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APPROACHABLE GRAPH (TREE) AND ITS APPLICATION IN HYPER (NETWORK)

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Abstract. A hypertree is a special type of connected hypergraph that removes any, its hyperedge then results in a disconnected hypergraph. Relation between hypertrees (hypergraphs) and trees (graphs) can be helpful to solve real problems in hypernetworks and networks and it is the main tool in this regard. The purpose of this paper is to introduce a positive relation (as α-relation) on hypertrees that makes a connection between hypertrees and trees. This relation is dependent on some parameters such as path, length of a path, and the intersection of hyperedges. For any $q \in \mathbb{N}$, we introduce the concepts of a derivable tree, (α, q)-hypergraph, and fundamental (α, q)-hypertree for the first time in this study and analyze the structures of derivable trees from hypertrees via given positive relation. In the final, we apply the notions of derivable trees, (α, q)-trees in real optimization problems by modeling hypernetworks and networks based on hypertrees and trees, respectively.

1. Introduction

The structure of hypergraphs has been presented by Berge as an extension of the theory of graphs with this the motivation that hypergraphs cover problems and shortcomings of structures of the graph around 1960 [3]. Hypertrees or tree hypergraphs are specially connected hypergraphs that are a generalization of trees. Moreover, hypertrees structures have very important applications in the modeling of complex hypernetworks. If we want to connect the elements in several groups, we can not use the

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structure of trees because of the limited connection of more than two elements. Hypertrees can connect some group of elements such that we don't any limitation in the connection between elements of an underlying set. Therefore, this algebraic property becomes important for the structure of hypertrees in this regard. Today, hypertrees have important applications and are used in hypernetworks such as computer science, wireless sensor hypernetwork, and machine learning such as extending factorizations of complete uniform hypergraphs [1], rainbow fractional matchings [2], hypergraph-based Berge hypergraphs [4], prime 3-uniform hypergraphs [5], in search of hyperpaths [6], on the spectral radius of uniform hypertrees [7], α-derivable digraphs and its application in wireless sensor networking [8], affine planes and transversals in 3-uniform linear hypergraphs [10], on derivable Tree [11], topology and geometry of random 2−dimensional hypertrees [12], biochemical and phylogenetic networks-I: hypertrees and corona products [13], finding a minimal spanning hypertree of a weighted hypergraph [14] and partitions of the complete hypergraph K_6^3 and a determinant-like function [15].

Motivation and advantage: Modeling based on (hyper) trees (sometimes called tree hypergraphs) is a clustering or grouping of elements based on certain properties, in which the properties of its elements are checked in each cluster, and play an important role in optimization problems. The most important property of hypertrees is that as a connected hypergraph, it becomes a disconnected hypergraph by removing any of its hyperedges and covers the limited connection of more than two elements in trees. This property allows us to use hypertrees in the design of hypernetworks, where removing any information in it causes the hypernetwork to fall apart. These properties make a main motivation in the study and extension of hypertrees theory. Most problems related to group classification of interdependent elements such as groups of animals and food groups, groups of machines and groups of tools, groups of educated people, and groups of jobs can be modeled with hyper trees. Designing and modeling the mentioned problems with the help of hypertrees becomes more important when this type of design is optimized. One of the most important motivations of this study is that the design of hypernetworks should be optimal in terms of parameters such as time, distance, raw and produced materials, consumed and produced energy, speed, and other similar parameters. Therefore, with the help of basic relationships, we calculate the trees extracted from these hypertrees, whose vertices are an equivalence class of elements with similar properties. In fact, these trees are dependent on the optimizer parameters apart from the basic relationship, and we can obtain the value of these parameters with the necessary calculations. If we want to look at the problem from another perspective, assuming that our optimal tree in the target network should have what properties, we can build the target hypertree based on the detailed information of the target-dependent sets and the available data. In the type of construction of this hypertree, it is possible to classify the groups in the desired way so that it is optimal in terms of our goal. This advantage leads to obtaining, optimal results in the hypernetwork, based on our mathematical methods and computations.

Regarding these points, we try to make a connection between of structure of hypertrees and trees via algebraic tools, so introduce a positive relation between hypertrees. Positive relation on hypertrees converts hypertrees to trees and so it is a fundamental tool for connecting the category of hypergraphs and graphs. The main motivation of this work is the application of hypertrees in hypernetwork and networks, so we presented a technique for modeling hypertrees in hypernetwork. Indeed, it is tried to design hypertrees based on real problems and use positive relations to apply trees to the real problem that their nodes are a group of elements. This view leads to apply of mathematics tools in hypernetworks and networks that deal with groups and families, such as telephone, email, social networks, food chains, transportation networks, traffic networks, and water and sewage networks. The notions of valued-part, the intersection of hyperedges, valued-hypertrees, and the fundamentally valued hypertrees are introduced and presented in this work to reach the goal. To illustrate the importance of this work, we have presented some practical examples in the real world such as the application of hypernetwork and networks in social networks and the food web.

2. Preliminaries

In this section, we recall some definitions and results, which we need in what follows.

Definition 2.1. [3, 9] *Let X be a finite set and $P^*(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$. A hypergraph on X is a pair $H = (X, \{E_i\}_{i=1}^m)$ such that for all $1 \leq i \leq m, E_i \in P^*(X)$ and $\bigcup_{i=1}^{m} E_i = X$. The elements $x_1, x_2, \ldots, x_n$ of X are called hypervertices, and the sets $E_1, E_2, \ldots, E_m$ are called the hyperedges of the hypergraph H. In hypergraphs, hyperedges can contain an element (loop) two elements (edge) or more than three elements. A hypergraph $H = (X, \{E_i\}_{i=1}^m)$ is called a complete hypergraph, if for any $x, y \in X$ there is $1 \leq i \leq m$ such that $\{x, y\} \subseteq E_i$. A hypergraph $H = (X, \{E_i\}_{i=1}^n)$ is called as a joint complete hypergraph, if $|X| = n$ for all $1 \leq i \leq n, |E_i| = i$ and $E_i \subseteq E_{i+1}$ element (loop). If for all $1 \leq k \leq m$ $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph.*

Definition 2.2. [11] *Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypergraph. A walk of length l in a hypergraph H is a sequence $x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_l\ E_l\ x_{l+1}$ such that for all $i \in \{1, 2, \ldots, l\}, x_i, x_{i+1} \in E_i$. A walk of length l in a H is said to be a path if, (i) all the vertices $x_1, x_2, \ldots, x_{l+1}$ except x_1 and x_{l+1} are distinct and (ii), all the edges $E_1, E_2, \ldots, E_l$ are distinct. If $l > 1$ and $x_1 = x_{l+1}$, the path $x_l\ E_1\ x_2\ E_2\ x_3 \cdots\ x_l\ E_l\ x_{l+1}$ is called a cycle of length l. A hypergraph H is connected if for any two vertices $a, b \in X$ there is a path joining the vertices a and b. A hypergraph $H = (X, \{E_i\}_{i=1}^m)$ is called a hypertree if it is connected, non-trivial and cycle-free.*

3. α-Relation on Hypergraphs

In this section, we introduce a positive relation on hypergraphs such that its quotient on the given positive relation to a binary operation is a graph. Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypergraph and $x \in X$. From now on, set $\mathcal{E} = \{(E_i \cap E_j) \in P^*(X) \mid 1 \leq i \neq j \leq m\}$ and have the following results.

Lemma 3.1. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a connected hypergraph. Then for all $1 \leq t \leq m$, there exists $1 \leq s \leq m$, such that $E_t \cap E_s \neq \emptyset$.*

Proof. Let $1 \leq t \leq m$. Since $H = (X, \{E_i\}_{i=1}^m)$ is a hypergraph, there exists $x \in E_t$ and so for all $y \in X$, there exists $E_{t'}$ such that $y \in E_{t'}$. Now, $H = (X, \{E_i\}_{i=1}^m)$ is a connected hypergraph, thus there is a sequence $x = x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_l\ E_l\ x_{l+1} = y$ such that for all $i \in \{1, 2, \ldots, l\}, x_i, x_{i+1} \in E_i$. It follows that for all $1 \leq t \leq m$, there exists $1 \leq s \leq m$, such that $E_t \cap E_s \neq \emptyset$. $\square$

Definition 3.2. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypergraph. Define a relation α on H by $(x, y) \in \alpha$, if there exists a unique $1 \leq i \leq m$ such that $\{x, y\} \subseteq E_i (\nexists\ 1 \leq j \neq k \neq i \leq m$ such that $x \in E_j, y \notin E_j, y \in E_k, x \notin E_k)$ or there exist $1 \leq i_1 \neq i_2 \neq \cdots i_k \leq m$ that $\{x, y\} \subseteq \bigcap_{j=1}^{k} E_{i_j}$. Clearly, α is an equivalence relation on H and, we will denote $H/\alpha = \{\alpha(x) \mid x \in H\}$ as the set of all equivalence class of α on H.*

Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypergraph and $x, y \in X$. From now on, will call $P(x, y)$ as a path from x to y and $|P(x, y)|$ as its length.

Theorem 3.3. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypergraph. Then there exists a binary operation $*$ on H/α such that $(H/\alpha, *)$ is a graph.*

Proof. Let $\alpha(x), \alpha(y) \in H/\alpha$ and $q \in \mathbb{N}$. Define a binary operation $*^{(q)}$ on H/α, as follows

$$\alpha(x) *^{(q)} \alpha(y) = \begin{cases} \widehat{\alpha(x), \alpha(y)} & \text{if } \exists\ P \text{ s.t } |P(x, y)| = q \\ \widehat{\emptyset} & \text{o.w} \end{cases},$$

where, $\widehat{\alpha(x), \alpha(y)}$ shows that there exists an edge between of $\alpha(x), \alpha(y)$ and $\widehat{\emptyset}$ shows that dose not exist any edge between of $\alpha(x), \alpha(y)$. Let $\alpha(x) = \alpha(x')$ and $\alpha(y) = \alpha(y')$. Then there exists unique $1 \leq i \leq m$ such that $\{x, x'\} \subseteq E_i$ or there exists $1 \leq i_1 \neq i_2 \neq \cdots i_k \leq m$ that $\{x, x'\} \subseteq \bigcap_{j=1}^{k} E_{i_j}$ and there exists unique $1 \leq r \leq m$ such that $\{y, y'\} \subseteq E_r$ or there exists $1 \leq r_1 \neq r_2 \neq \cdots r_l \leq m$ that $\{y, y'\} \subseteq \bigcap_{s=1}^{l} E_{r_s}$. Since $H = (X, \{E_i\}_{i=1}^m)$ is a connected hypergraph, there exist path P from x to y and path P' from x' to y' such that $|P| = u$ and $|P'| = v$, where $u, v \in \mathbb{N}$. Because there exists unique $1 \leq i \leq m$ such that $\{x, x'\} \subseteq E_i$ or there exists $1 \leq i_1 \neq i_2 \neq \cdots i_k \leq m$ that $\{x, x'\} \subseteq \bigcap_{j=1}^{k} E_{i_j}$ and there exists unique $1 \leq r \leq m$ such that $\{y, y'\} \subseteq E_r$ or there exists $1 \leq r_1 \neq r_2 \neq \cdots r_l \leq m$ that

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$\{y,y'\}\subseteq \bigcap\limits_{s=1}^{l} E_{r_s}$, get P is related for both paths from x to y and from x' to y'. In similar a way P' is related for both paths from x to y and a path from x' to y', so $\alpha(x)*^{(u)}\alpha(y)=\alpha(x')*^{(u)}\alpha(y')$ and $\alpha(x)*^{(v)}\alpha(y)=\alpha(x')*^{(v)}\alpha(y')$. It follows for each $i\in\mathbb{N}$, binary operation $*^{(i)}$ is well defined on H/α. Clearly, $H/\alpha=(V(H/\alpha),E(H/\alpha),*^{(i)})$ is a graph. □

Theorem 3.4. *Let $H=(X,\{E_i\}_{i=1}^m)$ be a hypergraph and $q\in\mathbb{N}$. If $(H/\alpha,*^{(q)})$ is a connected graph, then*

- *(i) $(H/\alpha,*^{(p)})$ has a cycle, where $p<q$,*
- *(ii) $(H/\alpha,*^{(p)})$ is a complete graph, where $p\leq\lfloor\frac{q}{2}\rfloor$,*

Proof. (i) Let $x,y\in X$. Since $(H/\alpha,*^{(q)})$ is a connected graph, there exists a path

$$\alpha(x_0)e_1\alpha(x_1)e_2\alpha(x_2)\cdots\alpha(x_{n-1})e_n\alpha(x_n)$$

such that for all $0\leq i\leq n$, $\alpha(x_i)$ and e_i are distinct. In addition, for all $0\leq i\leq n$, there exists a sequence $x_i=x_{i1}\ E_{i1}\ x_{i2}\ E_{i2}\ x_{i3}\ E_{i3}\cdots\ x_{iq}\ E_{iq}\ x_{iq+1}=x_{i+1}$ such that for all $j\in\{1,\ldots,q\}$, we have $x_{ij},x_{ij+1}\in E_{ij}$. Hence for any $0\leq i\leq n$, $x_{is}\ E_{is}\ x_{is+1}\ E_{is+1}\ x_{is+2}\ E_{is+2}\cdots\ x_{is+q}\ E_{is+q}\ x_{is+q+1}=x_{i+1}$ is a subsequence with length of $q-s<q$ and for any $j\in\{s,\ldots,q\}$, we have $x_{ij},x_{ij+1}\in E_{ij}$. It follows that there exists a path $\alpha(x_s)e_s\alpha(x_{s+1})e_{s+2}\alpha(x_{s+2})\cdots\alpha(x_{n-1})e_n\alpha(x_n)$, such that for all $s\leq i\leq n$, $\alpha(x_i)$ and e_i are distinct. Therefore, the two above paths make a cycle and so $(H/\alpha,*^{(p)})$ has at least one cycle, where $p<q$. (ii) Immediate by item 3.4 (i). □

Theorem 3.5. *Let $H=(X,\{E_i\}_{i=1}^m)$ be a connected hypergraph and $x,y\in X$ and $q\in\mathbb{N}$. Then in $H=(X,\{E_i\}_{i=1}^m)$ and $(H/\alpha,*^{(q)})$:*

- *(i) If $q=1$, then $|P(\alpha(x),\alpha(y))|\leq|P(x,y)|$.*
- *(ii) If for some $1\leq i\neq j\leq m$, that $\emptyset\neq E_i\cap E_j$, have $|E_i\cap E_j|=1$, then $|P(\alpha(x),\alpha(y))|=\lfloor\frac{|P(x,y)|}{q}\rfloor$.*

Proof. (i) Let $x,y\in X$ and $|P(x,y)|=l$. Then there exists a sequence

$$x=x_1\ E_1\ x_2\ E_2\ x_3\ E_3\cdots\ x_l\ E_l\ x_{l+1}=y$$

such that for all $i\in\{1,2,\ldots,l\},x_i,x_{i+1}\in E_i$, where x_i and E_i are distinct. If for all $i\in\{1,2,\ldots,l\}$, $E_i\cap E_j\neq\emptyset$, concludes that $|E_i\cap E_j|=1$, then $|\alpha(x_i)|=1$ and so so in $(H/\alpha,*^{(q)})$, for all $1\leq i\leq l,\alpha(x_i)*^{(1)}\alpha(x_{i+1})=\widehat{\alpha(x_i),\alpha(x_{i+1})}$. Hence there exists a sequence

$$\alpha(x_1),\widehat{\alpha(x_1),\alpha(x_2)},\alpha(x_2),\widehat{\alpha(x_2),\alpha(x_3)},\alpha(x_3),\ldots,\alpha(x_l),\widehat{\alpha(x_l),\alpha(x_{l+1})},\alpha(x_{l+1}).$$

Thus $=|P(\alpha(x),\alpha(y))|=l$ and so $|P(\alpha(x),\alpha(y))|=|P(x,y)|$. Now, if for some $i\in\{1,2,\ldots,l\}$, $E_i\cap E_j\neq\emptyset$, concludes that $|E_i\cap E_j|\geq 2$, then $|\alpha(x_i)|\geq 2$. Thus in sequence

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$x = x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_l\ E_l\ x_{l+1} = y$, there exists some $i \in \{1, 2, \ldots, l\}$, such that $x_i, x_{i+1} \in E_i \cap E_{i+1}$. It follows that $\alpha(x_i) = \alpha(x_{i+1})$ and so in $(H/\alpha, *^{(1)})$ are be considered, while in sequence $x = x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_l\ E_l\ x_{l+1} = y$, $x_iE_ix_{i+1}E_{i+1}x_{i+2}$ are distinct. Thus in this case we get that $|P(\alpha(x), \alpha(y))| < |P(x, y)|$.

(ii) Let $x, y \in X$ and $|P(x, y)| = l$. Since $|P(x, y)| = l$, there exists a sequence

$$x = x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_l\ E_l\ x_{l+1} = y$$

such that for all $i \in \{1, 2, \ldots, l\}, x_i, x_{i+1} \in E_i$, where x_i and E_i are distinct. Hence for all $i \in \{1, 2, \ldots, l\}$, we get that $\alpha(x_i) \neq \alpha(x_{i+1})$, so in $(H/\alpha, *^{(q)})$, for all $1 \leq i \leq l, \alpha(x_i) *^{(1)} \alpha(x_{i+q}) = \widehat{\alpha(x_i), \alpha(x_{i+q})}$. Hence there exists a sequence

$$\alpha(x_1), \widehat{\alpha(x_1), \alpha(x_{q+1})}, \alpha(x_{q+1}), \widehat{\alpha(x_{q+1}), \alpha(x_{2q+1})}, \alpha(x_{2q+1}), \ldots, \alpha(x_{\lfloor \frac{l}{q} \rfloor}), \widehat{\alpha(x_{\lfloor \frac{l}{q} \rfloor}), \alpha(x_{\lfloor \frac{l+1}{q} \rfloor})}, \alpha(x_{\lfloor \frac{l+1}{q} \rfloor}).$$

Thus by item 3.5(i), get that $|P(\alpha(x), \alpha(y))| = \lfloor \dfrac{l}{q} \rfloor$ and so $|P(\alpha(x), \alpha(y))| = \lfloor \dfrac{|P(x, y)|}{q} \rfloor$. □

Example 3.6. *Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and consider the hypergraph $H = (X, \{E_i\}_{i=1}^{3})$ as shown in Figure 1.*

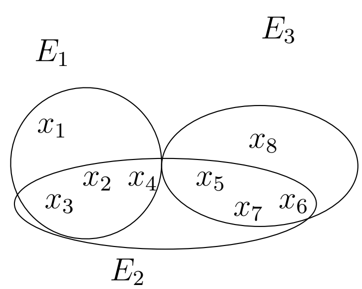

FIGURE 1. Connected Hypergraph $H = (X, \{E_i\}_{i=1}^{3})$

One can see that $\alpha(x_1) = \{x_1\}, \alpha(x_2) = \{x_2, x_3, x_4\}, \alpha(x_5) = \{x_5, x_6, x_7\}, \alpha(x_8) = \{x_8\}$. In hypergraph $H = (X, \{E_i\}_{i=1}^{3}), |P(x_1, x_6)| = 3$, because of $x_1E_1x_2, E_2x_5, E_3x_6$, but for $i = 1$, in the tree $(H/\alpha, *^{(1)}) \cong P_4$ as shown in Figures 2, $|P(\alpha(x_1), \alpha(x_6))| = 2$, because of $\alpha(x_5) = \alpha(x_6)$.

Corollary 3.7. *Let $H = (X, \{E_i\}_{i=1}^{m})$ be a connected hypergraph and $x, y \in X$, $q \in \mathbb{N}$ and for some $1 \leq i \neq j \leq m$, $\emptyset \neq E_i \cap E_j$ implies that $|E_i \cap E_j| = 1$. Then in $H = (X, \{E_i\}_{i=1}^{m})$ and $(H/\alpha, *^{(q)})$, $|P(\alpha(x), \alpha(y))| = |P(x, y)|$ if and only if $q = 1$.*

Theorem 3.8. *Let $H = (X, \{E_i\}_{i=1}^{m})$ be a hypergraph and $q \in \mathbb{N}$.*

(i) *If for all $1 \leq i \leq m$, there exists $1 \leq j \neq i \leq m$ such that $E_i \subseteq E_j \subseteq E_m$, then $(H/\alpha, *^{(1)}) \cong K_{|E|}$(complete graph),*

(ii) *If for all $1 \leq i \neq j \leq m, |E_i \cap E_j| = 0$, then $(H/\alpha, *^{(q)}) \cong N_{|E|}$(null graph);*

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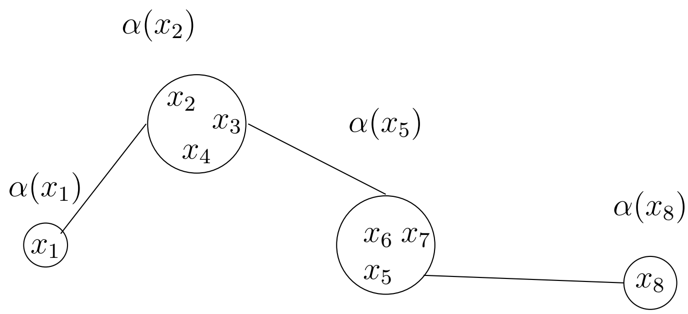

FIGURE 2. Tree $(H/\alpha, *^{(1)})$

(*iii*) *If for all* $1 \leq i \neq j \leq m, \emptyset \neq E_i \cap E_j = \bigcap\limits_{i=1}^{m} E_i$, *then* $(H/\alpha, *^{(1)}) \cong S_{|E|+1}$, (*star graph*);

(*iv*) *If for all* $1 \leq i \neq j \leq m, \emptyset \neq E_i \cap E_j = \bigcap\limits_{i=1}^{m} E_i$ *and there exists* $1 \leq t \leq m$ *that* $E_i \subseteq E_t$, *then* $(H/\alpha, *^{(1)}) \cong W_{|E|+1}$, (*wheel graph*);

Proof. (*i*) Since for all $1 \leq i \leq m$, there exists $1 \leq j \neq i \leq m$ such that $E_i \subseteq E_j \subseteq E_m$, for all $1 \leq i \leq m$ we can rearrange $E_i \subseteq E_{i+1} \subseteq E_m$. It follows that, for all $x, y \in X$, there exists $1 \leq i_1 \neq i_2 \neq \cdots i_k \leq m$ such that $\{x, y\} \subseteq \bigcap\limits_{j=1}^{k} E_{i_j}$.Thus for all $x \in E_i, \alpha(x) = E_1$, and for all $2 \leq i \leq m$, $x \in E_{i+1} \setminus E_i$ implies that $\alpha(x) = E_{i+1} \setminus E_i$. Hence by Theorem 3.3, for all $x, y \in X, \alpha(x) *^{(1)} \alpha(y) = \widehat{\alpha(x), \alpha(y)}$ and so $(H/\alpha, *^{(1)}) \cong K_{|E|}$.
(*ii*) Since for all $1 \leq i \neq j \leq m, |E_i \cap E_j| = 0$, we get that $E_i = \alpha(x)$, where $x \in E_i$. In addition, for all $1 \leq i \neq j \leq m, E_i \cap E_j = \emptyset$, imply that $\alpha(x) *^{(1)} \alpha(y) = \widehat{\emptyset}$. Hence by Theorem 3.3, $(H/\alpha, *^{(q)}) \cong N_{|E|}$.
(*iii*) Let $x \in \bigcap\limits_{i=1}^{m} E_i$. Since $\emptyset \neq \bigcap\limits_{i=1}^{m} E_i$, we get $\alpha(x) = \{x\}$. Moreover, for all $1 \leq i \neq j \leq m, \emptyset \neq E_i \cap E_j = \bigcap\limits_{i=1}^{m} E_i$ imply that $\alpha(x) = E_i \setminus \bigcap\limits_{i=1}^{m} E_i$, where $x \in E_i \setminus \bigcap\limits_{i=1}^{m} E_i$. By Theorem 3.3, it follows that for all $y \in X, \alpha(x) *^{(1)} \alpha(y) = \widehat{\alpha(x), \alpha(y)}$, where $x \in \bigcap\limits_{i=1}^{m} E_i$ and $\alpha(x) *^{(1)} \alpha(y) = \widehat{\emptyset}$, where $x \notin \bigcap\limits_{i=1}^{m} E_i$. Thus $(H/\alpha, *^{(1)}) \cong S_{|E|+1}$.
(*iv*) Since for all $1 \leq i \leq m$, there exists $1 \leq t \leq m$ that $E_i \subseteq E_t$, we get that for all $\alpha(x), \alpha(y) \in V(H/\alpha), \alpha(x) *^{(1)} \alpha(y) = \widehat{\alpha(x), \alpha(y)}$. In addition, using item 3.8 (*iii*), $(H/\alpha, *^{(1)}) \cong W_{|E|+1}$. □

Theorem 3.9. *Let* $H = (X, \{E_i\}_{i=1}^{m})$ *be a connected hypergraph. Then* $(H/\alpha, *^{(1)})$ *is a connected graph.*

Proof. Let $\alpha(x), \alpha(y) \in H/\alpha$. Since $H = (X, \{E_i\}_{i=1}^{m})$ is a connected hypergraph and $x, y \in H$, there exists a path $x = x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_l\ E_l\ x_{l+1} = y$ in connected hypergraph H, where edges

$E_1, E_2, \ldots, E_l$ and vertices $x_1, x_2, \ldots, x_{l+1}$ except x_1 and x_{l+1} are distinct. So for all $1 \leq i \leq l, x_i E_i x_{i+1}$ and $E_i \cap E_{i+1} \neq \emptyset$. It follows that for $q = 1, \alpha(x_i) *^{(q)} \alpha(x_{i+1}) = \widehat{\alpha(x_i), \alpha(x_{i+1})}$ and as Figure 10, there exists a path $\alpha(x_1)e_1\alpha(x_2)e_2\alpha(x_3)\cdots\alpha(x_{l-1})e_{l-1}\alpha(x_l)$, where $e_i = \widehat{\alpha(x_i), \alpha(x_{i+1})}$ in graph H/α. Thus $(H/\alpha, *^{(1)})$ is a connected graph. □

Theorem 3.10. *Let $q \in \mathbb{N}$, $H = (X, \{E_i\}_{i=1}^m)$ be a connected hypergraph such that for all $1 \leq i \leq m, |E_i| \geq 2$. If for any $E_i \in E$ there exists $E_j \neq E_i$ such that $|E_i \cap E_j| \neq 0$, then $(H/\alpha, *^{(\lceil \frac{m}{2} \rceil)})$ is a connected graph.*

Proof. Since for any $E_i \in E$ there exist $E_j \neq E_i$ such that $|E_i \cap E_j| \neq 0$ and $|E_i| + |E_j| \geq 4$, we get that there exists a path $a_i\ E_i\ b_{ij}\ E_j\ a_j$, where $b_{ij} \in E_i \cap E_j$ and $a_i \in E_i, a_j \in E_j$. It follows that there exists a sequence $x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_m\ E_m\ x_{m+1}$ such that for all $i \in \{1, 2, \ldots, m\}, x_i, x_{i+1} \in E_i$ and $E_i \cap E_{i+1} \neq \emptyset$. Thus for all $1 \leq i \neq j \leq m, |P(x_i, x_j)| = \lceil \frac{j-i}{2} \rceil$, and so we consider the sequence

$$x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \cdots\ x_{\lceil \frac{j-i}{2} \rceil}\ E_{\lceil \frac{j-i}{2} \rceil}\ x_{1+\lceil \frac{j-i}{2} \rceil}\ E_{1+\lceil \frac{j-i}{2} \rceil} \cdots\ x_m\ E_m x_{m+1}.$$

Hence by Theorem 3.3, for any $1 \leq i \leq m$, get that

$$\alpha(x_i), \alpha(x_{i+\lceil \frac{m}{2} \rceil}) \in V(H/\alpha), \alpha(x_i) *^{(\frac{m}{2} \rceil)} \alpha(x_{i+\lceil \frac{m}{2} \rceil}) = \alpha(x_i), \widehat{\alpha(x_{i+\lceil \frac{m}{2} \rceil}})).$$

Thus $(H/\alpha, *^{(\lceil \frac{m}{2} \rceil)})$ is a connected graph. □

Let X be a nonempty set and $k \in \mathbb{N}$. We will call X is a k-part, if $X = (\bigcup_{\substack{Y_i \in P^*(X) \\ 1 \leq i \leq k}} Y_i)$ such that for all $i \neq j$, have $Y_i \cap Y_j = \emptyset$.

Theorem 3.11. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a connected hypergraph and $k \geq 3$.*

(i) If there exist some $1 \leq i \leq m$ such that E_i has a k-part, then K_k is a subgraph of $(H/\alpha, *^{(1)})$.

(ii) If $|\{E_i \in P^*(X) \mid E_i$ is a $k-$part$\}| = r$, then $|E(H/\alpha, *^{(1)})| \geq (r(\binom{k}{2} - 1) + m)$.

(iii) If $|\{E_i \in P^*(X) \mid E_i$ is a $k-$part$\}| = m$, then $|E(H/\alpha, *^{(1)})| \geq m\binom{k}{2}$.

Proof. (i) Let E_s is a k-part hyperedge in the connected hypergraph $H = (X, \{E_i\}_{i=1}^m)$. Then there exist at least $k-1$ hyperedges $E_{i_1}, E_{i_2}, \cdots, E_{i_{k-1}}$, and $x_s \in E_s$ such that $E_s = \{x_s\} \cup \bigcup_{j=1}^{k-1}(E_s \cap E_{i_j})$, for all $1 \leq j \leq k-1, x_s \notin E_{i_j}$ and for all $1 \leq j' \neq j \leq k-1, E_s \cap (E_{i_j} \cap E_{i_{j'}}) = \emptyset$. Hence for all $x \in E_s$ there exists $1 \leq j \leq k-1$ such that $\alpha(x) = \{x_s\}$ or $\alpha(x) = E_s \cap E_{i_j}$. Since for all $1 \leq j \leq k-1, (E_s \cap E_{i_j}) \subseteq E_s$, for all $x, y \in E_s$, we get that $\alpha(x) *^{(1)} \alpha(y) = \widehat{\alpha(x), \alpha(y)}$. Thus the complete graph K_k is a subgraph of $(H/\alpha, *^{(1)})$, because of $|E_s| = k$.

(ii) Since $|\{E_i \in P^*(X) \mid E_i$ is a k$-part\}| = r$, by 3.11 (i), the number of subgraphs K_k of $(H/\alpha, *^{(1)})$ is

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r. In addition, $H = (X, \{E_i\}_{i=1}^m)$ is a connected hypergraph, so there exists at least $\alpha(z_1), \alpha(z_2), \ldots, \alpha(z_{m-r})$ such that by Theorem 3.9, there exist some $\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_{m-r})$, which $\alpha(z_i) *^{(1)} \alpha(w_{i'}) = \widehat{\alpha(z_i), \alpha(w_{i'})}$ and $1 \leq i \neq i' \leq m - r$. Hence $|E(H/\alpha, *^{(1)})| \geq r\binom{k}{2} + m - r$.
(iii) Since $|\{E_i \in P^*(X) \mid E_i \text{ is a k}-part\}| = m$, by 3.11 (ii), the number of subgraphs K_k of $(H/\alpha, *^{(1)})$ is m and so $|E(H/\alpha, *^{(1)})| \geq m\binom{k}{2}$. □

Theorem 3.12. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypertree and $k \in \mathbb{N}$.*

- (i) *If $|\{E_i \in P^*(X) \mid E_i$ is a $k-part\}| = r$, then $k \geq 2$.*
- (ii) *If $k = 2$ and $|\{E_i \in P^*(X) \mid E_i$ is a $k-part\}| = r$, then $|E(H/\alpha, *^{(1)})| = m$.*
- (iii) *If $k \geq 3$ and $|\{E_i \in P^*(X) \mid E_i$ is a $k-part\}| = m$, then $|E(H/\alpha, *^{(1)})| = m\binom{k}{2}$.*

Proof. (i) Let $k = 1$. If $|\{E_i \in P^*(X) \mid E_i \text{ is a } 1-part\}| = r$, then $H = (X, \{E_i\}_{i=1}^m)$ is not a connected hypergraph and so it is a contradiction, because of $H = (X, \{E_i\}_{i=1}^m)$ is a hypertree.
(ii) Let $k = 2$. Then for all $x \in X$, we have $\alpha(x) = \{x\}$ and $\alpha(x) *^{(1)} \alpha(y) = \widehat{\alpha(x), \alpha(y)}$, where $y \in E_i \cap E_j$ and $x \in E_i, y \in E_j$. Hence $|E(H/\alpha, *^{(1)})| = m$.
(iii) Let $k \geq 3$ and $|\{E_i \in P^*(X) \mid E_i \text{ is a k}-part\}| = m$. Using Theorem 3.11, $|E(H/\alpha, *^{(1)})| \geq m\binom{k}{2}$. If $|E(H/\alpha, *^{(1)})| > m\binom{k}{2}$, then there exist $\alpha(x), \alpha(y) \in H/\alpha$ such that $|P(\alpha(x), \alpha(y))| \geq k + 1$ and $\alpha(x) *^{(1)} \alpha(y) = \widehat{\alpha(x), \alpha(y)}$. It follows that there exist two sequences

$$x\ E_{i'}\ y \text{ and } x = x_{i1}\ E_{i1}\ x_{i2}\ E_{i2}\ x_{i3}\ E_{i3} \cdots\ x_{is}\ E_{is}\ x_{is+1} = y$$

such that for all $j \in \{1, \ldots, s\}$, we have $x_{ij}, x_{ij+1} \in E_{ij}$, $s \geq k + 1$ and $1 \leq i' \leq m$. Thus there exist a sequence

$$x\ E_{i'}\ y \text{ and } x = x_{i1}\ E_{i1}\ x_{i2}\ E_{i2}\ x_{i3}\ E_{i3} \cdots\ x_{is}\ E_{is}\ x_{is+1} = y\ E_{i'}\ x$$

such that for all $j \in \{1, \ldots, s\}$, we have $x_{ij}, x_{ij+1} \in E_{ij}$, $s \geq k + 1$ and $1 \leq i' \leq m$. Therefore, $H = (X, \{E_i\}_{i=1}^m)$ has a cycle, which is a contradiction. Thus $|E(H/\alpha, *^{(1)})| = m\binom{k}{2}$. □

Example 3.13. *Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$. Consider the hypertree $H = (X, \{E_i\}_{i=1}^5)$, in Figure 3.*

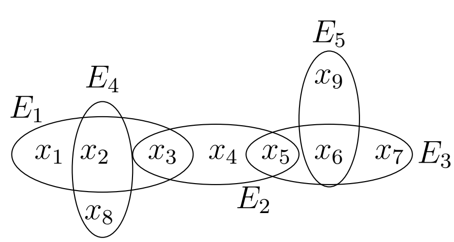

Figure 3. Hypertree $H = (X, \{E_i\}_{i=1}^5)$

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Computations show that for all $i \in \{1,3,4,5,7\}$, $\alpha(x_i) = \{x_i\}, \alpha(x_2) = \{x_2, x_8\}, \alpha(x_6) = \{x_6, x_9\}$ *and a graph* $(H/\alpha, *^{(1)})$ *is obtained in Figure 4.*

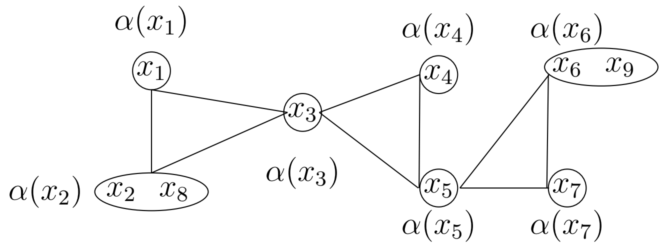

FIGURE 4. Graph $(H/\alpha, *^{(1)})$

Definition 3.14. *Let* $H = (X, \{E_i\}_{i=1}^m)$ *be a hypergraph. Then* H *is called*

(i) *an* (α, q)*-hypergraph, if* $(H/\alpha, *^{(q)})$ *is a graph,*

(i) *a fundamental* (α, q)*-hypergraph, if* $(H/\alpha, *^{(q)}) \cong H = (X, \{E_i\}_{i=1}^m)$.

Corollary 3.15. *Let* $H = (X, \{E_i\}_{i=1}^m)$ *be a hypertree and* $k \in \mathbb{N}$. *Then* $H = (X, \{E_i\}_{i=1}^m)$ *is a fundamental* $(\alpha, 1)$*-hypertree if and only if* $k = 2$ *and* $|\{E_i \in P^*(X) \mid E_i \text{ is a } k\text{−part}\}| = m$.

Theorem 3.16. *Let* $k \geq 3$. *Then every hypertree that is free of* k*-part is an* $(\alpha, 1)$*-hypertree.*

Proof. Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypertree. Since every hypertree is a connected hypergraph, by Theorem 3.9, $(H/\alpha, *^{(1)})$ is a connected graph. Using Theorem 3.3, $(H/\alpha, *^{(1)})$ is an $(\alpha, 1)$-hypertree, because of $(H/\alpha, *^{(1)})$ is free of k-part. □

Example 3.17. *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. *Then a hypertree* $H = (X, \{E_i\}_{i=1}^3)$, *is an* $(\alpha, 1)$*-hypertree but not an* $(\alpha, 2)$*-hypertree in Figure 5.*

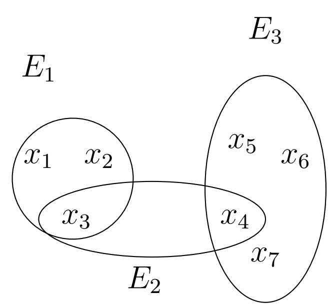

FIGURE 5. Hypertree $H = (X, \{E_i\}_{i=1}^3)$

One can see that $\alpha(x_1) = \{x_1, x_2\}, \alpha(x_3) = \{x_3\}, \alpha(x_4) = \{x_4\}, \alpha(x_5) = \{x_5, x_6, x_7\}$. *Since* $x_1E_1x_3, x_3E_2x_4, x_4E_3x_5$ *and* $x_1E_1x_3E_2x_4, x_3E_2x_4E_3x_5$ *for* $i = 1$, *a tree* $(H/\alpha, *^{(1)}) \cong P_4$ *and for* $i = 2$, *a unconnected graph* $(H/\alpha, *^{(2)})$ *are obtained in Figures 6, 7, respectively.*

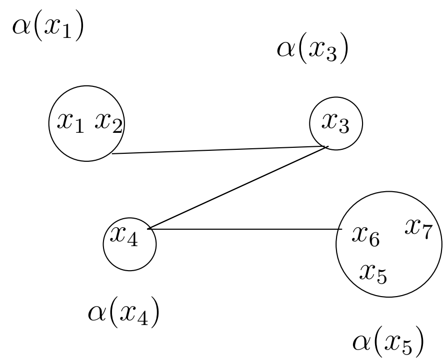

FIGURE 6. Tree $(H/\alpha, *^{(1)})$

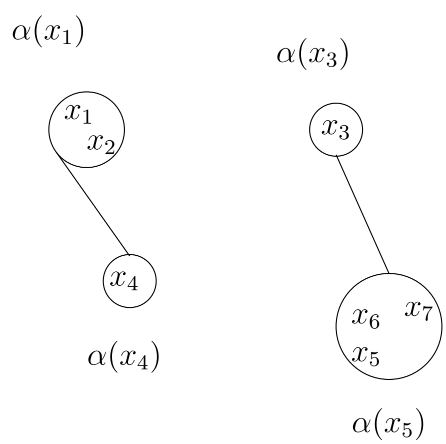

FIGURE 7. Nonconnected Graph $(H/\alpha, *^{(2)})$ & Graphs $(H/\alpha, *^{(i)})$ for $i \in \{1, 2\}$

The following example shows that the converse of Theorem 3.16, is not necessarily true.

Example 3.18. *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. *Then a hypergraph* $H = (X, \{E_i\}_{i=1}^{4})$, *is an* $(\alpha, 1)$-*hypertree in Figure 8.*

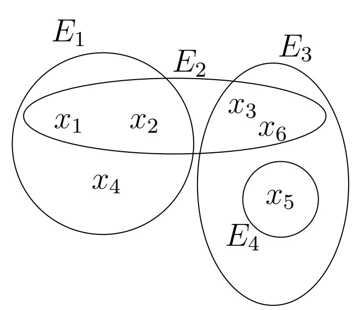

FIGURE 8. Hypergraph (non-hypertree)$H = (X, \{E_i\}_{i=1}^{4})$

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Obviously $\alpha(x_1)=\{x_1,x_2\},\alpha(x_3)=\{x_3,x_6\},\alpha(x_4)=\{x_4\},\alpha(x_5)=\{x_5\}$ *and* $H/\alpha\cong T_3$ *is obtained in Figure 9.*

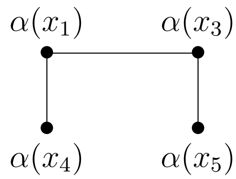

FIGURE 9. Tree $(H/\alpha,*_1)\cong T_3$

Let $H=(X,E=\{E_i\}_{i=1}^m)$ be a hypergraph. Then $H=(X,\{E_i\}_{i=1}^m)$ is called a complete hypergraph, if for any $x,y\in X$, there exists a hyperedge $E_i\in E$ such that $x,y\in E_i$.

Theorem 3.19. *Let* $H=(X,\{E_i\}_{i=1}^m)$ *be a complete hypergraph.*

(*i*) *If for all* $1\leq i\neq j\ \leq m, |E_i\cap E_j|=k$, *then* $(H/\alpha,*^{(1)})\cong K_{\lfloor\frac{|X|}{k}\rfloor}$;

(*ii*) *If for all* $1\leq i\neq j\ \leq m, |E_i\cap E_j|\neq\emptyset$, *then* $(H/\alpha,*^{(1)})\cong K_{|\mathcal{E}|}$;

Proof. (*i*) Let $x\in X$. Then for any $y\in X$, there exists $E_i\in E$ such that $x,y\in E_i$, because of $H=(X,\{E_i\}_{i=1}^m)$ is a complete hypergraph. Thus for any given $\alpha(x),\alpha(y)\in H/\alpha$, there exists a path $x\ E_i\ y$, which $x,y\in E_i$, and so $\alpha(x)*^{(1)}\alpha(y)=\widehat{\alpha(x),\alpha(y)}$. In addition, for all $1\leq i\neq j\ \leq m, |E_i\cap E_j|=k$, we get that $\alpha(x)=E_i\cap E_j$, which $x,y\in E_i\cap E_j$, hence $|V(H/\alpha)|=\lfloor\frac{|X|}{k}\rfloor$. □

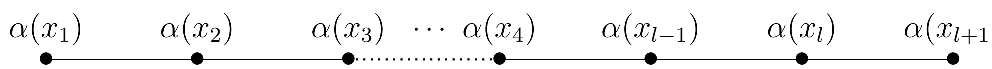

FIGURE 10. A path from $\alpha(x)$ to $\alpha(y)$ in graph H/α

Let $k_1,k_2,\ldots,k_m\in\mathbb{N}^{\geq2}$. A hypertree $H=(X,\{E_i\}_{i=1}^m)$ is called a $(k_1,k_2,\ldots,k_m)$-hypertree, if for all $1\leq i\leq m, |E_i|=k_i$ and $E_i\cap E_{i+1}\neq\emptyset(E_i\cap E_j=\emptyset, j\neq i+1)$. Clearly for any $n\in\mathbb{N}$, path graph P_n is a $(k_1,k_2,\ldots,k_{n-1})$-hypertree.

Theorem 3.20. *Let* $H=(X,E=\{E_i\}_{i=1}^m)$ *be a* $(k_1,k_2,\ldots,k_m)$*-hypertree.*

(*i*) $|X|=\sum\limits_{i=1}^m k_i-\sum\limits_{1\leq i\neq j\leq m}|E_i\cap E_j|$.

(*ii*) *If for all* $1\leq i\leq m, k_i=t$, *then* $|E|=\dfrac{|X|+\sum\limits_{1\leq i\neq j\leq m}|E_i\cap E_j|}{t}$.

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Proof. (*i*) Since $\bigcup_{i=1}^{m} E_i = X$ and for all $1 \leq i \neq j \leq m, E_i \cap E_{i+1} \neq \emptyset$, get $\sum_{i=1}^{m} k_i = |X| + \sum_{1\leq i\neq j\leq m} |E_i \cap E_j|$.
(*ii*) Immediate by (*i*). □

Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypergraph. From now on, for simplify, $(E_i \setminus E_{i-1}) \setminus (E_i \cap E_{i+1}) \neq \emptyset$, will denote by $(E_i \setminus E_{i-1}) \approx (E_i \cap E_{i+1})$ and $s = |\{1 \leq i \leq m \mid (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})\}|$.

Theorem 3.21. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a $(k_1, k_2, \ldots, k_m)$-hypertree.*

(*i*) *If for all $1 \leq i \leq m, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$, then $(H/\alpha, *^{(1)}) \cong P_{m+1}$.*
(*ii*) *If for all $1 \leq i \leq m, (E_{i+1} \setminus E_i) \approx (E_{i+1} \cap E_{i+2})$, then $(H/\alpha, *^{(1)}) \cong P_{2m-1}$.*
(*iii*) *If there exists $1 \leq i \leq m$ such that $(E_{i+1} \setminus E_i) \approx (E_{i+1} \cap E_{i+2})$, then $(H/\alpha, *^{(1)}) \cong P_{m+s+1}$.*

Proof. (*i*) Let $x \in X$. Since $H = (X, \{E_i\}_{i=1}^m)$ is a $(k_1, k_2, \ldots, k_m)$-hypertree and for all $1 \leq i \leq m, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$, we get that either $x \in E_1$ or $x \in E_m$ or for all $1 \leq i \leq m, x \in E_i \cap E_{i+1}$. It concludes $|H/\alpha| = 1 + (m-1) + 1 = m + 1$ and by Theorem 3.16, $(H/\alpha, *^{(1)}) \cong P_{m+1}$.
(*ii*) Let $x \in X$. Since $H = (X, \{E_i\}_{i=1}^m)$ is a $(k_1, k_2, \ldots, k_m)$-hypertree and for all $1 \leq i \leq m, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$, we get that $x \in E_1$ or $x \in E_m$ or for all $1 \leq i \leq m, x \in E_i \cap E_{i+1}$ or for all $1 \leq i \leq m, x \in [(E_{i+1} \setminus E_i) \setminus (E_{i+1} \cap E_{i+2})]$. It concludes $|H/\alpha| = 1 + m + (m-1) + 1 = m + 1$ and by Theorem 3.16, $(H/\alpha, *^{(1)}) \cong P_{2m-1}$.
(*iii*) Let $x \in X$. Since $H = (X, \{E_i\}_{i=1}^m)$ is a $(k_1, k_2, \ldots, k_m)$-hypertree and for all $1 \leq i \leq m, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$, we get that $x \in E_1$ or $x \in E_m$ or for all $1 \leq i \leq m, x \in E_i \cap E_{i+1}$ or there exist some $1 \leq i \leq m$, such that $x \in [(E_{i+1} \setminus E_i) \setminus (E_{i+1} \cap E_{i+2})]$. It concludes $|H/\alpha| = 1 + s + (m-1) + 1 = m + 1$ and by Theorem 3.16, $(H/\alpha, *^{(1)}) \cong P_{m+s+1}$. □

Corollary 3.22. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a $(k_1, k_2, \ldots, k_t, k_{t+1}, \ldots, k_m)$-hypertree. If $(E_{t+1} \setminus E_t) \not\approx (E_{t+1} \cap E_{t+2})$, then $(H/\alpha, *^{(1)}) \cong P_{t+m}$.*

Theorem 3.23. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a $(k_1, k_2, \ldots, k_m)$-hypertree, for all $1 \leq i \leq m-2, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$ and for all $1 \leq i \leq m-1, |E_i \cap E_{i+1}| = k \geq 2$.*

(*i*) *If $(H/\alpha, *^{(q)})$ is a connected graph, then $\lceil \frac{m}{2} \rceil \geq q - 1$.*
(*ii*) *If $m \geq 2$, then $(H/\alpha, *^{(2)})$ has* **Eulerian** *path and $|E(H/\alpha, *^2)| = 2m - 1$.*

Proof. (*i*) Since H has m hyperedges, by Theorem 3.21, there exist $x_1, x_2, \ldots, x_{m+1}$ such that $H/\alpha = \{\alpha(x_i) \mid 1 \leq i \leq m+1\}$. Since H/α is connected, for any $1 \leq i \neq j \leq m+1$, get that $|P(\alpha(x_i), P(\alpha(x_j))| \geq 1$. Thus in $(H/\alpha, *^{(q)})$, for any given path $\alpha(x_i)\ E_i \alpha(x_{i+1})\ E_{i+1} \cdots \alpha(x_j) E_j$ must be $j = q + i - 1 \geq q - 1$, where $j \leq \lceil \frac{m}{2} \rceil$. It concludes that $\lceil \frac{m}{2} \rceil \geq q - 1$.
(*ii*) Since $H = (X, \{E_i\}_{i=1}^m)$ is a $(k_1, k_2, \ldots, k_m)$-hypertree and for all $1 \leq i \leq m, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$, by Theorem 3.21, there exist $x_1, x_2, \ldots, x_{m+1}$ such that $H/\alpha = \{\alpha(x_i) \mid 1 \leq i \leq m\}$. Thus for all $1 \leq i \leq m-2, x_i\ E_i\ x_{i+1}\ E_{i+1}\ x_{i+2}$ and $x_i\ E_i\ x_{i+1}\ E_{i+1}\ x_{i+3}$. It follows that in $H/\alpha, \alpha(x_i) *^{(2)} \alpha(x_{i+1}) =$

$\widehat{\alpha(x_i),\alpha(x_{i+1})}$ and $\alpha(x_i) *^{(2)} \alpha(x_{i+2}) = \widehat{\alpha(x_i),\alpha(x_{i+2})}$. So in $H/\alpha, deg(\alpha(x_1)) = deg(\alpha(x_{m+1})) = 2, deg(\alpha(x_2)) = deg(\alpha(x_m)) = 3$ and for any $i \in \{x_3, x_4, \ldots, x_{m-1}\}$, $deg(\alpha(x_i))) = 4$. Hence $(H/\alpha, *^2)$ has **Eulerian** path and $\sum_{i=1}^{m} deg(\alpha(x_i)) = (2\times 2) + (2\times 3) + ((m-3)\times 4) = 4m-2$ and so $|E(H/\alpha)| = 2m-1$. □

Theorem 3.24. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a $(k_1, k_2, \ldots, k_m)$-hypertree, for all $1 \le i \le m-2, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$ and for all $1 \le i \le m-1, |E_i \cap E_{i+1}| = k \ge 2$.*

- (i) *If $m \ge 3$, then $|E(H/\alpha, *^{(3)})| = 3m-5$.*
- (ii) *If $m \ge 5$, then $|E(H/\alpha, *^{(4)})| = 3m-8$.*
- (iii) *If $m \ge 7$, then $|E(H/\alpha, *^{(5)})| = 3m-12$.*
- (iv) *If $m \ge 9$, then $|E(H/\alpha, *^{(6)})| = 2m-3$.*

Proof. (i) Since H has m hyperedges, by Theorem 3.21, there exist $x_1, x_2, \ldots, x_{m+1}$ such that $H/\alpha = \{\alpha(x_i) \mid 1 \le i \le m+1\}$. Since H/α is connected, for any $1 \le i \ne j \le m+1$, get that $|P(\alpha(x_i), P(\alpha(x_j))| \ge 1$. Thus in $(H/\alpha, *^{(q)})$, for any given path $\alpha(x_i)\ E_i\alpha(x_{i+1})\ E_{i+1}\cdots\alpha(x_j)E_j$ must be $j = q+i-1 \ge q-1$, where $j \le \lceil \frac{m}{2} \rceil$. It concludes that $\lceil \frac{m}{2} \rceil\ \ge q-1$. Applying Theorem 3.23, $\lceil \frac{m}{2} \rceil\ \ge 3-1$, implies that $m = 3$. By induction on m, for $m = 3$ we get a sequence $E(m = 3) = |E(H/\alpha, *^{(3)})| = 4$. Suppose that for $m = k$, $E(m = k) = |E(H/\alpha, *^{(3)})| = 3k-5$ and $V(H/\alpha, *^{(3)}) = \{\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_{k+1})\}$. Since by adding of any vertex $\alpha(x)$ to graph $(H/\alpha, *^{(3)})$, which $\alpha(x) \notin (H/\alpha, *^{(3)})$, we get that $deg(\alpha(x)) = 2$ thus by $V(H/\alpha, *^{(3)}) \cup \{\alpha(x)\}$, we get that $\sum_{y\ne x} deg(\alpha(y)) = \sum_{y\ne x} deg(\alpha(y)) + deg(\alpha(y)) = 6k-10+6 = 6k-4$. Hence $E(m = k+1) = |E(H/\alpha, *^{(3)})| = 3k-2$ and so $|E(H/\alpha, *^{(3)})| = 3m-5$.
$(ii), (iii), (iv)$ Are similar to item 3.24 (i). □

Theorem 3.25. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypertree. If for all $1 \le i \ne j\ \le m, |E_i \setminus E_j| = |E_j \setminus E_j| = |E_i \cap E_j| = k$ and $E_i \not\subseteq E_j$, then $m = \lfloor \frac{|X|}{k} \rfloor - 1$;*

Proof. Since for all $1 \le i \ne j\ \le m, |E_i \setminus E_j| = |E_j \setminus E_j| = |E_i \cap E_j| = k$, we get that for all $1 \le i \ne j\ \le m, |E_i| = |E_j| = 2k$. It follows there exists $t \in \mathbb{N}$ such that $|X| \le 2k + tk$ and so $t \le \frac{|X|-2k}{k}$. Thus $m \le \frac{|X|}{k} - 2 + 1 = \frac{|X|}{k} - 1$. Now, if add a hyperedge to H, it makes at least a cycle which is a contradiction. Therefore, $m = \frac{|X|}{k} - 1$. □

Corollary 3.26. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a $(k_1, k_2, \ldots, k_m)$- hypertree. If for all $1 \le i \ne j\ \le m, |E_i \setminus E_j| = |E_j \setminus E_j| = |E_i \cap E_j| = k$ and $E_i \not\subseteq E_j$, then $(H/\alpha, *^{(1)}) \cong P_{\lfloor \frac{|X|}{k} \rfloor}$.*

Theorem 3.27. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypertree. If for all $1 \le i \ne j\ \le m, |E_i \setminus E_j| = |E_j \setminus E_j| = |E_i \cap E_j| = k$ and $E_i \not\subseteq E_j$. If $(H/\alpha, *^{(1)}) = (V(H/\alpha), E(H/\alpha))$, then*

- (i) *for all $x \in X$, we have $deg(\alpha(x)) = deg(x)$;*

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(ii) $|V(H/\alpha)| = 1 + \lceil \frac{1}{2k} \sum_{i=1}^{m} |E_i| \rceil$ *and* $|E(H/\alpha)| = \lceil \frac{1}{2k} \sum_{i=1}^{m} |E_i| \rceil$.

Proof. (i) Let $x \in X$. Then there exists some $1 \leq i \leq m$ such that $x \in E_i$. If for all $1 \leq j \neq i \leq m, x \notin E_i \cap E_j$, then $deg(x) = 1$. In this case since $H = (X, E = \{E_i\}_{i=1}^m)$ is a hypertree, by Lemma 3.1, there exists $j \neq i$ such that $E_i \cap E_j \neq \emptyset$ and so there exists a hyperedge $E_j, y \in E_j$ and a path $x(E_i \cap E_j)y$. Thus by Theorem 3.3, $\alpha(x) *^{(1)} \alpha(y) = \alpha\widehat{(x), \alpha(}y)$ and so $deg(\alpha(x)) = 1$. Now, if there exists some $1 \leq j \neq i \leq m, x \in E_i \cap E_j$, since $H = (X, E = \{E_i\}_{i=1}^m)$ is a hypertree and for all $1 \leq i \neq j \ \leq m, |E_i \setminus E_j| = |E_j \setminus E_j| = |E_i \cap E_j| = k$ and $E_i \nsubseteq E_j$, we get that $deg(x) = k$. In this case, there exists $1 \leq j_1 \neq j_2 \neq \cdots \neq j_k \leq m$ and hyperedges $E_{j_1}, E_{j_2}, \ldots, E_{j_k}, y_{j_1} \in E_{j_1}, y_{j_2} \in E_{j_2}, \ldots, y_{j_k} \in E_{j_k}$ such that $x(E_i \cap E_{j_1})y_{j_1}, x(E_i \cap E_{j_2})y_{j_2}, \ldots, x(E_i \cap E_{j_k})y_{j_k}$. Thus by Theorem 3.3, for all $l \in \{j_1 \neq j_2 \neq \ldots \neq j_k \leq m\}$, $\alpha(x) *^{(1)} \alpha(y_{j_l}) = \alpha\widehat{(x), \alpha(}y_{j_l})$ and so $deg(\alpha(x)) = k$.
(ii) By Theorem 3.21, $(H/\alpha, *^{(1)}) = (V(H/\alpha), E(H/\alpha))$, Let $x \in X$. Then there exist E_i, E_j such that either $x \in (E_i \setminus E_j) \setminus (E_j \setminus E_i), 1 \leq i \neq j \ \leq m, |E_i \setminus E_j| = |E_j \setminus E_j| = k$ or $x \in E_i \cap E_j$ that $|E_i \cap E_j| = k$. Thus we get that $|\alpha(x)| = \frac{|E_i|}{2k}$. In addition $V/\alpha = \{\alpha(x) \mid x \in H\}$ is the set of all equivalence class of α on X, $H = (X, \{E_i\}_{i=1}^m)$ is a hypertree and so $|V(H/\alpha)| = 2 + \lceil \frac{1}{2k} \sum_{i=1}^{m-1} |E_i| \rceil = 1 + \lceil \frac{1}{2k} \sum_{i=1}^{m} |E_i| \rceil$. By Theorem 3.21, $(H/\alpha, *^{(1)}) = (V(H/\alpha), E(H/\alpha))$ is a tree and for all $x \in X$, $deg(\alpha(x)) = deg(x)$, hence $|E(H/\alpha)| = \lceil \frac{1}{2k} \sum_{i=1}^{m} |E_i| \rceil$. □

Corollary 3.28. *Let* $H = (X, E = \{E_i\}_{i=1}^m)$ *be a hypertree. If for all* $1 \leq i \neq j \ \leq m, |E_i \setminus E_j| = |E_j \setminus E_j| = |E_i \cap E_j| = k$ *and* $E_i \nsubseteq E_j$*. Then* $\sum_{x \in X} deg(x) = 2|E|$.

4. Application of (Hyper)tree in (Hyper)networks

In this section, we consider the concepts of hypertree and α-relation and apply them to hypernetworks and networks.

Social Network: A social network is modelified by a graph whose nodes represent the actors of the network (people, organizations), and the links between the nodes show relationships. Let $X = \{a := BOB, b := LISA, d := ANDY, e := ALEX, f := ROY, h := MARK, t := RICK, s := KIM, r := JAIN, d := MARY, l := KATE\}$ be a set of users that make an organizational social hypernetwork that is presented in Figure 11. In this hypernetwork, hyperedges reflect contacts (relationships) between users, and the relationships are extracted from social communication or common activities of users by WhatsApp messenger.

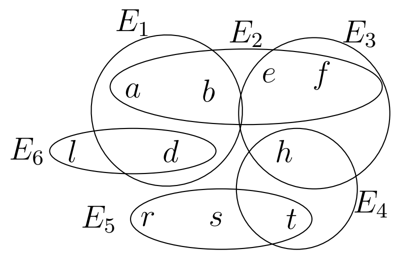

FIGURE 11. Hypernetwork $H = (X, \{E_i\}_{i=1}^{6})$

Now, by Theorem 3.3, we obtain a network $(H/\alpha, *^{(2)})$ as shown in Figure 12, where $\alpha(a) = \{a, b\}, \alpha(d) = \{d\}, \alpha(e) = \{e, f\}, \alpha(h) = \{h\}, \alpha(t) = \{t\}, \alpha(s) = \{r, s\}$ and $\alpha(l) = \{l\}$. In the network, $(H/\alpha, *^{(2)})$, $q = 2$ means that users decide to have a decision tree for how to communications after two hours of discussion.

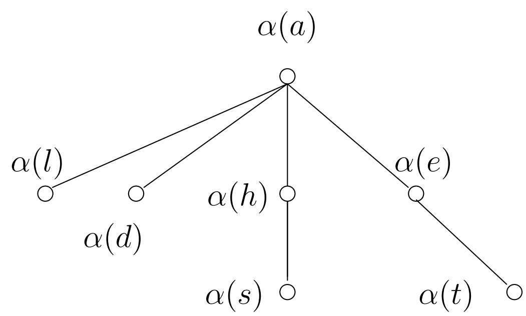

FIGURE 12. Network $(H/\alpha, *^{(2)})$

In the network $(H/\alpha, *^{(2)})$, anyone see that the group user $\alpha(a)$ can connect to four group of users and connected to user h, while in hypernetwork $H = (X, \{E_i\}_{i=1}^{6})$, there is a gap between of Bon and Mark.

Food Web: A food chain is the interdependence of food between living organisms, which resembles chains in which energy is transferred from one ring to another, and each creature derives its energy from the previous being while supplying the existing energy. When several food chains are shared by one or more living things, they come together to form a food web. Let $X = \{C := Coyote, V := Vulture, V_1 := Vulture\text{-}1, H := Hawk, M := Mouse, S := Snake, S_1 := Snake\text{-}1, G := Grasses, G_1 := Grasshopper\text{-}1, G_2 := Grasshopper\text{-}2\}$ be a set that contains grasses and some type of animals. We consider a hypernetwork of animals with accessible food as shown in Figure 13, where

$$E_1 = \{Mouse, Grasses\}, E_2 = \{Vulture, Vultore - 1, Grasses\}, E_3 = \{Coyote, Grasses\},$$

$$E_4 = \{Snake, Snake - 1, Grasses\}, E_5 = \{Mouse, Grasshoper - 1, Grasshoper - 2\},$$

$$E_6 = \{Mouse, Hawk\} \text{ and } E_7 = \{Mouse, Hawk, Coyote\}.$$

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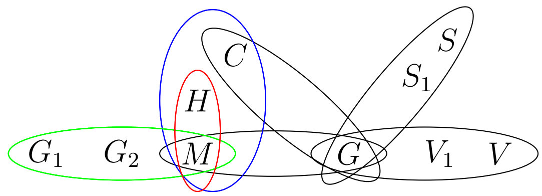

FIGURE 13. Hypernetwork $H = (X, \{E_i\}_{i=1}^{7})$

Now we want to investigate secondary consumer in this hypernetwork, so obtain a food web $(H/\alpha, *^{(2)})$ as shown in Figure 14, where

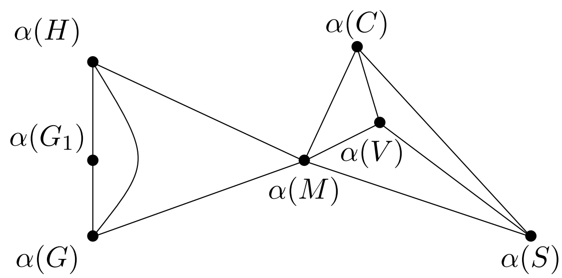

FIGURE 14. Food web $(H/\alpha, *^{(2)})$

$$\alpha(M) = \{Mouse\}, \alpha(V) = \{Vulture, Vultore - 1\}, \alpha(C) = \{Coyote\},$$
$$\alpha(S) = \{Snake, Snake - 1\}, \alpha(G_1) = \{Grasshoper - 1, Grasshoper - 2\},$$
$$\alpha(H) = \{Hawk\}.$$

In the food web $(H/\alpha, *^{(2)})$, $q = 2$ means that we investigate the secondary consumer and for instance, Snakes eat Mouse and don't eat Grasses, while in the Hypernetwork $H = (X, \{E_i\}_{i=1}^{7})$, Snakes eat Grasses. Let $H = (X, \{E_i\}_{i=1}^{m})$ be a $(k_1, k_2, \ldots, k_m)$-hypertree, for all $1 \leq i \leq m - 2, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$ and for all $1 \leq i \leq m - 1, |E_i \cap E_{i+1}| = k \geq 2$. For any given m, q, don't compute the $|E(H/\alpha, *^{(q)})|$ in Theorem 3.24.

Open Problem: Let $H = (X, \{E_i\}_{i=1}^{m})$ be a $(k_1, k_2, \ldots, k_m)$-hypertree, for all $1 \leq i \leq m - 2, (E_{i+1} \setminus E_i) \not\approx (E_{i+1} \cap E_{i+2})$ and for all $1 \leq i \leq m - 1, |E_i \cap E_{i+1}| = k \geq 2$. If $\lceil \frac{m}{2} \rceil \geq q - 1$, then $|E(H/\alpha, *^{(q)})| =$????.

Acknowledgments

This study considers a real problem in the world in which a group of elements participates and models are based on a hypertree as a hypernetwork. After the modeling problem on the desired hypertree, according to the desired optimization problem, we obtain a tree model as a network using mathematical tools,

especially the induction relation of the hypertree. The extracted tree is applied with all mathematical calculations to optimize the problem. In this article, an attempt has been made to use path symbols, path lengths, and subscriptions.

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