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THE FIRST AND SECOND ZAGREB INDICES OF HYPERGRAPHS

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ABSTRACT. Let $\mathcal{H}$ be a hypergraph on the non-empty finite vertex set $V(\mathcal{H})$ with the hyperedge set $E(\mathcal{H})$, where each hyperedge $e \in E(\mathcal{H})$ is a subset of $V(\mathcal{H})$ with at least two vertices. The bounds on the first and second Zagreb indices of hypergraphs, weak bipartite hypergraphs, hypertrees, k-uniform hypergraphs, k-uniform weak bipartite hypergraphs, and k-uniform hypertrees are discussed.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $d_G(v)$ be its degree. The first and second Zagreb indices, that were introduced by Gutman and Trinajstić [7] in 1972, defined as

$$\mathcal{M}_1 = \mathcal{M}_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)),$$

$$\mathcal{M}_2 = \mathcal{M}_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

are widely studied topological indices. The Zagreb indices and their variants have been used to study molecular complexity, chirality, ZE-isomerism and heterosystems whilst the overall Zagreb indices exhibited a potential applicability for deriving multilinear regression models. Zagreb indices are also used by various researchers in their QSPR and QSAR studies. For details of the mathematical theory and chemical applications of the Zagreb indices, see the surveys [6, 10] and the literatures there.

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A hypergraph $\mathcal{H}$ consists of a non-empty finite vertex set $V(\mathcal{H})$ and a hyperedge set $E(\mathcal{H})$, where each hyperedge $e \in E(\mathcal{H})$ is a subset of $V(\mathcal{H})$ with at least two vertices. Let $\mathcal{H}$ be a hypergraph, $v \in V(\mathcal{H})$ and $e \in E(\mathcal{H})$. If $v \in e$ then we say e contains v or v is contained in e. The degree of v in $\mathcal{H}$, denoted by $d_{\mathcal{H}}(v)$, is the number of hyperedges that contain v. If $d_{\mathcal{H}}(v) = 1$, then v is called a pendent vertex.

Hypergraphs find application in chemistry when modeling molecules or chemical reactions involving multiple atoms bonding simultaneously. Unlike graphs, hypergraphs can represent interactions involving more than two atoms, which is particularly relevant for reactions with complex bonding patterns or in capturing molecular properties that arise from multiple atom groupings. Hypergraphs offer a more accurate depiction of certain chemical scenarios, such as transition states in reactions, which involve multiple atoms simultaneously changing their bonding configurations [8]. Recently, the idea of topological indices is extended from graph to hypergraph [1, 3, 4, 5, 11, 12, 13, 14, 15, 16]. In particular, the degree-based indices of hypergraphs were first extensively studied in [14], and the expected generalisation of some vertex degree based topological indices from graphs to hypergraphs has been listed in [12].

For a hypergraph $\mathcal{H}$, the first and second Zagreb indices of $\mathcal{H}$ are defined as [12]

$$\mathcal{M}_1(\mathcal{H}) = \sum_{e \in E(\mathcal{H})} \sum_{v \in e} d_{\mathcal{H}}(v), \quad \mathcal{M}_2(\mathcal{H}) = \sum_{e \in E(\mathcal{H})} \prod_{v \in e} d_{\mathcal{H}}(v),$$

respectively.

In this paper, the bounds for the first and second Zagreb indices of hypergraphs and hypertrees are discussed. This paper is organized as follows. In Section 2, some terminologies and definitions are introduced. In Section 3, the sharp upper and lower bounds on the first and second Zagreb indices of connected hypergraphs and weak bipartite hypergraphs with n vertices are obtained, and the extremal hypergraphs are characterized, respectively. The sharp upper bounds on the first and second Zagreb indices of connected k-uniform hypergraphs and k-uniform weak bipartite hypergraphs with n vertices are obtained, and the extremal hypergraphs are characterized, respectively. In Section 4, the sharp upper and lower bounds on the first and second Zagreb indices of hypertrees with $n = m + p$ vertices ($p \geq 1$) and m hyperedges are obtained, and the extremal hypertrees are characterized, respectively. The sharp upper and lower bounds on the first and second Zagreb indices of k-uniform hypertrees with m hyperedges are obtained, and the extremal hypertrees are characterized, respectively.

2. Terminologies and definitions

Let $\mathcal{H}$ be a hypergraph. For a hyperedge $e \in E(\mathcal{H})$, the degree of e, denoted by $|e|$, is the number of vertices that are contained in e. If e contains exactly $|e| - 1$ pendent vertices, then e is called a pendent hyperedge. Two hyperedges in a hypergraph are said to be adjacent if they have at least one vertex in common.

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A walk in a hypergraph $\mathcal{H}$ is a sequence of vertices and hyperedges, $(v_0, e_1, v_1, e_2, v_2, \ldots, e_t, v_t)$, with $\{v_{i-1}, v_i\} \subseteq e_i$ and $v_{i-1} \neq v_i$ for $i = 1, 2, \ldots, t$. A walk w in $\mathcal{H}$ is called a path if all e_i's and all v_i's are distinct in w and, is called a cycle if all e_i's and all v_i's are distinct except $v_0 = v_t$. A hypergraph $\mathcal{H}$ is connected, if there exists a path between any two vertices in $\mathcal{H}$.

A hypergraph $\mathcal{H}$ is said to be linear, if any two hyperedges in $\mathcal{H}$ have at most one vertex in common. A hypergraph with $|e| = k$ ($k \geq 2$) for every $e \in E(\mathcal{H})$ is called a k-uniform hypergraph. Especially, 2-uniform hypergraph is the ordinary graph.

A sunflower hypergraph $\mathcal{S}(m, p, k)$ is a k-uniform hypergraph, with $m \geq 1$, and $1 \leq p < k$ is defined as follows [2]. Let A be a set of p vertices called seeds and define m disjoint sets $\{B_i\}_{i=1}^{m}$ of $k - p$ vertices each, called petals. The hyperedge set of $\mathcal{S}(m, p, k)$ consists of $A \cup \{B_i\}$, $1 \leq i \leq m$.

A hypergraph on the vertex set V with $|V| = n \geq 2$ and hyperedge set being the collection of all possible subsets of V with at least two vertices is a complete hypergraph and is denoted by $\mathcal{K}_n$.

A weak bipartite hypergraph $\mathcal{H}(V = V_1 \cup V_2, E)$ is a hypergraph whose vertex set V can be partitioned into non-empty subsets V_1 and V_2 such that, every hyperedge in E contains at least one vertex from each of the partition V_1 and V_2. A complete weak bipartite hypergraph, denoted by $\mathcal{K}_{p,q}$, is the weak bipartite hypergraph $\mathcal{H}(V_1 \cup V_2, E)$, with $|V_1| = p$, $|V_2| = q$, and all possible hyperedges such that every hyperedge contains at least one vertex from each of the partition V_1 and V_2.

A hypertree $\mathcal{T}$ is a connected hypergraph in which removal of any hyperedge in $\mathcal{T}$ disconnects the hypergraph and it is important to note that a hypertree can contain cycle [9]. A hyperpath is a hypertree with conditions that the degree of every vertex is at most two and a hyperedge is adjacent to at most two other hyperedges. A hyperstar is defined as a hypertree whose all hyperedges are pendent hyperedges.

3. Hypergraph and weak bipartite hypergraph

We agree that for a positive integer n, $\binom{n}{0} = 1$, and $\binom{n}{m} = 0$ if $m > n$.

Lemma 3.1. *Let $n \geq 2$. Then*

$$\mathcal{M}_1(\mathcal{K}_n) = n\left(2^{n-1} - 1\right)^2,$$
$$\mathcal{M}_2(\mathcal{K}_n) = 2^{n(n-1)} - n\left(2^{n-1} - 1\right) - 1.$$

Proof. It is easy to observe that there is $\binom{n}{i}$ hyperedges of degree i in $\mathcal{K}_n$ for $2 \leq i \leq n$, and for any vertex v in $\mathcal{K}_n$,

$$d_{\mathcal{K}_n}(v) = \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1} = 2^{n-1} - 1.$$

Then

$$\mathcal{M}_1(\mathcal{K}_n) = \sum_{e \in E(\mathcal{K}_n)} \sum_{v \in e} d_{\mathcal{K}_n}(v)$$

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$$
\begin{aligned}
&= \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=2}} \sum_{v\in e} d_{\mathcal{K}_n}(v) + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=3}} \sum_{v\in e} d_{\mathcal{K}_n}(v) + \cdots + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=n}} \sum_{v\in e} d_{\mathcal{K}_n}(v) \\
&= \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=2}} 2\left(2^{n-1}-1\right) + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=3}} 3\left(2^{n-1}-1\right) + \cdots + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=n}} n\left(2^{n-1}-1\right) \\
&= \left(2^{n-1}-1\right)\sum_{i=2}^{n} i\binom{n}{i} = n\left(2^{n-1}-1\right)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}_2(\mathcal{K}_n) &= \sum_{e\in E(\mathcal{K}_n)} \prod_{v\in e} d_{\mathcal{K}_n}(v) \\
&= \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=2}} \prod_{v\in e} d_{\mathcal{K}_n}(v) + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=3}} \prod_{v\in e} d_{\mathcal{K}_n}(v) + \cdots + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=n}} \prod_{v\in e} d_{\mathcal{K}_n}(v) \\
&= \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=2}} \left(2^{n-1}-1\right)^2 + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=3}} \left(2^{n-1}-1\right)^3 + \cdots + \sum_{\substack{e\in E(\mathcal{K}_n)\\|e|=n}} \left(2^{n-1}-1\right)^n \\
&= \sum_{i=2}^{n} \binom{n}{i} \left(2^{n-1}-1\right)^i = 2^{n(n-1)} - n\left(2^{n-1}-1\right) - 1.
\end{aligned}
$$

□

Lemma 3.2. *Let $p, q \geq 1$. Then*

$$
\begin{aligned}
\mathcal{M}_1(\mathcal{K}_{p,q}) &= \sum_{k=2}^{p+q} \sum_{i=1}^{k-1} \binom{p}{i} \binom{q}{k-i} \left(i\cdot 2^{p-1}(2^q-1) + (k-i)\cdot 2^{q-1}(2^p-1)\right), \\
\mathcal{M}_2(\mathcal{K}_{p,q}) &= \sum_{k=2}^{p+q} \sum_{i=1}^{k-1} \binom{p}{i} \binom{q}{k-i} \left(2^{p-1}(2^q-1)\right)^i \left(2^{q-1}(2^p-1)\right)^{k-i}.
\end{aligned}
$$

Proof. Let $V(\mathcal{K}_{p,q}) = V_1 \cup V_2$ with $|V_1| = p$ and $|V_2| = q$. Then

$$
\begin{aligned}
\mathcal{M}_1(\mathcal{K}_{p,q}) &= \sum_{e\in E(\mathcal{K}_{p,q})} \sum_{w\in e} d_{\mathcal{K}_{p,q}}(w) \\
&= \sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=2}} \sum_{w\in e} d_{\mathcal{K}_{p,q}}(w) + \sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=3}} \sum_{w\in e} d_{\mathcal{K}_{p,q}}(w) + \cdots + \sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=p+q}} \sum_{w\in e} d_{\mathcal{K}_{p,q}}(w),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}_2(\mathcal{K}_{p,q}) &= \sum_{e\in E(\mathcal{K}_{p,q})} \prod_{w\in e} d_{\mathcal{K}_{p,q}}(w) \\
&= \sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=2}} \prod_{w\in e} d_{\mathcal{K}_{p,q}}(w) + \sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=3}} \prod_{w\in e} d_{\mathcal{K}_{p,q}}(w) + \cdots + \sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=p+q}} \prod_{w\in e} d_{\mathcal{K}_{p,q}}(w).
\end{aligned}
$$

For any $u \in V_1$ and $v \in V_2$,

$$d_{\mathcal{K}_{p,q}}(u) = \left(\sum_{i=0}^{p-1}\binom{p-1}{i}\right)\left(\sum_{j=1}^{q}\binom{q}{j}\right) = 2^{p-1}(2^q-1),$$
$$d_{\mathcal{K}_{p,q}}(v) = \left(\sum_{i=0}^{q-1}\binom{q-1}{i}\right)\left(\sum_{j=1}^{p}\binom{p}{j}\right) = 2^{q-1}(2^p-1).$$

For any $e \in E(\mathcal{K}_{p,q})$ with $|e| = k$, we use $|e \cap V_1|$ and $|e \cap V_2|$ to represent the number of vertices in V_1 and V_2 contained in the hyperedge e, respectively. Then $1 \le |e \cap V_1| \le k-1$, $1 \le |e \cap V_2| \le k-1$, and $|e \cap V_1| + |e \cap V_2| = k$. So

$$\begin{aligned}
\sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=k}}\sum_{w\in e} d_{\mathcal{K}_{p,q}}(w) &= \sum_{i=1}^{k-1}\sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=k\\|e\cap V_1|=i}}\sum_{w\in e} d_{\mathcal{K}_{p,q}}(w)\\
&= \sum_{i=1}^{k-1}\sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=k\\|e\cap V_1|=i}} \left(i\cdot 2^{p-1}(2^q-1) + (k-i)\cdot 2^{q-1}(2^p-1)\right)\\
&= \sum_{i=1}^{k-1}\binom{p}{i}\binom{q}{k-i}\left(i\cdot 2^{p-1}(2^q-1) + (k-i)\cdot 2^{q-1}(2^p-1)\right)\\
&= \sum_{i=1}^{k-1}\binom{p}{i}\binom{q}{k-i}\left(i\left(2^{q-1}-2^{p-1}\right) + k\left(2^p-1\right)2^{q-1}\right),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=k}}\prod_{w\in e} d_{\mathcal{K}_{p,q}}(w) &= \sum_{i=1}^{k-1}\sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=k\\|e\cap V_1|=i}}\prod_{w\in e} d_{\mathcal{K}_{p,q}}(w)\\
&= \sum_{i=1}^{k-1}\sum_{\substack{e\in E(\mathcal{K}_{p,q})\\|e|=k\\|e\cap V_1|=i}} \left(2^{p-1}(2^q-1)\right)^i\left(2^{q-1}(2^p-1)\right)^{k-i}\\
&= \sum_{i=1}^{k-1}\binom{p}{i}\binom{q}{k-i}\left(2^{p-1}(2^q-1)\right)^i\left(2^{q-1}(2^p-1)\right)^{k-i}\\
&= \sum_{i=1}^{k-1}\binom{p}{i}\binom{q}{k-i}2^{i(p-q)+k(q-1)}(2^q-1)^i(2^p-1)^{k-i}.
\end{aligned}$$

Thus

$$\mathcal{M}_1(\mathcal{K}_{p,q}) = \sum_{k=2}^{p+q}\sum_{i=1}^{k-1}\binom{p}{i}\binom{q}{k-i}\left(i\cdot 2^{p-1}(2^q-1) + (k-i)\cdot 2^{q-1}(2^p-1)\right),$$

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$$\mathcal{M}_2(\mathcal{K}_{p,q}) = \sum_{k=2}^{p+q}\sum_{i=1}^{k-1}\binom{p}{i}\binom{q}{k-i}\left(2^{p-1}(2^q-1)\right)^i\left(2^{q-1}(2^p-1)\right)^{k-i}.$$

The lemma holds. □

Let $\mathcal{K}_{p,q}^{(k)}$ be the weak bipartite hypergraph with bipartition $V = V_1 \cup V_2$ and hyperedge set as all possible k-element subsets of V such that every subset contains at least one vertex from each of V_1 and V_2, where $p, q \geq 1$ and $2 \leq k \leq p+q$. We call $\mathcal{K}_{p,q}^{(k)}$ the complete k-uniform weak bipartite hypergraph.

Lemma 3.3. *Let $p, q \geq 1$ and $2 \leq k \leq p+q$. Then*

$$\mathcal{M}_1(\mathcal{K}_{p,q}^{(k)}) = \sum_{j=1}^{k-1}\binom{p}{j}\binom{q}{k-j}\left(j\sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q}{k-1-i} + (k-j)\sum_{i=0}^{k-2}\binom{q-1}{i}\binom{p}{k-1-i}\right),$$

$$\mathcal{M}_2(\mathcal{K}_{p,q}^{(k)}) = \sum_{j=1}^{k-1}\binom{p}{j}\binom{q}{k-j}\left(\sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q}{k-1-i}\right)^j\left(\sum_{i=0}^{k-2}\binom{q-1}{i}\binom{p}{k-1-i}\right)^{k-j}.$$

Proof. Let $V(\mathcal{K}_{p,q}^{(k)}) = V_1 \cup V_2$ with $|V_1| = p$ and $|V_2| = q$. It is easy to see that for any $u \in V_1$ and $v \in V_2$,

$$d_{\mathcal{K}_{p,q}^{(k)}}(u) = \sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q}{k-1-i},$$

$$d_{\mathcal{K}_{p,q}^{(k)}}(v) = \sum_{i=0}^{k-2}\binom{q-1}{i}\binom{p}{k-1-i}.$$

For any $e \in E(\mathcal{K}_{p,q}^{(k)})$, we use $|e \cap V_1|$ and $|e \cap V_2|$ to represent the number of vertices in V_1 and V_2 contained in the hyperedge e, respectively. Then $1 \leq |e \cap V_1| \leq k-1$, $1 \leq |e \cap V_2| \leq k-1$, and $|e \cap V_1| + |e \cap V_2| = k$. Then

$$\begin{aligned}\mathcal{M}_1(\mathcal{K}_{p,q}^{(k)}) &= \sum_{e\in E(\mathcal{K}_{p,q}^{(k)})}\sum_{w\in e} d_{\mathcal{K}_{p,q}^{(k)}}(w) = \sum_{j=1}^{k-1}\sum_{\substack{e\in E(\mathcal{K}_{p,q}^{(k)})\\ |e\cap V_1|=j}}\sum_{w\in e} d_{\mathcal{K}_{p,q}^{(k)}}(w)\\ &= \sum_{j=1}^{k-1}\binom{p}{j}\binom{q}{k-j}\left(j\sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q}{k-1-i} + (k-j)\sum_{i=0}^{k-2}\binom{q-1}{i}\binom{p}{k-1-i}\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_2(\mathcal{K}_{p,q}^{(k)}) &= \sum_{e\in E(\mathcal{K}_{p,q}^{(k)})}\prod_{w\in e} d_{\mathcal{K}_{p,q}^{(k)}}(w) = \sum_{j=1}^{k-1}\sum_{\substack{e\in E(\mathcal{K}_{p,q}^{(k)})\\ |e\cap V_1|=j}}\prod_{w\in e} d_{\mathcal{K}_{p,q}^{(k)}}(w)\\ &= \sum_{j=1}^{k-1}\binom{p}{j}\binom{q}{k-j}\left(\sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q}{k-1-i}\right)^j\left(\sum_{i=0}^{k-2}\binom{q-1}{i}\binom{p}{k-1-i}\right)^{k-j}.\end{aligned}$$

The lemma holds. □

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Theorem 3.4. *Let $\mathcal{H}$ be a connected hypergraph with $n \geq 2$ vertices. Then*

$$n \leq \mathcal{M}_1(\mathcal{H}) \leq n\left(2^{n-1}-1\right)^2,$$
$$1 \leq \mathcal{M}_2(\mathcal{H}) \leq 2^{n(n-1)} - n\left(2^{n-1}-1\right) - 1,$$

the lower bounds are attained by the hypergraph with the hyperedge set $E = \{V\}$, and the upper bounds are attained by the complete hypergraph $\mathcal{K}_n$.

Proof. Since the hypergraph $\mathcal{H}$ is connected, the lower bounds are trivial and are attained when $\mathcal{H}$ has only one hyperedge which contains all the vertices of $\mathcal{H}$.

It is easy to observe that the upper bounds are attained by the complete hypergraph $\mathcal{K}_n$. So by Lemma 3.1, the theorem holds. □

The complete k-uniform hypergraph on n vertices, denoted by $\mathcal{K}_n^{(k)}$, is a hypergraph on the vertex set V with $|V| = n$ and hyperedge set as all possible k-element subsets of V, where $2 \leq k \leq n$. By [14, Theorem 2.1], the following theorem holds.

Theorem 3.5. *Let $2 \leq k \leq n$, and $\mathcal{H}$ be a connected k-uniform hypergraph with n vertices. Then*

$$\mathcal{M}_1(\mathcal{H}) \leq k\binom{n}{k}\binom{n-1}{k-1},$$
$$\mathcal{M}_2(\mathcal{H}) \leq \binom{n}{k}\binom{n-1}{k-1}^k,$$

the equalities hold if and only if $\mathcal{H}$ is $\mathcal{K}_n^{(k)}$.

Theorem 3.6. *Let $\mathcal{H} = \mathcal{H}(V_1 \cup V_2, E)$ be a connected weak bipartite hypergraph on $p+q$ vertices, where $|V_1| = p \geq 1$ and $|V_2| = q \geq 1$. Then*

$$p+q \leq \mathcal{M}_1(\mathcal{H}) \leq \mathcal{M}_1(\mathcal{K}_{p,q}),$$
$$1 \leq \mathcal{M}_2(\mathcal{H}) \leq \mathcal{M}_2(\mathcal{K}_{p,q}),$$

the lower bounds are attained by the hypergraph with the hyperedge set $E = \{V\}$, and the upper bounds are attained by the complete weak bipartite hypergraph $\mathcal{K}_{p,q}$.

Proof. The lower bounds are clear. Since the maximum number of possible hyperedges in a weak bipartite hypergraph $\mathcal{H} = \mathcal{H}(V_1 \cup V_2, E)$ is attained by the complete weak bipartite hypergraph $\mathcal{K}_{p,q}$. Then $\mathcal{M}_i(\mathcal{H}) \leq \mathcal{M}_i(\mathcal{K}_{p,q})$ for $i = 1, 2$, and the equality holds if and only if $\mathcal{H}$ is isomorphic to $\mathcal{K}_{p,q}$. □

Theorem 3.7. *Let $\mathcal{H} = \mathcal{H}(V_1 \cup V_2, E)$ be a connected k-uniform weak bipartite hypergraph on $p+q$ vertices, where $|V_1| = p \geq 1$, $|V_2| = q \geq 1$, and $2 \leq k \leq p+q$. Then for $i = 1, 2$,*

$$\mathcal{M}_i(\mathcal{H}) \leq \mathcal{M}_i(\mathcal{K}_{p,q}^{(k)}),$$

the upper bounds are attained by the complete k-uniform weak bipartite hypergraph $\mathcal{K}_{p,q}^{(k)}$.

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Proof. Since the maximum number of possible hyperedges in a k-uniform weak bipartite hypergraph $\mathcal{H} = \mathcal{H}(V_1 \cup V_2, E)$ is attained by the complete k-uniform weak bipartite hypergraph $\mathcal{K}_{p,q}^{(k)}$. Then $\mathcal{M}_i(\mathcal{H}) \leq \mathcal{M}_i(\mathcal{K}_{p,q}^{(k)})$ for $i = 1, 2$, and the equality holds if and only if $\mathcal{H}$ is isomorphic to $\mathcal{K}_{p,q}^{(k)}$. □

4. Hypertree

Firstly, we consider some special hypertrees, such as the hyperstar, hyperpath, and so on.

Lemma 4.1. *Let $n > m \geq 2$, and $\mathcal{S}_{n,m}$ be a hyperstar on n vertices with m hyperedges. Then*

$$\mathcal{M}_1(\mathcal{S}_{n,m}) = n + m^2 - 1,$$
$$\mathcal{M}_2(\mathcal{S}_{n,m}) = m^2.$$

Proof. In $\mathcal{S}_{n,m}$, every hyperedge is a pendent hyperedge, the degree of the central vertex is m, and the degree of every vertex except the central vertex, is one. Thus

$$\begin{aligned}
\mathcal{M}_1(\mathcal{S}_{n,m}) &= \sum_{e \in E(\mathcal{S}_{n,m})} \sum_{v \in e} d_{\mathcal{S}_{n,m}}(v) \\
&= \sum_{e \in E(\mathcal{S}_{n,m})} (m + |e| - 1) = m(m-1) + \sum_{e \in E(\mathcal{S}_{n,m})} |e| \\
&= m(m-1) + (n + m - 1) = n + m^2 - 1, \\
\mathcal{M}_2(\mathcal{S}_{n,m}) &= \sum_{e \in E(\mathcal{S}_{n,m})} \prod_{v \in e} d_{\mathcal{S}_{n,m}}(v) = \sum_{e \in E(\mathcal{S}_{n,m})} m = m^2.
\end{aligned}$$

□

Lemma 4.2. *Let $\mathcal{S}_m^{(k)}$ be the k-uniform hyperstar with m hyperedges, where $k, m \geq 2$. Then*

$$\mathcal{M}_1(\mathcal{S}_m^{(k)}) = m(m + k - 1),$$
$$\mathcal{M}_2(\mathcal{S}_m^{(k)}) = m^2.$$

Proof. It is easy to see the number of the vertices of $\mathcal{S}_m^{(k)}$ is $(k-1)m + 1$. Then by Lemma 4.1, the lemma holds. □

Recall that a linear hyperpath $\overline{\mathcal{P}}$ is a hyperpath such that any two hyperedges in $\overline{\mathcal{P}}$ have at most one vertex in common. A k-uniform linear hyperpath $\overline{\mathcal{P}}^{(k)}$ is a linear hyperpath such that every hyperedge contains exactly k vertices.

Lemma 4.3. *Let $n > m \geq 2$, and $\overline{\mathcal{P}}_{n,m}$ be a linear hyperpath on n vertices with m hyperedges. Then*

$$\mathcal{M}_1(\overline{\mathcal{P}}_{n,m}) = n + 3m - 3,$$
$$\mathcal{M}_2(\overline{\mathcal{P}}_{n,m}) = 4(m - 1).$$

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Proof. In $\overline{\mathcal{P}}_{n,m}$, the degree of every vertex is at most two, a hyperedge is adjacent to at most two other hyperedges, and any two hyperedges have at most one vertex in common. Then each non-pendent hyperedge contains two vertices of degree two, and there are exactly two pendent hyperedges, say e_1 and e_2, and each of e_1 and e_2 contains one vertex of degree two. So

$$\begin{aligned}
\mathcal{M}_1(\overline{\mathcal{P}}_{n,m}) &= \sum_{e\in E(\overline{\mathcal{P}}_{n,m})}\sum_{v\in e} d_{\overline{\mathcal{P}}_{n,m}}(v)\\
&= \sum_{v\in e_1} d_{\overline{\mathcal{P}}_{n,m}}(v) + \sum_{v\in e_2} d_{\overline{\mathcal{P}}_{n,m}}(v) + \sum_{e\in E(\overline{\mathcal{P}}_{n,m})\setminus\{e_1,e_2\}}\sum_{v\in e} d_{\overline{\mathcal{P}}_{n,m}}(v)\\
&= (|e_1|+1) + (|e_2|+1) + \sum_{e\in E(\overline{\mathcal{P}}_{n,m})\setminus\{e_1,e_2\}} (|e|+2)\\
&= 2 + 2(m-2) + \sum_{e\in E(\overline{\mathcal{P}}_{n,m})} |e|\\
&= 2m-2+(n+m-1) = n+3m-3,\\
\mathcal{M}_2(\overline{\mathcal{P}}_{n,m}) &= \sum_{e\in E(\overline{\mathcal{P}}_{n,m})}\prod_{v\in e} d_{\overline{\mathcal{P}}_{n,m}}(v)\\
&= \prod_{v\in e_1} d_{\overline{\mathcal{P}}_{n,m}}(v) + \prod_{v\in e_2} d_{\overline{\mathcal{P}}_{n,m}}(v) + \sum_{e\in E(\overline{\mathcal{P}}_{n,m})\setminus\{e_1,e_2\}}\prod_{v\in e} d_{\overline{\mathcal{P}}_{n,m}}(v)\\
&= 2+2+\sum_{e\in E(\overline{\mathcal{P}}_{n,m})\setminus\{e_1,e_2\}} 4\\
&= 4+4(m-2) = 4(m-1).
\end{aligned}$$

□

Lemma 4.4. *Let $\overline{\mathcal{P}}^{(k)}_m$ be the k-uniform linear hyperpath with m hyperedges, where $k,m\geq 2$. Then*

$$\mathcal{M}_1(\overline{\mathcal{P}}^{(k)}_m) = m(k+2)-2,$$
$$\mathcal{M}_2(\overline{\mathcal{P}}^{(k)}_m) = 4(m-1).$$

Proof. It is easy to see the number of the vertices of $\overline{\mathcal{P}}^{(k)}_m$ is $km-(m-1)$. Then by Lemma 4.3, the lemma holds. □

Lemma 4.5. *Let $n>m\geq 2$, and $\mathcal{P}_{n,m}$ be a hyperpath on n vertices with m hyperedges. Then*

$$\mathcal{M}_1(\mathcal{P}_{n,m}) \geq n+3m-3,$$
$$\mathcal{M}_2(\mathcal{P}_{n,m}) \geq 4(m-1),$$

the equalities hold if and only if $\mathcal{P}_{n,m}$ is isomorphic to a linear hyperpath $\overline{\mathcal{P}}_{n,m}$.

Proof. In $\mathcal{P}_{n,m}$, the degree of every vertex is at most two, and a hyperedge is adjacent to at most two other hyperedges. Then in order to minimise $\mathcal{M}_1(\mathcal{P}_{n,m})$ and $\mathcal{M}_2(\mathcal{P}_{n,m})$, the number vertices of degree two must be minimal. So any two adjacent hyperedges have exactly one vertex in common. It implies that $\mathcal{P}_{n,m}$ is a linear hyperpath. By Lemma 4.3, the lemma holds. □

Using calculus, we can get the following lemma.

Lemma 4.6. *Let $a \geq 1$ and $c \geq 4$ be two given positive integers. Then for any $x \geq 2, y \geq 2$ with $x + y = c$, we have*

(1) $2^x(1 + 2^y) + 2^{y+a} < 2(1 + 2^{c-1}) + 2^{(c-1)+a}$;

(2) $2^x(2 + 2^y) + 2^{y+a} < 2(2 + 2^{c-1}) + 2^{(c-1)+a}$; *and*

(3) $2^{x+2} + 2^{y+2} < 2^3 + 2^{c+1}$.

Let $\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})$ be a hyperpath on n vertices with m hyperedges as shown in Figure 3.1, where $k_i \geq 1$ is the number of common vertices of the hyperedges e_i and e_{i+1} for $i = 1, 2, \ldots, m-1$, and $k_1 + k_2 + \cdots + k_{m-1} + 2 = n$.

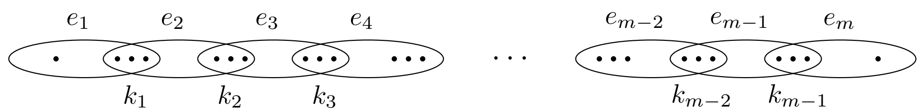

FIGURE 3.1. The hyperpath $\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})$

Lemma 4.7. *Let $n > m \geq 2$, and $\mathcal{P}_{n,m}$ be a hyperpath on n vertices with m hyperedges. Then*

$$n + 3m - 3 \leq \mathcal{M}_1(\mathcal{P}_{n,m}) \leq 4n - 6, \tag{4.1}$$

$$4(m-1) \leq \mathcal{M}_2(\mathcal{P}_{n,m}) \leq \begin{cases} 2^{n-1}, & \text{if } m = 2, \\ 2^{n-2} + 2^{n-3} + 2, & \text{if } m = 3, \\ 2^{n-m+2} + 4(m-3), & \text{if } m \geq 4. \end{cases} \tag{4.2}$$

The left equalities in (4.1) and (4.2) hold if and only if $\mathcal{P}_{n,m}$ is isomorphic to a linear hyperpath $\overline{\mathcal{P}}_{n,m}$. The right equality in (4.1) holds if and only if $\mathcal{P}_{n,m}$ is isomorphic to the hyperpath $\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})$ as shown in Figure 3.1. The right equality in (4.2) holds if and only if

- *for $m = 2$, $\mathcal{P}_{n,2}$ is isomorphic to $\widetilde{\mathcal{P}}_{n,2}(n-2)$;*
- *for $m = 3$, $\mathcal{P}_{n,3}$ is isomorphic to $\widetilde{\mathcal{P}}_{n,3}(1, n-3)$;*
- *for $m \geq 4$, $\mathcal{P}_{n,m}$ is isomorphic to the hyperpath $\widetilde{\mathcal{P}}_{n,m}(1, \ldots, 1, k_j, 1, \ldots, 1)$, where $2 \leq j \leq m-2$ and $k_j = n - m$.*

Proof. From Lemma 4.5, we only need to prove the result about the upper bounds.

Note that the degree of every vertex in $\mathcal{P}_{n,m}$ is at most two and a hyperedge of $\mathcal{P}_{n,m}$ is adjacent to at most two other hyperedges. Then in order to maximize $\mathcal{M}_1(\mathcal{P}_{n,m})$ and $\mathcal{M}_2(\mathcal{P}_{n,m})$, the number of vertices of degree two must be maximum and is equal to $n-2$, that is, $\mathcal{P}_{n,m}$ is isomorphic to the hyperpath $\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})$ as shown in Figure 3.1. Then

$$\mathcal{M}_1(\mathcal{P}_{n,m}) \leq \mathcal{M}_1(\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})) = \sum_{i=1}^{m} \sum_{v \in e_i} d_{\widetilde{\mathcal{P}}_{n,m}(k_1,k_2,\ldots,k_{m-1})}(v)$$

$$= (2k_1+1) + \sum_{i=2}^{m-1} 2(k_{i-1}+k_i) + (2k_{m-1}+1)$$

$$= 4(k_1+k_2+\cdots+k_{m-1}) + 2 = 4n-6, \tag{4.3}$$

and

$$\mathcal{M}_2(\mathcal{P}_{n,m}) \leq \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1,k_2,\ldots,k_{m-1})) = \sum_{i=1}^{m} \prod_{v\in e_i} d_{\widetilde{\mathcal{P}}_{n,m}(k_1,k_2,\ldots,k_{m-1})}(v)$$

$$= 2^{k_1} + \sum_{i=2}^{m-1} 2^{k_{i-1}+k_i} + 2^{k_{m-1}}$$

$$= 2^{k_1} + 2^{k_1+k_2} + 2^{k_2+k_3} + \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}}. \tag{4.4}$$

By (4.3), the upper bound of $\mathcal{M}_1(\mathcal{P}_{n,m})$ holds. In the following, we determine the upper bound of $\mathcal{M}_2(\mathcal{P}_{n,m})$.

If $m=2$, then $k_1 = n-2$, and by (4.4),

$$\mathcal{M}_2(\mathcal{P}_{n,2}) \leq \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,2}(n-2)) = 2^{n-2} + 2^{n-2} = 2^{n-1}.$$

If $m=3$, then $k_1+k_2 = n-2$, and by (4.4),

$$\mathcal{M}_2(\mathcal{P}_{n,3}) \leq \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,3}(k_1,k_2)) = 2^{k_1} + 2^{k_1+k_2} + 2^{k_2}$$

$$= 2^{k_1} + 2^{n-2} + 2^{n-k_1-2} \leq 2 + 2^{n-2} + 2^{n-3}. \tag{4.5}$$

The equalities in (4.5) hold if and only if $k_1 = 1$ or $k_1 = n-3$, and $\mathcal{P}_{n,3}$ is isomorphic to $\widetilde{\mathcal{P}}_{n,3}(1,n-3)$.

Now, we consider the case $m \geq 4$.

Claim 1. If $k_1 \geq 2$ (or $k_{m-1} \geq 2$), then there is $\mathcal{P}_{n,m}(\hat{k}_1,\hat{k}_2,\ldots,\hat{k}_{m-1})$ with $\hat{k}_1 = 1$ (or $k_{m-1} = 1$), such that $\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1,k_2,\ldots,k_{m-1})) < \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(\hat{k}_1,\hat{k}_2,\ldots,\hat{k}_{m-1}))$.

In fact, if $k_1 \geq 2$, then by Lemma 4.6(1),

$$\begin{aligned}
&\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1,k_2,\ldots,k_{m-1}))\\
&= 2^{k_1} + 2^{k_1+k_2} + 2^{k_2+k_3} + 2^{k_3+k_4} + \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}}\\
&= 2^{k_1}(1+2^{k_2}) + 2^{k_2+k_3} + 2^{k_3+k_4} + \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}}\\
&< 2(1+2^{k_1+k_2-1}) + 2^{k_1+k_2-1+k_3} + 2^{k_3+k_4} + \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}}\\
&= 2 + 2^{k_1+k_2} + 2^{k_1+k_2-1+k_3} + 2^{k_3+k_4} + \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}}\\
&= \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(1,k_1+k_2-1,k_3,\ldots,k_{m-1})).
\end{aligned}$$

Similarly, if $k_{m-1} \geq 2$, then

$$\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1,k_2,\ldots,k_{m-1})) < \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1,k_2,\ldots,k_{m-3},k_{m-2}+k_{m-1}-1,1)).$$

Thus, for $m=4$, then $\mathcal{M}_2(\mathcal{P}_{n,4}) \leq \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,4}(1,n-4,1)) = 2^{n-2}+4$.

Claim 2. Suppose that $m \geq 5$, $k_1 = k_2 = \cdots = k_{s-1} = 1$, $k_s \geq 2$ and $k_{s+1} \geq 2$, where $2 \leq s \leq m-3$. Then there is $\mathcal{P}_{n,m}(\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_{m-1})$ with $\hat{k}_1 = \hat{k}_2 = \cdots = \hat{k}_{s-1} = \hat{k}_s = 1$, such that $\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})) < \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_{m-1}))$.

In fact, by Lemma 4.6(2),

$$\begin{aligned}
&2^{k_{s-1}+k_s} + 2^{k_s+k_{s+1}} + 2^{k_{s+1}+k_{s+2}} \\
=\ &2^{1+k_s} + 2^{k_s+k_{s+1}} + 2^{k_{s+1}+k_{s+2}} \\
=\ &2^{k_s}(2 + 2^{k_{s+1}}) + 2^{k_{s+1}+k_{s+2}} \\
<\ &2(2 + 2^{k_s+k_{s+1}-1}) + 2^{(k_s+k_{s+1}-1)+k_{s+2}} \\
=\ &2^{k_{s-1}+1} + 2^{1+(k_s+k_{s+1}-1)} + 2^{(k_s+k_{s+1}-1)+k_{s+2}}.
\end{aligned}$$

Thus by (4.4), we have

$$\begin{aligned}
&\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})) \\
=\ &2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_{s-1}+k_s} + 2^{k_s+k_{s+1}} + 2^{k_{s+1}+k_{s+2}} \\
&+ \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}} \\
<\ &2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_{s-1}+1} + 2^{1+(k_s+k_{s+1}-1)} + 2^{(k_s+k_{s+1}-1)+k_{s+2}} \\
&+ \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}} \\
=\ &\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1, \ldots, k_{s-2}, k_{s-1}, 1, k_s + k_{s+1} - 1, k_{s+2}, \ldots, k_{m-1})) \\
=\ &\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(1, 1, \ldots, 1, k_s + k_{s+1} - 1, k_{s+2}, \ldots, k_{m-1})).
\end{aligned}$$

Claim 3. Suppose that $m \geq 5$, there are $2 \leq s, t \leq m-2$ with $t \geq s+2$ such that $k_1 = k_2 = \cdots = k_{s-1} = k_{s+1} = k_{t-1} = k_{t+1} = 1$, $k_s \geq 2$ and $k_t \geq 2$. Then there is $\mathcal{P}_{n,m}(\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_{m-1})$ with $\hat{k}_1 = \hat{k}_2 = \cdots = \hat{k}_{s-1} = \hat{k}_s = \hat{k}_{s+1} = 1$, and $\hat{k}_{t-1} = \hat{k}_{t+1} = 1$, such that $\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})) < \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_{m-1}))$.

In fact, by Lemma 4.6(3),

$$\begin{aligned}
&2^{k_{s-1}+k_s} + 2^{k_s+k_{s+1}} + 2^{k_{t-1}+k_t} + 2^{k_t+k_{t+1}} \\
=\ &2^{1+k_s} + 2^{k_s+1} + 2^{1+k_t} + 2^{k_t+1} \\
=\ &2^{k_s+2} + 2^{k_t+2} \\
<\ &2^3 + 2^{k_s+k_t+1} \\
=\ &2^{k_{s-1}+1} + 2^{1+k_{s+1}} + 2^{k_{t-1}+(k_s+k_t-1)} + 2^{(k_s+k_t-1)+k_{t+1}}.
\end{aligned}$$

Thus by (4.4), we have

$$\begin{aligned}
&\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})) \\
=\ &2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_{s-1}+k_s} + 2^{k_s+k_{s+1}} + \cdots + 2^{k_{t-1}+k_t} + 2^{k_t+k_{t+1}}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}} \\
&< 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_{s-1}+1} + 2^{1+k_{s+1}} + \cdots + 2^{k_{t-1}+(k_s+k_t-1)} \\
&\quad + 2^{(k_s+k_t-1)+k_{t+1}} + \cdots + 2^{k_{m-2}+k_{m-1}} + 2^{k_{m-1}} \\
&= \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(k_1, \ldots, k_{s-2}, 1, 1, 1, k_{s+2}, \ldots, k_{t-2}, 1, k_s + k_t - 1, 1, k_{t+2}, \ldots, k_{m-1})) \\
&= \mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(1, 1, \ldots, 1, k_{s+2}, \ldots, k_{t-2}, 1, k_s + k_t - 1, 1, k_{t+2}, \ldots, k_{m-1})).
\end{aligned}
$$

Considering the above three claims, it is easy to see that for $m \geq 4$, $\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})$ maximizes $\mathcal{M}_2(\mathcal{P}_{n,m})$ if and only if there is a index j with $2 \leq j \leq m-2$ such that $k_j = n - m$, and $k_i = 1$ for $i = 1, \ldots, m-1$ with $i \neq j$, that is, $\widetilde{\mathcal{P}}_{n,m}(k_1, k_2, \ldots, k_{m-1})$ is $\widetilde{\mathcal{P}}_{n,m}(1, \ldots, 1, k_j, 1, \ldots, 1)$, where $2 \leq j \leq m-2$ and $k_j = n - m$. By (4.4),

$$
\mathcal{M}_2(\widetilde{\mathcal{P}}_{n,m}(1, \ldots, 1, k_j, 1, \ldots, 1)) = 2 \times 2 + (m-4) \times 2^2 + 2 \times 2^{n-m+1} = 2^{n-m+2} + 4(m-3).
$$

□

Lemma 4.8. *Let $m \geq 1$ and $1 \leq p < k$. Then*

$$
\begin{aligned}
\mathcal{M}_1(\mathcal{S}(m,p,k)) &= m(p(m-1)+k), \\
\mathcal{M}_2(\mathcal{S}(m,p,k)) &= m^{p+1}.
\end{aligned}
$$

Proof. Note that $\mathcal{S}(m,p,k)$ is a k-uniform hypergraph, and each hyperedge of $\mathcal{S}(m,p,k)$ has p vertices of degree m and $k-p$ vertices of degree of one. Then

$$
\begin{aligned}
\mathcal{M}_1(\mathcal{S}(m,p,k)) &= \sum_{e \in E(\mathcal{S}(m,p,k))} \sum_{v \in e} d_{\mathcal{S}(m,p,k)}(v) \\
&= \sum_{e \in E(\mathcal{S}(m,p,k))} (pm + k - p) \\
&= m(pm + k - p) = m(p(m-1)+k), \\
\mathcal{M}_2(\mathcal{S}(m,p,k)) &= \sum_{e \in E(\mathcal{S}(m,p,k))} \prod_{v \in e} d_{\mathcal{S}(m,p,k)}(v) \\
&= \sum_{e \in E(\mathcal{S}(m,p,k))} m^p = m^{p+1}.
\end{aligned}
$$

□

Theorem 4.9. *Let $\mathcal{T}_{n,m}$ be a hypertree on $n = m + p$ vertices ($p \geq 1$) with $m \geq 2$ hyperedges. Then*

$$
\begin{aligned}
n + 3m - 3 \leq \mathcal{M}_1(\mathcal{T}_{n,m}) &\leq m(p(m-1)+p+1), \\
4(m-1) \leq \mathcal{M}_2(\mathcal{T}_{n,m}) &\leq m^{p+1},
\end{aligned}
$$

the lower bounds are attained by a linear hyperpath $\overline{\mathcal{P}}_{n,m}$, and the upper bounds are attained by the sunflower hypergraph $\mathcal{S}(m,p,p+1)$.

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Proof. In order to minimise $\mathcal{M}_1(\mathcal{T}_{n,m})$ and $\mathcal{M}_2(\mathcal{T}_{n,m})$, the number vertices whose degrees are greater than or equal to 2 must be minimal. Note that in a linear hyperpath $\overline{\mathcal{P}}_{n,m}$ on n vertices with m hyperedges, any two hyperedges have at most one vertex in common (i.e. the number of vertices whose degrees are greater than or equal to 2 must be minimal among all hypertree on n vertices with m hyperedges), and the maximum degree of a vertex in $\overline{\mathcal{P}}_{n,m}$ is 2. So the lower bounds can be reached by a linear hyperpath $\overline{\mathcal{P}}_{n,m}$. Thus by Lemma 4.3, the lower bounds hold.

In order to maximize $\mathcal{M}_1(\mathcal{T}_{n,m})$ and $\mathcal{M}_2(\mathcal{T}_{n,m})$, the maximum number of vertices should have the maximum degree and these vertices has to be included uniformly among all the hyperedges. It is key to observe that a vertex can have maximum degree of m. To maximize the number of vertices with degree m, each hyperedge can have a maximum degree of $p+1$. So the hypertree that reaches the upper bounds of $\mathcal{M}_1(\mathcal{T}_{n,m})$ and $\mathcal{M}_2(\mathcal{T}_{n,m})$ must be the sunflower hypergraph $\mathcal{S}(m,p,p+1)$. By Lemma 4.8, the upper bounds follow. □

Theorem 4.10. *Let $\mathcal{T}_m^{(k)}$ be a k-uniform hypertree with m hyperedges, where $k, m \geq 2$. Then*

$$m(k+2)-2 \leq \mathcal{M}_1(\mathcal{T}_m^{(k)}) \leq m((k-1)(m-1)+k),$$
$$4(m-1) \leq \mathcal{M}_2(\mathcal{T}_m^{(k)}) \leq m^k,$$

the lower bounds are attained by the k-uniform linear hyperpath $\overline{\mathcal{P}}_m^{(k)}$, and the upper bounds are attained by the sunflower hypergraph $\mathcal{S}(m,k-1,k)$.

Proof. Similar to the proof of Theorem 4.9, the lower bounds can be reached by the k-uniform linear hyperpath $\overline{\mathcal{P}}_m^{(k)}$ with m hyperedges, and the k-uniform hypertree with m hyperedges that reaches the upper bounds must be the sunflower hypergraph $\mathcal{S}(m,k-1,k)$. Then by Lemmas 4.4 and 4.8, the theorem holds. □

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