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GROUPS WITH THE REAL CHAIN CONDITION ON NON-PRONORMAL SUBGROUPS

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Abstract. It is shown that a generalised radical group has no chain of non-pronormal subgroups with the same order type as the set $\mathbb{R}$ of the real numbers if and only if either the group is minimax or all subgroups are pronormal.

1. **Introduction**

The study of generalised soluble groups with a finiteness condition on the set of subgroups not verifying a given property is a standard problem in the theory of groups [4]. The properties which have received the most attention are probably chain conditions on the one hand and generalization of normality on the other hand.

Until now, the weakest chain condition considered is the so-called *deviation* [5, 15]. We do not state here the recursive definition of the deviation since, for our purposes, it is enough to recall the fact that a poset has deviation if and only if it contains no sub-poset order isomorphic to the poset D of all dyadic rationals $m/2^n$ in the interval 0 to 1 (see [10, 6.1.3]). Since D is a countable dense poset without endpoints, it is order-isomorphic to the rational numbers, by Cantor's isomorphism theorem. Therefore *a poset has deviation if and only if it contains no sub-poset order isomorphic to the poset* $\mathbb{Q}$ *of rational*

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numbers with their usual ordering. Note that this condition generalizes the so-called weak (double) chain conditions introduced by Zaicev [18] (see [1] for more details).

Very recenlty, it has been shown [5] that *if G is a radical group and the set of non-pronormal subgroups has deviation, then either G is minimax or all subgroups are pronormal.* Recall that a *radical group* is a group with an ascending (normal) series with locally nilpotent factors. Moreover a subgroup X of a group G is said to be *pronormal* when X and X^g are conjugate in $\langle X, X^g\rangle$ for every element g of G; clearly, every normal subgroup is pronormal. The structure of locally soluble groups in which all subgroups are pronormal has been long known [9] and groups with restrictions on the set of non-pronormal subgroups have been considered in many papers [5, 6, 7, 8, 16, 17]. Also, we must recall that a *soluble-by-finite minimax group* is a group with a finite series whose factors are cyclic, quasicyclic or finite. In fact they are just soluble-by-finite groups with deviation on all subgroups (see [15]). In this paper we extend the quoted result in [5] to *generalised radical* groups, i.e. groups with an ascending (normal) series with locally nilpotent or locally finite factors, and consider a chain condition even weaker than deviation.

A poset with the same order type as $\mathbb{R}$ will be called $\mathbb{R}$-*chain* and we will say that a poset has the *real chain condition (RCC)* if it contains no $\mathbb{R}$-*chain* as subposets. Generalised radical groups with *RCC* on (the poset of all) subgroups with – or without – some property have been recently considered in [1]. Along the same lines we obtain here the following result:

Theorem *Let G be a generalised radical group. Then G satisfies the real chain condition on non-pronormal subgroups if and only if either G is a soluble-by-finite minimax group or all subgroups of G are pronormal.*

Our notation and terminology are standard and can be found in [13].

2. Proof of the Theorem

We first recall some basic fact concerning pronormal subgroups that will be useful for our purposes. For more details about pronormal subgroups of infinite groups we refer to [6].

Lemma 2.1. *Let G be a group. Then:*

(i) *If H and K are pronormal subgroups of G such that $H^K = H$, then HK is pronormal in G.*
(ii) *A subgroup of G is normal if and only if it is pronormal and ascendant.*
(iii) *If G is locally nilpotent, then every pronormal subgroup of G is normal.*

Proof. See [6, Corollary 2.8, Corollary 2.3 and Theorem 2.4]. □

Let us now state a technical lemma which appears in [1] in a more general setting. However, for the sake of completeness, we give a proof also here.

Lemma 2.2. *Let G be a group having a section H/K which is the direct product of an infinite collection $(H_\lambda/K)_{\lambda\in\Lambda}$ of non-trivial subgroups, and let L be a subgroup of G such that $L\cap H \le K$ and $\langle H_\lambda, L\rangle = H_\lambda L$*

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for each λ. If there is no $\mathbb{R}$-chain of non-pronormal subgroups of G in the interval $[H/K]$, then there exists a normal subgroup H^* of H containing K such that $LH^* = H^*L$ is a pronormal subgroup of G.

Proof. Clearly the set Λ may be assumed to be countable, so that it can be replaced by the set $\mathbb{Q}$ of the rationals. Consider the subgroup $K_r = \underset{i<r}{\mathrm{Dr}} H_i$ for each $r \in \mathbb{R}$; then $\langle K_r, L\rangle = K_rL$ for each $r \in \mathbb{R}$. Let $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, then $K_{r_1} < K_{r_2}$. If were $K_{r_1}L = K_{r_2}L$, since $L \cap K_{r_2} \leq L \cap H \leq K \leq K_{r_1}$, Dedekind's Modular Law would give that

$$K_{r_2} = K_{r_2}L \cap K_{r_2} = K_{r_1}L \cap K_{r_2} = K_{r_1}(L \cap K_{r_2}) = K_{r_1};$$

this contradiction proves that $K_{r_1}L < K_{r_2}L$. Therefore $(K_rL)_{r\in\mathbb{R}}$ is an $\mathbb{R}$-chain and so K_rL must be a pronormal subgroup of G for some $r \in \mathbb{R}$; hence the lemma holds with $H^* = K_r$. □

Lemma 2.3. *Let G be a group with RCC on non-pronormal subgroups and let H/K be any section of G which is the direct product of an infinite family $(H_\lambda/K)_{\lambda\in\Lambda}$ of non-trivial subgroups. Then H is a pronormal subgroup of G; moreover, if H is an ascendant subgroup of G, then H_λ is normal in G for each $\lambda \in \Lambda$.*

Proof. Let λ be any element of Λ. Clearly we may write $H/K = (H_1/K)(H_2/K)$ where both H_1/K and H_2/K are the direct product of an infinite collection of non-trivial subgroups and $(H_1/K) \cap (H_2/K) = H_\lambda/K$. Application of Lemma 2.2 yields that there exist an H_1-invariant subgroup H_1^* in $[H_1/K]$ and an H_2-invariant subgroup H_2^* in $[H_2/K]$ such that both $H_1^*H_2$ and $H_1H_2^*$ are pronormal subgroups of G. Clearly $H_1^*H_2$ and $H_1H_2^*$ are both normal subgroups of H, so that $(H_1H_2^*)^{(H_1^*H_2)} = H_1H_2^*$ and hence $H = \langle H_1H_2^*, H_1^*H_2\rangle$ is a pronormal subgroup of G by Lemma 2.1.

Since H_1/K and H_2/K are direct product of infinitely many non-trivial subgroups, arguing as in the first part of this proof we have that H_1 and H_2 are pronormal in G. Therefore if we assume that H is ascendant, then H_1 and H_2 are likewise ascendant and so even normal by Lemma 2.1, and so $H_\lambda = H_1 \cap H_2$ is normal in G. □

In [1] it is proved that *a generalised radical groups with RCC on non-normal subgroups is either a soluble-by-finite minimax group or a Dedekind group*. We use this result in the next proof.

Lemma 2.4. *Let G be a group with RCC on non-pronormal subgroups, and let H be any locally nilpotent non-minimax subgroup of G. Then H is a Dedekind group; moreover, if H is periodic, then H is pronormal in G. In particular, the Hirsch-Plotkin subgroup R of G is hypercentral and all subgroups of R are ascendant in G.*

Proof. Il follows from Lemma 2.1 that the locally nilpotent subgroup H has *RCC* on non-normal subgroups so that H is a Dedekind group by the above quoted result from [1].

Assume that H is periodic, then $H = E \times A$ where A is abelian and E is either trivial or a quaternion group of order 8 (see [14, 5.3.7]). Clearly A is not minimax, so that it has a quotient A/B which is the direct product of infinitely many non-trivial groups (see for instance [5, Lemma 3.2]). Then H/B is likewise a direct product of infinitely many non-trivial groups, and hence Lemma 2.3 yields that H is pronormal in G.

Finally, the first part of this proof gives that R is either minimax or a Dedekind group, in particular, since locally nilpotent groups of finite rank are hypercentral (see [13, Part 2, p.38]), R is hypercental and so all subgroups of R are ascendant in G. □

Lemma 2.5. *Let G be a group with RCC on non-pronormal subgroups, and assume that the Hirsch-Plotkin radical R of G contains a subgroup which is the direct product of infinitely many cyclic non-trivial subgroups. Then all subgroups of R are normal in G.*

Proof. It follows from Lemma 2.4 that R is a Dedekind group; in particular, all subgroups of R are subnormal in G. Let X be a cyclic subgroup of R; the hypotesis imply that R contains a subgroup A which is the direct product of infinitely many cyclic non-trivial subgroups such that $X \cap A = \{1\}$. Write $A = A_1 \times A_2$ where both A_1 and A_2 are the direct product of an infinite collection of non-trivial cyclic subgroups. Application of Lemma 2.3 yields that both A_1X and A_2X are normal in G. Therefore $X = A_1X \cap A_2X$ is likewise a normal subgroup of G. Thus all subgroups of R are normal in G. □

Recall that the *total rank* of an abelian group is the sum of all p-ranks for $p = 0$ or p prime; in particular, an abelian group has finite total rank if and only if it is the direct sum of finitely many cyclic and quasicyclic groups and a torsion-free group of finite rank.

Lemma 2.6. *Let G be a group with RCC on non-pronormal subgroups. If A is an ascendant abelian subgroup of G which is not a minimax group, then all subgroups of A are normal in G and all cyclic subgroups of G/A are pronormal.*

Proof. Clearly A is contained in the Hirsch-Plotkin radical of G. In order to prove that all subgroups of A are normal in G, Lemma 2.5 allows us to suppose that A has finite total rank so that, in particular, since A is not minimax, we have that A is not periodic.

Let x be any element of infinite order of A. Consider any free subgroup E of A such that $x \in A$ and A/E is periodic; then E is finitely generated and A/E has infinitely many primary components. Let L be any subgroup of finite index on E containg x, then also L is finitely generated and A/L has infinitely many primary components; let us denote by A_p/L the p-component of A/L. By Lemma 2.3, A_p is G-invariant for each $p \in \pi(A/L)$. Thus

$$L = \bigcap_{p\in\pi(A/L)} A_p$$

is likewise normal in G. Since a well-know result by Mal'cev (see [14, 5.4.16]) yields that $\langle x \rangle$ is a closed subgroup of E, it follows that $\langle x \rangle$ is a normal subgroup of G. Hence all infinite cyclic subgroups of A are

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G-invariant, so that if $y \in A$ has finite order also $\langle x, y\rangle = \langle x, xy\rangle$ is normal in G and hence $\langle y\rangle$ is likewise normal in G since $\langle y\rangle$ is the torsion subgroup of $\langle x, y\rangle$. Therefore all subgroups of A are normal in G.

Consider now any cyclic subgroup X of G which is not contained in A, and let again E be a free subgroup of A such that A/E is periodic; then E is normal in G. Replacing A/E with a suitable direct product of (infinitely many) its primary components, we may clearly suppose that $(A/E) \cap (XE/E)$ is trivial and it is possible apply Lemma 2.2 to obtain that there exists a normal subgroup A^* of A containing E such that $(XE)A^* = A^*(XE)$ is pronormal in G. Thus $XA = ((XE)A^*)A$ is pronormal in G by Lemma 2.1 and so XA/A is likewise pronormal in G/A. Therefore all cyclic subgroups of G/A are pronormal. □

Recall that a group in which all subnormal subgroups are normal is called a *T-group*. The structure of soluble T-groups was described by Gaschütz and Robinson (see [12]). It turns out that soluble T-groups are metabelian and that finitely generated soluble T-groups are either finite or abelian; moreover, every subgroup of a finite soluble T-group is itself a T-group but this is no longer true for infinite soluble T-groups. Groups in which all subgroups are T-groups are called *$\bar{T}$-groups*. It is easy to see that finite $\bar{T}$-groups are soluble so that, in particular, *any locally (radical-by-finite) $\bar{T}$-group is always metabelian and even abelian if it is not periodic.*

The concept of pronormal subgroups arise naturally in the study of $\bar{T}$-groups. In fact, *if all cyclic subgroups of a group G are pronormal then G is a $\bar{T}$-group* (see [6, Lemma 3.2]) and Peng [11] proved that the converse also holds for finite soluble groups. Infinite (generalised) soluble infinite groups in which all subgroups are pronormal have been described in [9], where an example of infinite soluble periodic $\bar{T}$-group containing non-pronormal subgroups is also given (thus, in particular, Peng's theorem does not hold for infinite groups).

Lemma 2.7. *Let G be a group with RCC on non-pronormal subgroups, and let A be any subgroup of G which is the direct product of infinitely many cyclic non-trivial subgroups. If A is an ascendant subgroup of G, then G is a $\bar{T}$-group.*

Proof. Consider a subgroup X of G and a subnormal subgroup Y of X. Suppose first that $Y_0 = Y \cap A$ is not finitely generated; hence also Y_0 is a direct product of infinitely many non-trivial cyclic subgroups (see [14, 4.3.16]). Application of Lemma 2.6 yields that Y_0 is normal in G and all cyclic subgroups of G/Y_0 are pronormal. Thus, as just quoted above, G/Y_0 is a $\bar{T}$-group and hence Y is normal in X.

Assume now that Y_0 is finitely generated, thus Y_0 is contained in a direct product of finitely many direct factors of A and hence certainly there exist two subgroups A_1 and A_2 which are both the direct product of infinitely many direct factors of A such that $A_1 \cap A_2 = \langle A_1, A_2\rangle \cap Y = \{1\}$. Since all subgroups of A are normal in G by Lemma 2.6, application of Lemma 2.2 gives that there exist two subgroups $A_1^* \leq A_1$ and $A_2^* \leq A_2$ such that A_1^*Y and A_2^*Y are both pronormal in G. In particular, A_1^*Y and A_2^*Y are pronormal and subnormal in AX, so that they are both normal in AX by Lemma 2.1. Thus $Y = A_1^*Y \cap A_2^*Y$ is likewise normal in AX, and so also in X. Therefore X is a T-group and so G is a $\bar{T}$-group. □

Now we deal with locally (radical-by-finite) groups.

Lemma 2.8. *Let G be a locally (radical-by-finite) group with RCC on non-pronormal subgroups, and let A be an ascendant abelian subgroup of G which is not a minimax group. If G/A is not periodic, then G is abelian.*

Proof. By Lemma 2.6 all subgroups of A are normal in G. If B is a subgroup of A such that A/B contains non-minimax subgroups A_1/B and A_2/B such that $A_1 \cap A_2 = B$, Lemma 2.6 gives that G/A_1 and G/A_2 are groups whose cyclic subgroups are pronormal, thus they are $\bar{T}$-groups and hence they are abelian being locally (radical-by-finite) and non-periodic; thus $G' \leq A_1 \cap A_2 = B$. In particular, in order to prove that G is abelian it can be assumed that A cannot be decomposed as the direct product of two non-minimax subgroups; hence A does not contains subgroups which are the direct product of infinitely many non-trivial cyclic subgroups and so A has finite total rank. Let E be a free subgroup of A such that A/E is periodic. Since E is residually finite, it is enough to show that G/L is abelian for each subgroup L of finite index of E. Since A is not minimax but it has finite total rank, such a subgroup L is finitely generated and the periodic abelian group A/L has infinitely many primary components. Hence, A/L can be decomposed as the direct product of two non-minimax subgroups and so $G' \leq L$, as wished. □

For the sake of completeness let us prove the following elementary property of abelian groups.

Lemma 2.9. *Let A be any torsion-free abelian group and assume that $A^{p^n} = A^{p^{n+1}}$ for some prime p and $n \in \mathbb{N}$. Then there exists a subgroup B of A such that A/B is a group of type p^∞.*

Proof. Let x_0 be any element of A^{p^n}. Since $A^{p^n} = A^{p^{n+1}} = (A^{p^n})^p$, there exist elements $x_1, x_2, x_3, \ldots$ of A^{p^n} such that $x_0 = x_1^p$, $x_1 = x_2^p$, $x_2 = x_3^p, \ldots$. Put $P = \langle x_0, x_1, x_2, x_3, \ldots \rangle$, so that $P/\langle x_0 \rangle$ is a Prüfer p-group. Then a well-know result of Baer (see [14, 4.1.3]) yields that there exists a subgroup $B/\langle x_0 \rangle$ such that $A/\langle x_0 \rangle = P/\langle x_0 \rangle \times B/\langle x_0 \rangle$. Hence $A/B \simeq P/\langle x_0 \rangle$ is a group of type p^∞. □

Lemma 2.10. *Let G be a locally (radical-by-finite) group with RCC on non-pronormal subgroups, and let A be an ascendant abelian subgroup of G which is not a minimax group. If A is torsion-free, then $A \leq Z(G)$.*

Proof. By Lemma 2.6, all subgroups of A are normal in G and Lemma 2.8 allows us to suppose that G/A is periodic. Assume, by contradiction, that there exists $g \in G$ such that $[A, g] \neq \{1\}$. Since g acts as a power automorphism on A, g induces the inversion on A (see [12, Lemma 4.1.1]) and $g^2 \in C_G(A)$. Then, as G/A is periodic, if $g^n \in A$, we have that $g^n = g^{-n}$ and so g is periodic. In particular, $A \cap \langle g \rangle = \{1\}$.

Note that we may assume that A has finite rank. In fact, if not, A would contains two free subgroups of infinite rank X and Y such that $X \cap Y = \{1\}$ and Lemma 2.8 would imply that both factors G/X and G/Y are abelian, so that $G' = \{1\}$ and the statement would be true.

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Let B any subgroup of A such that A/B is a group of type 2^∞. Then $\langle g, A\rangle / \langle g^2, B\rangle$ is a locally dihedral 2-group, hence it is hypercentral; in particular, it follows that $\langle g, B\rangle$ is an ascendant subgroup of $\langle g, A\rangle$. On the other hand, B is not minimax and hence $\langle g, B\rangle /B$ is a pronormal subgroup of G/B by Lemma 2.6. Therefore Lemma 2.1 yields that $\langle g, B\rangle$ is even normal in $\langle g, A\rangle$, a contradiction. It follows that A has no quotients of type 2^∞, hence application of Lemma 2.9 gives that $A^2 \neq A^4$. Therefore A/A^4 contains some non-trivial element of order 4.

As A has finite rank, also A/A^4 has finite rank; on the other hand, any primary abelian group of finite rank satisfy the minimal condition, and hence we have that A/A^4 si finite. Thus A^4 is not minimax. Then follows from Lemma 2.6 that $\langle g, A^4\rangle$ is pronormal il G. Since $\langle g, A\rangle / \langle g^2, A^4\rangle$ is an abelian-by-finite 2-group of finite exponent, it is nilpotent (see [13, Part 2, Lemma 6.34]) and so $\langle g, A^4\rangle$ is pronormal and subnormal in $\langle g, A\rangle$. Thus $\langle g, A^4\rangle$ is normal in $\langle g, A\rangle$ by Lemma 2.1, and

$$\langle g, A\rangle / \langle g^2, A^4\rangle = \langle g, A^4\rangle / \langle g^2, A^4\rangle \times \langle g^2, A\rangle / \langle g^2, A^4\rangle$$

but this is not possible as g induces the inversion on $\langle g^2, A\rangle / \langle g^2, A^4\rangle$ and $\langle g^2, A\rangle / \langle g^2, A^4\rangle$ has exponent (exactly) 4. This contradiction proves that $A \leq Z(G)$. □

Lemma 2.11. *Let G be a locally (radical-by-finite) group with RCC on non-pronormal subgroups, and assume that the Hirsch-Plotkin radical R of G is not minimax. Then G is a $\bar{T}$-group.*

Proof. Assume, by contradiction, that G is not a $\bar{T}$-group. Lemma 2.4 yields that R is a Dedekind group, and so the torsion subgroup T of R is a Chernikov group as a consequence of Lemma 2.7 applied to the socle of R. Hence $T \neq R$, so that R is non periodic. Thus R is abelian and so $R = T \times A$ by a suitable torsion-free subgroup A (see for instance [14, 4.3.9]); in particular, A is not minimax. By Lemma 2.6, all subgroups of R are normal in G so that G acts on R either trivially or as the inversion map (see [12, Lemma 4.1.1]); on the other hand, A is a torsion-free (non-trivial) subgroup of R which is contained in $Z(G)$ by Lemma 2.10 and hence G must acts trivially on R. Thus $R \leq Z(G)$.

Clearly G is not abelian, hence G/A is periodic by Lemma 2.8. Let E be a free subgroup of A such that A/E is periodic. Then Lemma 2.7 yields that E is finitely generated, so that A/E does not satisfy the minimal condition. Since E is residually finite, in order to prove that G is soluble it is enough to prove that G'' is contained in each subgroup of finite index of E. Let L be any subgroup of finite index of E. Then L is a central subgroup of G and A/L does not satisfy the minimal condition, hence the socle of A/L is the direct product of infinitely many non-trivial cyclic subgroups and so G/L is a $\bar{T}$-group by Lemma 2.7; in particular, since G/L is locally (radical-by-finite), it follows that G/L is metabelian. Thus $G'' \leq L$ and so, as noted above, this is enough to guarantee that G soluble. Therefore $C_G(R) \leq R$ and so, as $R \leq Z(G)$, we have that $G = R$ is abelian. This contradiction proves the lemma. □

As already noted, a group whose (cyclic) subgroups are pronormal is a $\bar{T}$-group, moreover, any locally (radical-by-finite) $\bar{T}$-group is soluble (actually metabelian). Notice that also locally (radical-by-finite) minimax groups are soluble-by-finite. In fact, it is known that any radical minimax group is soluble (see [13, Part 2, Theorem 10.35]), while locally (soluble-by-finite) minimax groups are soluble-by-finite (see [3, Lemma 8]).

Lemma 2.12. *Let G be a locally (radical-by-finite) group with RCC on non-pronormal subgroups. Then either G is minimax or all subgroups of G are pronormal. In particular, G is soluble-by-finite.*

Proof. Assume first that G is a $\bar{T}$-group which is not minimax; in particular, G is metabelian. Since locally (radical-by-finite) non-periodic $\bar{T}$-groups are abelian, in order to prove that all subgroups of G are pronormal, we may further suppose that G is periodic. Thus, if $L = [G', G]$, we have $\pi(L) \cap \pi(G/L) = \emptyset$ and $2 \notin \pi(L)$ (see [12, Theorem 6.1.1]). Let $\pi = \pi(G/L)$ and assume, by contradiction, that there exists a Sylow π-subgroup P of G such that PL is a proper normal subgroup of G. Suppose that P is not a Chernikov group. Since $P \simeq PL/L$ is nilpotent, Lemma 2.4 yields that P is pronormal in G and hence $G = P^G N_G(P) = PLN_G(P)$. Since $N_G(P)/P$ is clearly a π'-group also

$$G/PL \simeq N_G(P)/(PL \cap N_G(P))$$

is a π'-group, a contradiction which proves that P is a Chernikov group. Let F/PL be a finite non-trivial subgroup of G/PL. Since G acts as a group of power automorphisms of L, the factor $G/C_G(L)$ is residually finite and so the largest divisible subgroup D of P is contained in $C_G(L)$. Therefore $F/C_F(L)$ is finite and so $F = XL$ for a suitable subgroup X such that $X \cap L = \{1\}$ (see [2, Theorem 2.4.5]). But F/DL is finite, hence F is abelian-by-finite and so its Sylow π-subgroups P and X are conjugate, which is not possible by the choice of X. Therefore $G = PL$ for each Sylow π-subgroup P of G. It follows that all subgroups of G are pronormal (see [9]) and hence the lemma holds when G is a $\bar{T}$-group.

Assume now that G is not a $\bar{T}$-group and, by contradiction, suppose that G is not minimax. Hence Lemma 2.11 yields the Hirsch-Plotkin radical of G is minimax, so that in particular all ascendant abelian subroups of G are minimax. Let N be any normal subgroup of G, and let H/N be the Hirsch-Plotkin radical of G/N. Then Lemma 2.11 yields that either G/N is a $\bar{T}$-group (and hence metabelian) or H/N is minimax (and hence soluble), thus the last non-trivial term of the derived series of H/N is a non-trivial abelian normal subgroup of G/N. It follows that G is hyperabelian and hence G is minimax (see [13, Part 2, p.175]), a contradiction which concludes the proof. □

Now we are in a position to prove a Theorem that contains the one stated in the introduction. In the next statement, *pronormal deviation* means that the set of non-pronormal subgroups has deviation (see [5]) and, as quoted in the introduction, this is equivalent to require that there are no subsets of non-pronormal subgroups order isomorphic to the set $\mathbb{Q}$ of rational numbers with their usual ordering.

Theorem 2.13. *Let G be a generalised radical group. Then the following are equivalent:*

(i) *G satisfies the real chain condition on non-pronormal subgroups;*

(ii) *G has pronormal deviation;*

(iii) *G satisfies the weak minimal condition on non-pronormal subgroups;*

(iv) *G satisfies the weak maximal condition on non-pronormal subgroups;*

(v) *G satisfies the weak double condition on non-pronormal subgroups.*

(vi) *either G is a soluble-by-finite minimax group or all subgroups of G are pronormal.*

Proof. Let G be a generalised radical group with the real chain condition on non-pronormal subgroups. Clearly, any section of G has likewise the real chain condition and so Lemma 2.12 yields that each locally (radical-by-finite) section of G is soluble-by-finite; therefore, since G is generalised radical, by transfinite induction can be obtained that G itself is soluble-by-finite and thus again Lemma 2.12 gives that either G is minimax or all subgroups of G are pronormal. Hence G satisfies both the weak minimal and weak maximal condition on non-pronormal subgroups.

On the other hand, the weak minimal condition on non-pronomal subgroups, as well as the weak maximal condition on non-pronormal subgroups, imply the weak double chain condition on non-pronormal subgroups and so also the pronormal deviation which in turn implies the real chain condition on non-pronormal subgroups (see [1]). □

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