www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 3 No. 1 (2014), pp. 57-64.

www.ui.ac.ir

CO-PROLONGATIONS OF A GROUP EXTENSION

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Communicated by Mark L. Lewis

Abstract. The aim of this paper is to study co-prolongations of central extensions. We construct the obstruction theory for co-prolongations and classify the equivalence classes of these by kernels of homomorphisms between 2-dimensional cohomology groups of groups.

1. Introduction

A description of group extensions by means of factor sets leads to a close relationship between the group extension problem and the group cohomology theory [1, 5, 9]. Analogously, the extension problems of a type of algebras (such as rings, **k**-split algebras) were dealt with by appropriate cohomology theories (such as Mac Lane cohomology, Hochschild cohomology) (see [4, 5]). Quang et al used the group cohomology to study the prolongation of central extensions in [8]. In this paper we consider the dual of that problem.

Let G be a group and A be a G-module. Consider a short exact sequences of groups

$$E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} G \longrightarrow 1,$$

in which the left action by $G = B/A$ on A induced by B-conjugation on the commutative normal subgroup A is the given G-module structure on A. We say that B is an *extension* of the group A by the group G. A *morphism* between two extensions is a triple of homomorphisms (α, β, γ) such that the following diagram commutes

MSC(2010): Primary: 20J05; Secondary: 20J06.

Keywords: Group extension, cohomology of groups, prolongation, obstruction.

Received: 25 February 2013, Accepted: 18 August 2013.

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$$
\begin{array}{lccccccccc}
E_0: & 0 & \longrightarrow & A_0 & \xrightarrow{i_0} & B_0 & \xrightarrow{p_0} & G_0 & \longrightarrow & 1 \\
 & & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
E: & 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & G & \longrightarrow & 1.
\end{array}
\tag{1.1}
$$

For a given extension E and a homomorphism γ, there exists an extension E_0 and a morphism (id_A, β, γ) (then, B is the fibred product, or the pull-back $A \times_{G_0} G$) (see [3] Ch. IV, [7]). This shows that $\mathrm{Ext}(G, A)$ is a contravariant functor in terms of the first variable, G, ([5] Ch. III). Given two extensions E_0, E and two morphisms α, γ, Proposition 5.1.1 [9] indicates a necessary and sufficient condition for there to exist a homomorphism β such that (α, β, γ) is a morphism.

Given an extension E_0 and a homomorphism $\gamma : G_0 \to G$, the problem here is that of finding whether there is any corresponding extension E such that $E_0 = E\gamma$. A particular case when E_0 is a central extension and $\gamma : G_0 \to G$ is a normal monomorphism is studied in [8]. In this case E is said to be a *prolongation* of the extension E_0. The obstruction theory and the classification of prolongations of E_0 were dealt with in the case of α is surjective.

The objective of this paper is to solve the problem of prolongations in the case γ is surjective and α is an identity. Then, E is termed a *co-prolongation* of the extension E_0. In Section 2 we show some necessary conditions for the existence of co-prolongations. We also prove that each such co-prolongation is just a central extension and it induces a crossed module. After stating the problem of γ-*co-prolongations* of a group extension, we construct the obstruction theory for this concept (Theorem 3.1) in Section 3, and then classify co-prolongations (Theorem 3.3) of the extension E_0 by the kernel of the inducing homomorphism

$$\overline{\gamma} : H^2(G, A) \to H^2(G_0, A).$$

2. Co-prolongations of a group extension

In this paper we fix the *system* (E_0, γ), where E_0 is the extension

$$0 \to A \overset{i_0}{\to} B_0 \overset{p_0}{\to} G_0 \to 0,$$

and $\gamma : G_0 \to G$ is injective. Further, for short, we identify the abelian group A with the normal subgroup $i_0(A)$ of B_0; the operations in B_0, G_0 are denoted by the addition, even though they are are non-necessarily abelian.

Now, suppose that $\alpha = id_A$ in the diagram (1.1). Then, the extension E is call a *co-prolongation* by γ (or γ-*co-prolongation*) of the extension E_0.

Remark. In the commutative diagram (1.1), since $\alpha = id_A$ and γ is surjective, β also is an *surjection.*

Proposition 2.1. *If the extension E is a γ-co-prolongation of the extension $E_0 \in Opext(G_0, A, \varphi_0)$, then there uniquely exists a homomorphism $\varphi : G \to \operatorname{Aut} A$ such that the following diagram commutes:*

$$\begin{array}{ccc} G_0 & \xrightarrow{\gamma} & G, \\ & {\scriptstyle\varphi_0}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \operatorname{Aut} A & \end{array} \qquad \varphi\gamma = \varphi_0. \tag{2.1}$$

Proof. The homomorphism $\varphi : G \to \operatorname{Aut} A$ induced by the extension E is determined by

$$i[(\varphi pb)a] = b + ia - b, \;\; b \in B, a \in A. \tag{2.2}$$

It is easy to check that φ satisfies (2.1). If $\varphi' : G \to \operatorname{Aut}(A)$ is a homomorphism satisfying (2.1), then $\varphi' = \varphi$ since γ is surjective. □

Proposition 2.2. *If the extension E is a γ-co-prolongation of the extension E_0, there exists an isomorphism $j : \operatorname{Ker}\gamma \to \operatorname{Ker}\beta$ such that $p_0 j = id_{\operatorname{Ker}\gamma}$.*

Proof. Since the right hand side square of the diagram (1.1) commutes,

$$p_0(\operatorname{Ker}\beta) \subset \operatorname{Ker}\gamma.$$

Thus, the homomorphism p_0 induces a homomorphism $p^\bullet : \operatorname{Ker}\beta \to \operatorname{Ker}\gamma$. We show that $p^\bullet$ is surjective. Take $c \in \operatorname{Ker}\gamma$, then $c = p_0(x_0)$, where $x_0 \in B_0$. Then,

$$0 = \gamma(c) = \gamma p_0(x_0) = p\beta(x_0).$$

It follows that $\beta(x_0) \in \operatorname{Ker} p = A$. Set $a = \beta(x_0)$. Since the left hand side square commutes, $\beta(a) = a = \beta(x_0)$, hence $x_0 - a \in \operatorname{Ker}\beta$. One obtains

$$c = p_0(x_0) = p_0(x_0 - a) \in p_0(\operatorname{Ker}\beta),$$

hence $p^\bullet$ is surjective. Also, $A \cap \operatorname{Ker}\beta = 0$, or $\operatorname{Ker}(p_0) \cap \operatorname{Ker}\beta = 0$, which implies that $p^\bullet$ is injective. Then, $j = (p^\bullet)^{-1} : \operatorname{Ker}\gamma \to \operatorname{Ker}\beta$ is the required isomorphism. □

A monomorphism $j : C \to D$ is said to be *normal* if jC is a normal subgroup in D.

Lemma 2.3. *If there exists a normal monomorphism $j : \operatorname{Ker}\gamma \to B_0$ such that $p_0 j = id_{\operatorname{Ker}\gamma}$, then $p_0^{-1}(\operatorname{Ker}\gamma) = A \times j(\operatorname{Ker}\gamma)$ and the following diagram commutes*

$$\begin{array}{ccccccccc} E' : 0 & \longrightarrow & A & \xrightarrow{i'} & A \times \operatorname{Ker}\gamma & \xrightarrow{p'} & \operatorname{Ker}\gamma & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\nu} & & \\ E_0 : 0 & \longrightarrow & A & \xrightarrow{i} & B_0 & \xrightarrow{p} & G_0 & \longrightarrow & 0, \end{array} \tag{2.3}$$

where $A \times \operatorname{Ker}\gamma$ is the direct product, ν is an inclusion, $i' : a \mapsto (a, 0)$, $p' : (a, c) \mapsto c$, and $\varepsilon : (a, c) \mapsto a + j(c)$.

Proof. Let $b \in p_0^{-1}(\operatorname{Ker}\gamma)$. Then there exists $c \in \operatorname{Ker}\gamma$ such that $p_0(b) = c = p_0 j(c)$. It follows that $b - j(c) = a \in \operatorname{Ker} p_0 = A$. Thus, $b = a + j(c) \in A + j(\operatorname{Ker}\gamma)$. It is easy to see that $A \cap j(\operatorname{Ker}\gamma) = 0$, hence $p_0^{-1}(\operatorname{Ker}\gamma) = A \times j(\operatorname{Ker}\gamma)$. The map $\alpha : (a, c) \mapsto a + j(c)$ is a homomorphism and diagram (2.3) commutes. □

Definition 2.4. *A crossed module is a quadruple $\mathcal{M} = (B, D, d, \theta)$ in which $d : B \to D$, $\theta : D \to AutB$ are group homomorphisms such that*

C_1. $\theta d = \mu$,

C_2. $d(\theta_x(b)) = \mu_x(d(b))$, $x \in D, b \in B$,

where μ_x is is an inner automorphism given by conjugation with x.

A crossed module (B, D, d, θ) is sometimes denoted by $B \xrightarrow{d} D$, or $B \to D$.

Crossed modules over groups are introduced by Whitehead [10] (see also [1] Ch. IV, [6]). The problem of group extensions of the type of a crossed module presented in [2]. This closely relates to the problem of prolongations (see [8]). Now, we show that each co-prolongation of a group extension determines a crossed module.

Proposition 2.5. *If there exists a γ-co-prolongation of E_0, then:*

1. E_0 is a central extension,

2. E_0 induces a homomorphism $\theta : G_0 \to \operatorname{Aut}(A \times \operatorname{Ker}\gamma)$ such that $(A \times \operatorname{Ker}\gamma, G_0, \nu p', \theta)$ is a crossed module.

Proof. It follows from Proposition 2.2 and Lemma 2.3 that the diagram (2.3) commutes. In this diagram, since $A \subset Z(A \times \operatorname{Ker}\gamma)$, E_0 is a (id_A, ν)-*prolongation* of the extension E' in the sense of [8]. According to Theorem 10 [8], E_0 is a central extension.

2) This follows from Proposition 2 [8]. The homomorphism $\theta : G_0 \to \operatorname{Aut}(A \times \operatorname{Ker}\gamma)$ is given by

$$\theta_g = \phi_{b_0},\ p_0(b_0) = g_0,$$
$$\phi_{b_0}(x) = \varepsilon^{-1}\mu_{b_0}(\varepsilon x),\ x \in A \times \operatorname{Ker}\gamma.$$

□

3. The obstruction of a co-prolongation

In this section, suppose that the isomorphism $\varphi : G \to \operatorname{Aut} A$ of the *system* (E_0, γ) satisfies (2.1). The "co-prolongation problem" is that of finding whether there is any extension E of A by G which is a co-prolongation of the extension E_0 and, if so, how many of these exist.

Let $\{u(x_0), x_0 \in G_0\}$, $u(0) = 0$, be a set of representatives of G_0 in B_0. This set induces a homomorphism $\varphi_0 : G_0 \to \operatorname{Aut} A$ by (2.2) and a factor set $f_0 : (G_0)^2 \to A$ by

$$f_0(x_0, y_0) = u(x_0) + u(y_0) - u(x_0 + y_0).$$

According to [5] Ch. IV, the function $f_0 : G_0 \times G_0 \to A$ is uniquely defined in terms of $B^2_{\varphi_0}(G_0, A)$, that means the element

$$\overline{f_0} = f_0 + B^2_{\varphi_0}(G_0, A)$$

is completely determined. Thanks to the relation (2.1), the homomorphism $\gamma : G_0 \to G$ induces one

$$\overline{\gamma} : H^2_{\varphi}(G, A) \to H^2_{\varphi_0}(G_0, A)$$
$$\overline{h} \mapsto \overline{\gamma^* h},$$

where $(\gamma^* h)(x_0, y_0) = h(\gamma x_0, \gamma y_0), \forall x_0, y_0 \in G_0$. Then, the element

$$\widetilde{f_0} = \overline{f_0} + \mathrm{Im}(\overline{\gamma}) \in \mathrm{Coker}(\overline{\gamma}),$$

is not dependent on the choice of the representative $u(x_0)$. We call $\widetilde{f_0}$ the *obstruction* of γ-co-prolongation of the extension E_0.

To prove Theorem 3.1 and Corollary 3.2, one represents the factor set f_0 with respect to the factor set $s : G^2 \to \mathrm{Ker}\,\gamma$ of the extension $G_0 \xrightarrow{\gamma} G$. Under the hypothesis of Lemma 2.3, choose a set of representatives $\{u(x_0), x_0 \in G_0\}$ in B_0 as follows. Firstly, choose a set of representatives $\{v(x), x \in G\}, v(1) = 0$, of G in G_0 whose the corresponding factor set is $s : G^2 \to \mathrm{Ker}\,\gamma$. For each $x \in G$, choose an element u_x in B_0 such that

$$p_0(u_x) = v(x),\ u_1 = 0.$$

Since the element $x_0 \in G_0$ is uniquely written as

$$x_0 = c + v(x), \quad c \in \mathrm{Ker}\,\gamma, \quad x \in G,$$

we set

$$u(x_0) = jc + u_x. \tag{3.1}$$

Let f_0 be the factor set of B_0 corresponding to this factor set. For $x_0, y_0 \in G_0$, one has

$$x_0 = c + v(x),\ y_0 = d + v(y),\ c, d \in \mathrm{Ker}\,\gamma,\ x, y \in G.$$

It follows that

$$\begin{aligned} x_0 + y_0 &= (c + v(x)) + (d + v(y)) \\ &= c + \mu_{v(x)}(d) + v(x) + v(y) \\ &= c + \mu_{v(x)}(d) + s(x, y) + v(xy). \end{aligned}$$

Then,

$$c_0 = c + \mu_{v(x)}(d) + s(x, y) \in \mathrm{Ker}\,\gamma, \tag{3.2}$$

hence the relation (3.1) implies

$$u(x_0 + y_0) = jc_0 + u_{xy}.$$

Simple calculations lead to

$$f_0(x_0, y_0) = jc + \mu_{u_x}(jd) + (u_x + u_y - u_{xy}) - jc_0. \tag{3.3}$$

Theorem 3.1. *Co-prolongations of E_0 exist if and only if $\widetilde{f_0}$ vanishes on* $\mathrm{Coker}(\overline{\gamma})$.

Proof. Necessary condition. Let E be a γ-co-prolongation of E_0. Choose in B_0 a set of representatives $u(x_0), x_0 \in G_0$, such that the induced obstruction $\widetilde{f_0}$ vanishes in $\operatorname{Coker}(\overline{\gamma})$.

If $j : \operatorname{Ker}\gamma \to \operatorname{Ker}\beta$ is the isomorphism mentioned in Proposition 2.2, then $p_0 j = id_{\operatorname{Ker}\gamma}$. The set of representatives $u(x_0), x_0 \in G_0$, chosen by (3.1) in B_0, gives a factor set f_0 satisfying (3.3).

Since

$$p\beta(u_x) = \gamma p_0(u_x) = \gamma(v(x)) = x,$$

hence $\{r(x) = \beta(u_x), x \in G\}$ is a set of representatives of G in B (clearly, $r(1) = 0$). Let f be a factor set of B corresponding to this set of representatives, we prove that

$$f_0 = \gamma^* f.$$

Since $jc, jc_0, \mu_{u_x}(jd)$ are in $\operatorname{Ker}\beta$, act β on two sides of the equality (3.3) (note that $\beta|_A = id_A$), one has

$$\begin{aligned} f_0(x_0, y_0) &= \beta u_x + \beta u_y - \beta u_{xy} \\ &= r(x) + r(y) - r(xy) = f(x, y). \end{aligned}$$

It follows that

$$(\gamma^* f)(x_0, y_0) = f(\gamma x_0, \gamma y_0) = f(x, y) = f_0(x_0, y_0),$$

that is $\gamma^* f = f_0$, and hence $\widetilde{f_0}$ vanishes in $\operatorname{Coker}(\overline{\gamma})$.

Sufficient condition. Let $\widetilde{f_0} = 0 \in \operatorname{Coker}(\overline{\gamma})$, where f_0 is a factor set of E_0. There exists $f \in Z^2(G, A)$ such that

$$f_0 = \gamma^*(f) + \delta t,\ \ \delta t \in B^2(G_0, A).$$

If $\{u(x_0), x_0 \in G_0\}$ is a set of representatives corresponding to the factor set f_0, then one can choose a set of representatives $u'(x_0) = u(x_0) - t(x)$ so that one obtains a new factor set

$$f_0'(x_0, y_0) = (\gamma^* f)(x_0, y_0).$$

According to [5] Ch. IV, there exists an extension E of the crossed product $\overline{B} = [A, \varphi, f, G]$. This is a γ-co-prolongation of the extension E_0. Indeed, consider the diagram

$$\begin{array}{cccccccccc} E_0: & 0 & \longrightarrow & A & \xrightarrow{i_0} & B_0 & \xrightarrow{p_0} & G_0 & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ E: & 0 & \longrightarrow & A & \xrightarrow{i} & \overline{B} & \xrightarrow{p} & G & \longrightarrow & 1 \end{array}$$

where $i : a \mapsto (a, 1)$; $p : (a, x) \mapsto x$; $\beta : a + u'(x_0) \mapsto (a, \gamma x_0)$. Clearly, β is a group homomorphism making the above diagram commute, that means E is a co-prolongation of E_0. □

Corollary 3.2. *Let* (E_0, γ) *and* $\varphi : G \to \operatorname{Aut} A$ *satisfy* (2.1). *If the onto-homomorphism* $\gamma : G_0 \to G$ *is split and there is a normal monomorphism* $j : \operatorname{Ker}\gamma \to B_0$ *such that* $p_0 j = id_{\operatorname{Ker}\gamma}$, *then* γ-*co-prolongations of the extension* E_0 *exist.*

Proof. Since the onto-homomorphism $\gamma : G_0 \to G$ is split, there is a normal monomorphism $v : G \to G_0$ such that $\gamma v = id_G$. Then,

$$G_0 = \mathrm{Ker}\,\gamma \times \mathrm{Im}v.$$

We choose a set of representatives $\{v(x) \mid x \in G\}$ of G in G_0 respect to v. The corresponding factor set in G_0 is $s = 0$. Choose a set of representatives $\{u(x_0), x_0 \in G_0\}$ by (3.1) as in the proof of Theorem 3.1. Also, by (3.1)

$$u(x_0 + y_0) = j(c + d) + u_{xy}.$$

Since j is a normal monomorphism,

$$u_x + jd - u_x = j(d'),\ d, d' \in \mathrm{Ker}\,\gamma.$$

Act p_0 on two sides of the above equality, we have $v(x) + d - v(x) = d'$. Since $G_0 = \mathrm{Ker}\,\gamma \times \mathrm{Im}v$, $d' = d$, that means $\mu_{u_x}(jd) = jd$. Since $s = 0$, the relation (3.2) becomes $c_0 = c + d$. Besides,

$$u_x + u_y - u_{xy} = u(v(x)) + u(v(y)) - u(v(x)v(y)) = f_0(v(x), v(y)) \in A.$$

Then, it follows from $A \cap j(\mathrm{Ker}\,\gamma) = 0$ that each element of A commutes with each element of $j(\mathrm{Ker}\,\gamma)$. Thus, equality (3.3) turns into

$$f_0(x_0, y_0) = f_0(v(x), v(y)). \tag{3.4}$$

Now, define a function $f : G^2 \to A$ by

$$f(x, y) = f_0(v(x), v(y)),\ x, y \in G. \tag{3.5}$$

The relation (2.1) and the fact that $f_0 \in Z^3_{\varphi_0}(G_0, A)$ imply $f \in Z^3_{\varphi}(G, A)$. Clearly,

$$(\gamma^* f)(x_0, y_0) = f(\gamma x_0, \gamma y_0) = f((x, y) \overset{(3.5)}{=} f_0(v(x), v(y)) \overset{(3.4)}{=} f_0(x_0, y_0).$$

Thus, $\widetilde{f_0} = 0$ in $\mathrm{Coker}(\overline{\gamma})$, and hence by Theorem 3.1, there exist co-prolongations of E_0 by γ. □

Theorem 3.3 (Classification theorem)**.** *If the system* (E_0, γ) *together with the homomorphism* $\varphi : G \to \mathrm{Aut}\,A$ *satisfying the relation* (2.1) *have* γ*-co-prolongations, then the set of equivalence classes of* γ*-co-prolongations is a torseur under the group* $K = \mathrm{Ker}(\overline{\gamma})$*, where*

$$\overline{\gamma} : H^2_{\varphi}(G, A) \to H^2_{\varphi_0}(G_0, A)$$

is a homomorphism induced by γ*.*

Proof. Firstly, observe that each extension of G by A inducing φ is isomorphic to the extension of the crossed product $[A, \varphi, f, G]$, where f is uniquely determined up to a coboundary $\delta t \in B^2_{\varphi}(G, A)$.

Let U be the set of equivalence classes of co-prolongations of the extension E. To prove that U is a torseur under $K = \mathrm{Ker}(\overline{\gamma})$, one constructs a map

$$\Lambda : \mathrm{Ker}(\overline{\gamma}) = K \to \mathrm{Aut}(U)$$

by the formula

$$\Lambda(\overline{h})(cls[A, \varphi, f, G]) = cls[A, \varphi f, h, G].$$

Thanks to the above observation, this formula is not dependent on the representative element of the class $\overline{h}$, as well as the representative f. Thus, Λ is well defined. Further, $\Lambda(\overline{h})$ is actually an element of the group of transformations of U. Clearly, Λ is a group homomorphism.

It remains to prove that for any two co-prolongations E_1, E_2 of the extension E, there uniquely exists an element $\overline{h} \in \mathrm{Ker}(\overline{\gamma})$ such that

$$clsE_2 = \Lambda(\overline{h})(clsE_1).$$

Indeed, one has $clsE_1 = cls[G, \varphi, f_i, A], i = 1, 2$, where $\overline{\gamma f_i} = \overline{f}$. By setting $h = -f_1 + f_2$, the proof is completed. □

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