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DOUBLE CHAIN CONDITION ON NON-ASCENDANT SUBGROUPS

JIA ZHANG

ABSTRACT. If θ is a subgroup property, a group G is said to satisfy the double chain condition on θ-subgroups if it admits no infinite double chain

$$\cdots < X_{-n} < \cdots < X_{-1} < X_0 < X_1 < \cdots < X_n < \cdots$$

consisting of θ-subgroups. Here we want to describe the structure of locally finite and locally nilpotent groups satisfying the double chain condition on non-ascendant subgroups in term of chain conditions and of ascendant subgroups.

1. **Introduction**

A group theoretical property is called a *finiteness condition* if it is satisfied by every finite group. A particularly successful class of finiteness condition are the so-called *chain conditions*, which impose some particular restrictions to the lattice of subgroups of groups. Among these, the first to appear were the maximal and the minimal conditions on subgroups. A group G is said to satisfy the maximal (resp. the minimal) condition on subgroups if the lattice of the subgroups of G contains no sublattice ordered as the set of the positive (resp. negative) integers. Starting in 1938, Hirsch [9] proved that a solvable group satisfies the maximal condition on subgroups if and only if it has a subnormal series with cyclic factors, while Chernikov determined the structure of solvable groups satisfying the minimal condition on subgroups in 1940 [8]. The former groups are called *polycyclic groups* and the latter *Chernikov groups*.

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Moving to more general conditions, in 1951, Malcev [Malcev(1951)] proved that a solvable group satisfying the maximal condition on abelian subgroups is a polycyclic group and in 1945 Schmidt [13] proved that a solvable group satisfying the minimal condition on abelian subgroups is a Chernikov group. After these results, many papers appeared revolving around the study of the maximal and the minimal condition on specific kind of subgroups and the subject has proved to be a very rich one.

In 1973 and 1974, Shores [14] and Zaicev [16], respectively, introduced independently what is now called the *double chain condition*. They in fact proved that a solvable group satisfying the double chain condition on its subgroups, namely a group whose lattice of subgroups contains no chains of subgroups with the same order type of the set of the integers, satisfies either the minimal or the maximal condition on its subgroups. Such conditions gave new life to the study of chain conditions and, together with their generalizations, proved to be a very fruitful subject. Let θ be a subgroup theoretical propery, as it can be being abelian or being subnormal in the whole group. Then one can consider groups whose lattices do not contain infinite double chains of θ-subgroups. In other words, following [1] a group G is said to satisfy the double chain condition on θ-subgroups if for each chain

$$\cdots \leq X_{-n} \leq \cdots \leq X_{-1} \leq X_0 \leq X_1 \leq \cdots \leq X_n \leq \cdots$$

of θ-subgroups of G there exists an integer k such that either $X_n = X_k$ for all $n \leq k$ or $X_n = X_k$ for all $n \geq k$. It is clear that the double chain condition on θ-subgroups is a generalization of both the maximal and the minimal condition on θ-subgroups, but it not always reduces to just the two of them. In some cases, just like in Shores and Zaicev, in [1], in [4, Theorem 3.3], in [7] and in [5], satisfying the double chain condition on θ-subgroups just means satisfying either the maximal or the minimal condition on these subgroups, while in other case, as in [3], [6] and [2] one finds some additional conditions. Our result will basically fall into the first type of results.

The aim of the present paper is giving a contribution to this theory by studying groups satisfying the double chain condition on non-ascendant subgroups. Our main results, which will follow from slightly general results, are the following. The first one is about locally finite groups. One should notice that a locally finite group satisfying the minimal condition on its subgroups is known to be a Chernikov group.

Theorem 1.1. *Let G be a locally finite group satisfying the double chain condition on non-ascendant subgroups. Then G satisfies either the minimal condition on its subgroups or is a Gruenberg group.*

The second main theorem deals with locally nilpotent groups. Also in this case, it is known that a locally nilpotent group satisfying the minimal condition on its subgroups, although it can fail to be nilpotent.

Theorem 1.2. *Let G be a locally nilpotent group satisfying the double chain condition on non-ascendant subgroups. Then G satisfies either the minimal condition on its subgroups or is a Gruenberg group.*

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A Gruenberg group is a group in which every cyclic subgroup is ascendant (see [12]), so our results give good descriptions of the groups in question by chain conditions and properties attaining ascendancy.

2. **Preliminary lemmas**

Recall that if G is a group, the subgroup generated by every ascendant cyclic subgroup of G is called the Gruenberg subgroup (or the Gruenberg radical) of G. This is in analogy with the concept of Gruenberg group, which is a group in which every cyclic subgroup is ascendant. The name takes its origin from a result of K. Gruenberg stating that for any couple of finitely generated nilpotent ascendant subgroups H and K of a group G, then $\langle H, K\rangle$ is still ascendant in G.

Lemma 2.1. *Let G be a group satisfying the double chain condition on non-ascendant subgroups and let W be the Gruenberg subgroup of G. Let $A = \mathrm{Dr}_{\lambda\in\Lambda}A_\lambda$ be the direct product of an infinite collection of non-trivial subgroups of G. Let g be an element of G.*

(i) *If A_λ is $\langle g\rangle$-invariant for every $\lambda \in \Lambda$, then $g \in W$. In particular, $A \leq W$;*

(ii) *If A_λ is cyclic for every $\lambda \in \Lambda$, $g \in N_G(A)$ and there exists a positive integer n such that $g^n \in C_G(A)$, then $g \in W$.*

Proof. **(i)** By hypothesis, if $\lambda \neq \mu$ and $\lambda, \mu \in \Lambda$, then A_λ and A_μ have trivial intersection, so that there exists a finite subset Λ_0 of Λ such that $\langle g\rangle \cap A \leq \mathrm{Dr}_{\lambda\in\Lambda_0}A_\lambda$. Then, without loss of generality, we assume that $\langle g\rangle \cap A = \{1\}$. Further, we can also assume that Λ is countable. We can then split Λ in two countable disjoint subsets, say M and N, ordered as $\mathbb{Z}$. So we have $A = U_1 \times U_2$ where $U_1 = \mathrm{Dr}_{\mu\in M}A_\mu$, $U_2 = \mathrm{Dr}_{\nu\in N}A_\nu$. Since M and N are ordered as $\mathbb{Z}$, we may put $M = \{m_i | i \in \mathbb{Z}\}$, where $m_i < m_j$ if and only if $i < j$, and equally $N = \{n_i | i \in \mathbb{Z}\}$, where $n_i < n_j$ if and only if $i < j$. Define now $B_n = \mathrm{Dr}_{i<n}A_{m_i}$ and $C_n = \mathrm{Dr}_{i<n}A_{n_i}$. Then

$$\cdots < B_{-n} < \cdots < B_{-1} < B_0 < B_1 < \cdots < B_n < \cdots$$

and

$$\cdots < C_{-n} < \cdots < C_{-1} < C_0 < C_1 < \cdots < C_n < \cdots$$

are two (infinite) double chains whose subgroups are clearly $\langle g\rangle$-invariant. Then,

$$\cdots < \langle g\rangle B_{-n} < \cdots < \langle g\rangle B_{-1} < \langle g\rangle B_0 < \langle g\rangle B_1 < \cdots < \langle g\rangle B_n < \cdots$$

and

$$\cdots < \langle g\rangle C_{-n} < \cdots < \langle g\rangle C_{-1} < \langle g\rangle C_0 < \langle g\rangle C_1 < \cdots < \langle g\rangle C_n < \cdots$$

are still infinite double chains of G and hence, by the double chain condition hypothesis, we can find two integers m and n such that $\langle g\rangle B_m$ and $\langle g\rangle C_n$ are ascendant subgroups of G. Since it is easy to show that the intersection between ascendant subgroup of a group is again ascendant, we have that $\langle g\rangle = B_m\langle g\rangle \cap C_n\langle g\rangle$ is ascendant in G.

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(ii) Without loss of generality, assume that $\Lambda = \mathbb{N}$ and let $A_1 = \langle a_1 \rangle^{\langle g \rangle}$. Since g^n centralizes A, A_1 is generated by finitely many conjugates of $\langle a_1 \rangle$ and hence is finitely generated. Since A is not finitely generated, there exists a $\langle g \rangle$-invariant subgroup B_1 of A such that $A_1 \cap B_1 = 1$ and A/B_1 is finitely generated. Assume to have constructed two sequences $A_1, A_2, \dots, A_n$ and $B_1 > B_2 > \dots > B_n$ such that every A_i is finitely generated and A/B_i is finitely generated, $A_i \cap A_j = \{1\}$ when i and j are different, every $B_i^{\langle g \rangle} = B_i$ is not finitely generated and $\langle A_1, A_2, \dots, A_n \rangle \cap B_n = \{1\}$. Since $\langle A_1, A_2, \dots, A_n \rangle$ and A/B_n are finitely generated, there exists a subgroup of B_n, which we call B_{n+1}, which has trivial intersection with $\langle A_1, A_2, \dots, A_n \rangle$ and such that A/B_{n+1} is still finitely generated. As stated before, we can take B_{n+1} to be $\langle g \rangle$-invariant. Finally, take inside B_{n+1} a finitely generated $langlegangle$-invariant subgroup A_{n+1}. So, by induction, we have constructed an infinite sequence $(A_n)_{n \in \mathbb{N}}$ of $\langle g \rangle$-invariant subgroups of A which have pairwise trivial intersection. Now the thesis follows from point **(i)** □

We quote here two known results we need in the following. The first one is due to Zaicev.

Lemma 2.2. *Let G be a periodic locally solvable group and let F be a finite subgroup of $AutG$. If every F-invariant abelian subgroup of G satisfies the minimal condition on abelian subgroups, then G satisfies the minimal condition on abelian subgroups.*

Next lemma is due to B. I. Plotkin (see [10]). Recall that a group is said to be radical if it has an ascending series with locally nilpotent factors.

Lemma 2.3. *Let G be a radical group satisfying the minimal condition on abelian subgroup. Then it is a Chernikov group.*

Lemma 2.4. *Let G be a group satisfying the double chain condition on non-ascendant subgroups, let W be the Gruenberg radical of G and let $A = \mathrm{Dr}_{\lambda \in \Lambda} A_\lambda$ be an infinite periodic subgroup of G. Then every periodic element of G is an element of W.*

Proof. Let first a be an element of A, which has finite order by hypothesis. Then we can find an infinite subset Λ_1 of Λ such that $\langle a \rangle$ has trivial intersection with $D = \mathrm{Dr}_{\lambda \in \Lambda_1} A_\lambda$. Clearly every direct factor of D is invariant under the action of $\langle a \rangle$, so by Lemma 2.1**(i)**, a is in W. By the generality of a, it follows that $A \leq W$. Since W is locally nilpotent, its elements of finite order form a subgroup, so that we can take the torsion subgroup T of W. Let now g be a periodic element of G. If T satisfies the minimal condition on abelian subgroups, then it would satisfy also the minimal condition on subgroups by the quoted result of Plotkin (Lemma 2.3); however, T contains A and this proves that T does not satisfy Min-ab. By Lemma 2.2, T contains a periodic abelian $\langle g \rangle$-invariant subgroup which does not satisfy Min-ab and which therefore contains a direct product of infinitely many non-trivial cyclic subgroups. The result follows from Lemma 2.1(i). □

With the help of these preliminary results, we can now prove the main results of this paper.

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3. Proof of the Theorems

Theorem 1.1 is a particular case of the following proposition, which follows easily from what just proved plus a well-known theorem of Šunkov.

Proposition 3.1. *Let G be a group satisfying the double chain condition on non-ascendant subgroups, let H be a locally finite subgroup of G and let W be the Gruenberg radical of G. If H is not Chernikov, then H is a subgroup of W. In particular, if G is locally finite, then G is either Chernikov or a Gruenberg group.*

Proof. Since H is not a Chernikov group and it is locally finite, it follows from a theorem of Šunkov (see [15]) that it does not satisfy the minimal condition on abelian subgroups, which is equivalent to say that H contains an abelian subgroup A which has infinite rank. Now, inside A it can be found an infinite direct product of cyclic subgroups and hence H, by its periodicity, is a subgroup of W by Lemma 2.4. □

Theorem 1.2 is an immediate, particular case of the next proposition, which applies to a wider class of groups.

Proposition 3.2. *Let G be a group satisfying the double chain condition on non-ascendant subgroups, let H be a locally nilpotent subgroup of G and let W be the Gruenberg radical of G. If H is not Chernikov and not finitely generated, then H is a subgroup of W. In particular, if G is locally nilpotent, then G is a Gruenberg group.*

Proof. Since H is locally nilpotent, it is known that the set of the elements of finite order of H is a subgroup, call it T, namely the torsion subgroup of H. Assume that T does not satisfy the minimal condition on subgroups and let t be an element of T. By Lemma 2.2, H contains an abelian $\langle t\rangle$-invariant subgroup K which does not satisfy the minimal condition on subgroups, so that its socle S has infinite rank (see [12], p.110). As S is a characteristic subgroup of K, S is also $\langle t\rangle$-invariant. However, S has finite exponent and hence it is the direct product of infinitely many finite subgroups of prime order and this implies that t is an element of B, by Lemma 2.1(i). By the generality of t it follows that $T \leq B$ and this in particular settles the case in which H is periodic.

Assume now that H is not periodic and let C be an infinite cyclic subgroup of H, write $C = \langle c\rangle$ and assume for a contradiction that C is not ascendant in G. We first claim we can construct an infinite descending chain of subgroups of C which are not ascendant in G. Indeed, if both C^2 and C^3 where ascendant in G, then C itself would be ascendant in G (see [12, 12.2.6]), so that we can assume that C^i is not ascendant in G for i being either 2 or 3. But C^2 and C^3 are again infinite cyclic subgroups and hence the same reasoning may be repeated inductively to construct the wanted descending chain of subgroups of C which are not ascendant in G. Now we apply the fact that H is not finitely generated

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to find a strictly ascending chain of subgroups of H

$$C < C_1 < C_2 < \cdots < C_n < \cdots$$

where every C_i is finitely generated for $i \in \mathbb{N}$. By the double chain condition on non-ascendant subgroups and since we have showed that C contains a strictly descending chain of subgroups which are not ascendant in G, we find a natural number m such that for every $k \geq m$, C_k is ascendant in G. In particular, C_m is ascendant in G. However, C_m is finitely generated, so by hypothesis is nilpotent and then we get that C itself is ascendant in G and this is a contradiction. The generality of C assures that every infinite cyclic subgroup of H is ascendant in G. Finally, let a be an element of infinite order of H and let b be an element of finite order of H. Since H is not finitely generated, in it we find an infinite family $\{h_i\}_{i\in\mathbb{N}}$ of elements of H with which we are able to construct the following double chain of subgroups of H, which are nilpotent because H is locally nilpotent

$$\cdots < \langle a^{2^n}, b\rangle < \cdots < \langle a, b\rangle < \langle a, b, h_1\rangle < \cdots < \langle a, b, h_1, \ldots, h_n\rangle < \cdots .$$

By the double chain condition on non-ascendant subgroups of G, we find out that at least one of the subgroups of this chain is ascendant in G. However, this subgroup is also nilpotent and hence $\langle b\rangle$ is ascendant in G. By the generality of b, we have shown that every periodic cyclic subgroup of H is ascendant in G, so that H is a subgroup of the Gruenberg subgroup of G. In the end, we notice that a group which is Chernikov and locally nilpotent is also a Gruenberg group (see [11]) and so our theorem is proved. □

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References

[1] M. Brescia, Groups satisfying the double chain condition on non-subnormal subgroups, *J. Group Theory*, **24** (2021) 95–108.

[2] M. Brescia, FC-groups with few subnormal non-normal subgroups, *Mediterr. J. Math.*, **19** (2022) 19 pp.

[3] M. Brescia and F. de Giovanni, Groups satisfying the double chain condition on subnormal subgroups, *Ric. Mat.*, **65** (2016) 255–261.

[4] M. Brescia and F. de Giovanni, Groups satisfying the double chain condition on non-pronormal subgroups, *Riv. Mat. Univ. Parma*, **8** (2017) 353–366.

[5] M. Brescia and F. de Giovanni, Groups satisfying the double chain condition on subnormal non-normal subgroups, *Adv. Group Theory Appl.*, **13** (2022) 83–102.

[6] M. Brescia and A. Russo, Groups satisfying the double chain condition on abelian subgroups, *Bull. Aust. Math. Soc.*, **99** (2019) 212–218.

[7] M. Brescia and A. Russo, Groups satisfying the double chain condition on non-abelian subgroups, *J. Algebra Appl.*, **19** (2020) 14 pp.

[8] S. N. Chernikov, Infinite locally soluble groups, *Mat. Sb.*, **7** (1940) 35–64.

[9] K. A. Hirsch, On infinite soluble groups I, *Proc. London Math. Soc. (2)*, **44** (1938) 53–60.

[Malcev(1951)] A. I. Malcev, On certain classes of infinite soluble groups, *Mat. Sb.*, **28** (1951) 567–588.

[10] B. I. Plotkin, On groups with finiteness conditions for abelian subgroups, *Dokl. Akad. Nauk. SSSR*, **107** (1956) 648–651.

[11] D. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin, 1972.

[12] D. Robinson, *A Course in the Theory of Groups*, Springer, Berlin, 1982.

[13] O. J. Schmidt, Infinite soluble groups, *Mat. Sb.*, **17** (1945) 145–162.

[14] T. S. Shores, A chain condition for groups, *Rocky Mountain J. Math.*, **3** (1973) 83–89.

[15] V. P. Šunkov, On locally finite groups with a minimality conditions for abelian subgroups,*Algebra and Logic*, **10** (1971) 199–225.

[16] D. I. Zaicev, On solvable subgroups of locally solvable groups, *Soviet Math. Dokl.*, **15** (1974) 342–345.

Jia Zhang
Pingxiang, China
Email: jiazhang@tutanota.com

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