G \setminus [v], \alpha - 1) \leq 2$ . This contradicts the fact that  $i(G \setminus [v], \alpha - 1) \geq 4$ .
- C) Let  $d_G(v) = 3$ . Thus  $G \setminus [v]$  has  $n - 4$  vertices. Since  $\alpha(G \setminus [v]) = n - 4$ ,  $G \setminus [v] \cong (n - 4)K_1$ . Therefore  $i(G \setminus [v], n - 4) = 1$ , a contradiction.

Thus  $i(G, \alpha) \leq 5$ . Now suppose that  $i(G, \alpha) = 5$ . Using the above method, we conclude that  $d_G(v) = 1$  and  $i(G \setminus [v], n - 4) \in \{3, 4\}$ . If  $i(G \setminus [v], n - 4) = 4$ , then the first part of Theorem 4.2 implies that  $G \setminus [v] \cong 2K_2 + (n - 4)K_1$  (see Case A). Therefore  $G \cong K(2, 2, 2)$ . Suppose that  $i(G \setminus [v], n - 4) = 3$ . Thus  $i(G \setminus v, n - 3) = 2$ . By the third part of Theorem 4.2,  $G \setminus v \cong K(2, 2, \underbrace{1, \dots, 1}_{n-5})$  or  $G \setminus v \cong K(3, \underbrace{1, \dots, 1}_{n-4})$ . On the other hand  $i(G \setminus [v], n - 4) = 3$ . Thus by the second part of Theorem 4.2 one can see that  $G \cong K(2, 2, 2)$ .

The proof is complete. □

Using Theorems 4.2 and 4.4 one can prove the following:

**Corollary 4.5.** *Let  $G$  be a disconnected graph of order  $n$  and  $\alpha(G) = \alpha = n - 3$ . Then  $i(G, \alpha) = 8$  or  $i(G, \alpha) \leq 6$  and the following hold:*

- 1)  $i(G, \alpha) = 8$  if and only if  $G \cong 3K_2 + (n - 6)K_1$  for some  $n \geq 6$ .
- 2)  $i(G, \alpha) = 6$  if and only if  $G \cong K_2 + P_4 + (n - 6)K_1$  for some  $n \geq 6$  or  $G \cong K_2 + K_3 + (n - 5)K_1$  for some  $n \geq 5$ .
- 3)  $i(G, \alpha) = 5$  if and only if  $G \cong K(3, 2) + (n - 5)K_1$  or  $G \cong C_5 + (n - 5)K_1$  for some  $n \geq 6$  or  $G \cong K(2, 2, 2) + (n - 6)K_1$  for some  $n \geq 7$ .

**Remark 4.6.** *Among all connected graphs of order  $n$  and independence number  $n - 3$ , in the proof of Theorem 4.4 we see that for the graphs  $G$ :  $K(3, 2, \underbrace{1, \dots, 1}_{n-5})$  ( $n \geq 6$ ),  $K(2, 2, 2, \underbrace{1, \dots, 1}_{n-6})$  ( $n \geq 7$ ) and  $F_n$  ( $n \geq 5$ ), where  $F_n$  is shown in Figure 3,  $i(G, n - 3) = 4$ . Also  $i(K_4, 1) = i(C_5 + e, 2) = 4$ , where  $C_5 + e$  is the graph whose obtained by adding a new edge to the cycle  $C_5$ .*

We finish the paper by the following problem.

**Problem 4.7.** *Among all connected graphs of order  $n$  and independence number  $n - 3$  find all graphs  $G$  such that  $i(G, n - 3) = 4$ .*

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