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ON GROUPS WITH MANY NORMAL SUBGROUPS

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ABSTRACT. The structure of groups which are rich in normal subgroups has been investigated by several authors. Here, we prove that if a radical group G has *normal deviation*, which means that the set of its non-normal subgroups satisfies a very weak chain condition, then either G is a minimax group or all its subgroups are normal.

1. **Introduction**

The structure of groups for which the set of non-normal subgroups has prescribed properties has been investigated by several authors. The first step was of course the description of groups in which all subgroups are normal; it is well known that such groups (called *Dedekind groups*) either are abelian or can be decomposed as a direct product of Q_8 (the quaternion group of order 8) and a periodic abelian group with no elements of order 4. In the theory of infinite groups a natural approach in dealing with groups in which the set of non-normal subgroups is somehow small was obtained by the imposition of finiteness conditions on the set of non-normal subgroups of an infinite group, with special emphasis to chain conditions (see for instance the interesting survey [4]).

Let G be a group, and denote by $n(G)$ the lattice of normal subgroups of G. We say that a non-empty set Ω of subgroups of G has *G-normal deviation* 0 if either $\Omega \subseteq n(G)$ or the set $\Omega \setminus n(G)$

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satisfies the minimal condition; moreover, if $\delta > 0$ is any ordinal, we use the induction to say that Ω has *G-normal deviation* δ if for every descending chain

$$H_1 > H_2 > \cdots > H_n > H_{n+1} > \cdots \qquad (\star)$$

of elements of the set $\Omega \setminus n(G)$ there exists a positive integer t such that the interval $[H_n/H_{n+1}]$ has G-normal deviation strictly smaller than δ for each $n \geq t$, and δ is the smallest ordinal with such a property. In particular, G is said to have *normal deviation* if the lattice $L(G)$ of all subgroups of G has G-normal deviation. Finally, if X is a subgroup of G and the set of its subgroups has G-normal deviation, we just say that X has G-normal deviation.

Clearly, a group satisfying the *minimal condition* on non-normal subgroups has normal deviation 0. This class of groups was firstly investigated by Černikov [2] within the universe of locally (soluble-by-finite) groups. Moreover, if G is group satisfying the *weak minimal condition* on non-normal subgroups (i.e. if for any descending chain $(\star)$ of non-normal subgroups of G, there exists a positive integer t such that $[H_n/H_{n+1}]$ is finite for all $n \geq t$) (see [8]), then G has normal deviation at most 1.

Recall that a group G is said to be *minimax* if it has a finite series each of whose factors satisfies either the minimal or the maximal condition on subgroups. It is well known (see [12] and [13]) that a *radical group* (i.e. a group admitting an ascending (normal) series with locally nilpotent factors) satisfies the weak minimal condition if and only if it is a soluble minimax group.

The aim of this short article is to prove that for a radical group with normal deviation, only the extreme and unavoidable cases can occur.

Theorem 1.1. *Let G be a radical group with normal deviation. Then either G is minimax or all its subgroups are normal. In particular, if G satisfies the minimal condition on non-normal subgroups, then either G is Černikov or all its subgroups are normal.*

Recall that a subgroup X of a group G is said to be *pronormal* if X and X^g are cojugate in $\langle X, X^g \rangle$ for every element g in G. It is easy to show that a subgroup of an arbitrary group which is pronormal and subnormal is also normal. It follows that a group G all of whose subgroups are pronormal is a $\bar{T}$*-group* (i.e. in every subgroup of G the normality is a transitive relation - see [9]). We note that in the definition of normal deviation, the replacement of $n(G)$ by the set $pn(G)$ of pronormal subgroups of G (by the set $sn(G)$ of subnormal subgroups of G, respectively) gives the corresponding concept of *pronormal deviation* (*subnormal deviation*, respectively). Radical groups with pronormal deviation have been investigated recently in [6], while soluble groups with subnormal deviation have been considered in [7]. Of course, as a group with normal deviation has both pronormal and subnormal deviation, the above theorem could be obtained immediately by using the results in [6] and [7]. In our proof we use only a part of the main result of [6].

Most of our notation is standard and can be found in [10].

2. Proof of the Theorem

Clearly, G has pronormal deviation and hence by results in [6] G is either minimax or a $\bar{T}$-group. Let F be the Fitting subgroup of G. Then $F = C_G(G')$ (see [9, Lemma 2.2.2]) and hence G acts on F as a power automorphism group. In particular, the factor group $G/C_G(F)$ is abelian (see [3, Theorem 2.1.1] or [1, Proposition 2.6]). Clearly, F coincides with the Hirsch-Plotkin radical of G and hence $C_G(F) = Z(F)$ since G is radical (see [10, Lemma 2.32]). It follows that G is soluble, and so we may suppose that G is periodic since a soluble non-periodic $\bar{T}$-group must be abelian. Furthermore, as a periodic automorphism group of a Černikov group is likewise Černikov (see [10, Theorem 3.29]), we may also assume that F is not Černikov. It follows that F contains an infinite abelian subgroup A of prime exponent or with infinitely many non-trivial primary components. Let g be an element of G. We will prove that the subgroup $\langle g\rangle$ is normal in G. As every subgroup of F is normal in G, there exists an infinite collection $(A_n)_{n\in\mathbb{N}}$ of non-trivial $\langle g\rangle$-invariant subgroups of A such that

$$A_0 = \langle A_n \mid n \in \mathbb{N}\rangle = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, A_n$$

and $\langle g\rangle \cap A_0 = \{1\}$. Denote by O and E the sets of odd and even numbers, respectively. Put

$$O = O_0, O_1 = O \setminus \{3^n \mid n \in \mathbb{N}\}, O_2 = O_1 \setminus \{5^n \mid n \in \mathbb{N}\}, O_3 = O_2 \setminus \{7^n \mid n \in \mathbb{N}\}$$

and so on. Moreover, put

$$E = E_0, E_1 = E \setminus \{2^n \mid n \in \mathbb{N}\}, E_2 = E_1 \setminus \{2\cdot 3^n \mid n \in \mathbb{N}\}, E_3 = E_2 \setminus \{2\cdot 5^n \mid n \in \mathbb{N}\}$$

and so on. In this way we obtain two infinite descending chains

$$O = O_0 > O_1 > O_2 > \cdots > O_n > O_{n+1} > \cdots$$

and

$$E = E_0 > E_1 > E_2 > \ldots > E_n > E_{n+1} > \cdots$$

such that$O_n \setminus O_{n+1}$ and $E_n \setminus E_{n+1}$ are infinite for all n.

Now consider the interval $[\langle g\rangle A_0/\langle g\rangle]$, and let δ be its G-normal deviation. For each positive integer n, put

$$X_n = \langle g\rangle \ltimes \underset{\alpha\in O_n}{\mathrm{Dr}}\, A_\alpha$$

and

$$Y_n = \langle g\rangle \ltimes \underset{\beta\in E_n}{\mathrm{Dr}}\, A_\beta.$$

Clearly,

$$X_0 > X_1 > \cdots > X_n > X_{n+1} > \cdots$$

and

$$Y_0 > Y_1 > \cdots > Y_n > Y_{n+1} > \cdots$$

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are infinite descending chains of elements of the interval $[\langle g\rangle A_0/\langle g\rangle]$. Assume first that $\delta = 0$. Then $[\langle g\rangle A_0/\langle g\rangle]$ satisfies the minimal condition on subgroups which are not normal in G, and hence there exist non-negative integers h and k such that X_h and Y_k are normal in G. It follows that $\langle g\rangle = X_h \cap Y_k$ is likewise normal in G, and so the statement is proved when $\delta = 0$.

Suppose that $\delta > 0$. By definition there exists a non-negative integer r such that the interval $[X_n/\langle g\rangle]$ has G-normal deviation strictly smaller than δ for each $n \geq r$. It follows, by induction on δ, that X_r is normal in G. A similar argument shows that there exists a non-negative integer s such that Y_s is normal in G. Therefore $\langle g\rangle = X_r \cap Y_s$ is likewise normal in G, and this completes the first part of the proof.

Consider now a group G satisfying the minimal condition on non-normal subgroups. From the first part of the proof, we may assume that G is not periodic. Let a be any element of infinite order of G, and let p and q be distinct prime numbers. Then

$$\langle a\rangle > \langle a^p\rangle > \langle a^{p^2}\rangle > \cdots > \langle a^{p^n}\rangle > \langle a^{p^{n+1}}\rangle > \cdots$$

and

$$\langle a\rangle > \langle a^q\rangle > \langle a^{q^2}\rangle > \cdots > \langle a^{q^n}\rangle > \langle a^{q^{n+1}}\rangle > \cdots$$

are strictly descending chains of subgroups of G, and by hypothesis there exist positive integers h and k such that both the subgroups $\langle a^{p^h}\rangle$ and $\langle a^{q^k}\rangle$ are normal in G. Then $\langle a\rangle = \langle a^{p^h}, a^{q^h}\rangle$ is likewise normal in G.

Let g be an element of finite order of G. If $a \in G$ has infinite order, we have that

$$\langle g\rangle\langle a\rangle > \langle g\rangle\langle a^2\rangle > \langle g\rangle\langle a^4\rangle > \cdots > \langle g\rangle\langle a^{2^n}\rangle > \langle g\rangle\langle a^{2^{n+1}}\rangle > \cdots$$

is a strictly descending chain of subgroups, and hence there exists a positive integer r such that $\langle g\rangle\langle a^{2^n}\rangle$ is normal in G for all $n \geq r$. It follows that also

$$\langle g\rangle = \bigcap_{n\geq r} \langle g\rangle\langle a^{2^n}\rangle$$

is normal in G.

The above considerations show that G is a Dedekind group and so even abelian since it is not periodic. The statement is proved.

We point out that the argument of the last part of the proof shows that an *arbitrary* non-periodic group satisfying the minimal condition on non-normal subgroups is abelian.

Recall that a subgroup X of a group G is said to be *transitively normal* if X is normal in any subgroup Y of G such that $X \leq Y$ and X is subnormal in Y. Clearly, a group G satisfies the property $\bar{T}$ if and only if all subgroups of G are transitively normal. In [5] it is proved that a group G with no infinite simple sections satisfies the minimal condition on subgroups that are not transitively normal if and only if either G is Černikov or a $\bar{T}$-group.

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Finally, it is certainly worth mentioning an article by Tuschev [11] in which a dual notion of normal deviation is investigated.

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