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GROUPS HAVING 11 CYCLIC SUBGROUPS

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Abstract. Let $c(G)$ denotes the number of cyclic subgroups of a finite group G. A group G is said to be *n-cyclic*, if $c(G) = n$. In this paper, we classify all 11-cyclic groups.

1. Introduction

Let G be a finite group, and $c(G)$ denotes the number of cyclic subgroups of G. A group G is said to be *n-cyclic*, if $c(G) = n$. Sometimes, the structure of a group can be determined by $c(G)$. For example, $c(G) = |G|$ if and only if G is an elementary abelian 2-group.

To elucidate further on this literature, we examined the incremental work done over time in counting the cyclic subgroups of a finite group G, beginning with the work by Tóth [13], which helps to count the number of cyclic subgroups of a finite abelian group. Later, Tărnăuceanu [11] classified all finite groups G having $|G| - 1$ number of cyclic subgroups. Belshoff, Dillstrom and Reid [2] found all the groups G with $c(G) = |G| - r$ for $r = 2, 3, 4$ and 5. Garonzi and Lima [5] and Tărnăuceanu [12] studied the function $\alpha(G) = \frac{c(G)}{|G|}$, investigated the basic properties of $\alpha(G)$ and its connection with probability of commutation.

In this article, we classify all 11-cyclic groups. The following are a few known results in this direction. It is easy to see that G is 1-cyclic and 2-cyclic if and only if $G \cong \{e\}$ and $G \cong \mathbb{Z}_p$, where p is a prime number respectively. Zohu [14] found all the groups, which are n-cyclic for $n = 3, 4$ and 5. Kalra [7]

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classified all the n-cyclic groups for $n \in \{6, 7, 8\}$. Recently, Ashrafi and Haghi [1] have classified all $9, 10$-cyclic groups. The main result of this paper is as follows.

Theorem 1.1. *If G is a finite group then $c(G) = 11$ if and only if $G \cong H$, where $H \in \{\mathbb{Z}_{p^{10}}, \mathbb{Z}_{27} \times \mathbb{Z}_3, \mathbb{Z}_{27} \rtimes \mathbb{Z}_3, Dic_7, \mathbb{Z}_7 \rtimes \mathbb{Z}_9, \mathbb{Z}_3 \times S_3, \mathbb{Z}_5 \rtimes \mathbb{Z}_8, \mathbb{Z}_3 \rtimes \mathbb{Z}_{16}\}$ and p is a prime number.*

The paper is organized as follows. The proof of the main result is in Section 3, and the background and notations are set in Section 2. In Section 4, some concluding remarks are given.

2. Notations and Preliminaries

The notations $\mathbb{Z}_n$, D_{2n}, Q_{2^n}, $M(p^a)$ and Dic_n denote the cyclic group of order n, dihedral group of order $2n$, generalized quaternion group of order 2^n, modular group of order p^a and dicyclic group of order $4n$ respectively. Throughout this paper, p, q and r are distinct prime numbers, and the Euler totient function is denoted by φ. The number of Sylow p-subgroups of G is denoted by $n_p(G)$. If G is a group, then $Z(G)$ denotes the center of G. A group G is *CLT* if it has subgroups corresponding to every divisor of $|G|$. A group G is *Dedekind* if every subgroup of G is normal. A non-cyclic group G is said to be *minimal non-cyclic* if all its proper subgroups are cyclic. In this paper, all the calculations are done by using GAP[4] and the SmallGroup(n, i) denotes the i^{th} group of order n in the *Small Group Library* of GAP.

Let $d(n), \omega(n)$ denote the number of positive divisors and the number of distinct prime divisors of n, respectively. It is easy to see that $c(G) \leq |G|$. Richard's Theorem [9] provides a lower bound on $c(G)$, in particular $c(G) \geq d(|G|)$ and the equality holds if and only if G is cyclic. Suppose G is an 11-cyclic group of order n. An immediate consequence of Richard's Theorem is that $\omega(n) \leq 3$ which implies that n is one of the form $p^k, pq, p^2q, pqr, p^3q, p^2q^2$ or p^4q, where $k \leq 10$.

Lemma 2.1. *If G is a finite n-cyclic group and $g \in G$, then $i_g := |G : N_G(\langle g \rangle)| < n - 1$, where $n > 2$ and $N_G(H)$ denotes the normalizer of H in G.*

Proof. One can observe that i_g is the number of conjugates of the subgroup $\langle g \rangle$. Since $c(G) = n$, then it is easy to see that $i_g \leq n - 1$. Now let us assume that there exists some element $g \in G$ such that $i_g = n - 1$. Then all the non-trivial cyclic subgroups of G are conjugate to $\langle g \rangle$. Therefore $i_h = n - 1$ for all $h \in G$ and every non-identity element has the same order in G. This shows that G is a p-group. Since every p-group has a non-trivial center, so there exists some $e \neq h \in Z(G)$ such that $i_h = 1$, which is a contradiction. Hence $i_g < n - 1$. □

Lemma 2.2. *Let G be a non-trivial n-cyclic group of order $p^a q^b$, where p and q are distinct prime numbers, and a Sylow p-subgroup of G denoted as P is Dedekind. Then either P is normal in G or $q < n - 1$.*

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Proof. If P is not normal in G, then there exists $g \in P$ such that $\langle g \rangle$ is not normal in G. Since P Dedekind, then $P \leq N_G(\langle g \rangle)$. Therefore $i_g := |G : N_G(\langle g \rangle)|$ is non-trivial power of q, also $q < n-1$ by Lemma 2.1. □

Lemma 2.3. [3] *If p is a prime, p divides $|G|$, and a_p denotes the number of subgroups of order p in G, then $a_p \equiv 1 (\mod p)$.*

Proposition 2.4. *Let G be a non-cyclic, n-cyclic group, and let M be a maximal subgroup of G. Then $c(M) < n-1$.*

Proof. One can easily check that $c(M) \leq n-1$. Let $|G| = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 < p_j$ for all $2 \leq j \leq k$ and let there exists a maximal subgroup M of G such that $c(M) = n-1$. Then the remaining $|G| - |M|$ elements of G generate a unique cyclic proper subgroup of G, which has at least $|G| - |M| + 1$ number of elements. Moreover, the maximum possible order of a proper subgroup of G is $m = p_1^{a_1 - 1} p_2^{a_2} \cdots p_k^{a_k}$. Now one can check that $(|G| - |M| + 1) > m$, which is a contradiction. This completes the proof. □

Let G be a finite group, and for any positive integer m, let $c(m)$ denotes the number of cyclic subgroups of order m in G. Since every element of G generates a cyclic subgroup, and $\varphi(d)$ is the number of generators of a cyclic group of order d, then

$$|G| = \sum_{m \mid |G|} c(m)\varphi(m). \tag{1}$$

Also, we have

$$T(G) = |G| - \sum_{m \mid |G|} c(m)\varphi(m). \tag{2}$$

In some cases, to find all 11-cyclic groups, we take different possibilities of $c(m)$ for all the divisors m of $|G|$ and then try to find all those groups G, which satisfies the equations $\sum_{m \mid |G|} c(m) = 11$ and $T(G) = 0$. In the next section, proof of the main result is given.

3. Proof of the main result

Let G be an 11-cyclic group. We prove the result casewise by taking different possibilities on $|G|$.

(1) $|\mathbf{G}| = \mathbf{p^a}$: We prove that

$$G \cong \begin{cases} \mathbb{Z}_{p^{10}} \text{ or } \mathbb{Z}_{27} \times \mathbb{Z}_3 & \text{if } G \text{ is abelian,} \\ \mathbb{Z}_{27} \rtimes \mathbb{Z}_3 & \text{otherwise.} \end{cases}$$

Let M be a maximal subgroup of G. Then we have the following two subcases:

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(a) Let M be a cyclic subgroup of G. First, assume that G is abelian. Since $|M| = p^{a-1}$, then either G is isomorphic to $\mathbb{Z}_{p^a}$ or $\mathbb{Z}_p \times \mathbb{Z}_{p^{a-1}}$. If $G \cong \mathbb{Z}_{p^a}$, then it is easy to see that $a = 10$. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{a-1}}$, then by [1, Theorem 1.1], $c(G) = (a-1)p+2$, consequently, $p = 3, a = 4$ and $|G| = 81$. By using a simple GAP program [4], $G \cong \mathbb{Z}_{27} \times \mathbb{Z}_3 =$*SmallGroup*$(81, 5)$.

Now, let G be a non-abelian group, then depending on p, we have the following two situations. If p is odd, then $G \cong M(p^a)$ and by [1, Theorem 1.1], $c(G) = (a-1)p + 2$. This shows that $p = 3, a = 4$. By [4], G is isomorphic to $\mathbb{Z}_{27} \rtimes \mathbb{Z}_3 =$*SmallGroup*$(81, 6)$. If $p = 2$, then by the classification theorem of finite non-abelian 2-groups containing cyclic maximal subgroup, $G \cong D_{2^a}, Q_{2^a}, M(2^a)$ or S_{2^a}. By using [1, Theorem 1.1], we have $c(D_{2^a}) = a + 2^{a-1}$, $c(Q_{2^a}) = a + 2^{a-2}$, $c(M(2^a)) = 2a$ and $c(S_{2^a}) = a + 3.2^{a-3}$. Now, by simple calculation, one can check that none of these groups are 11-cyclic.

(b) Let M be a non-cyclic subgroup of G. By Proposition 2.4, $c(M) \leq 9$. Let us consider the set $S = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_7 \times \mathbb{Z}_7, Z_9 \rtimes \mathbb{Z}_3, \mathbb{Z}_8 \rtimes \mathbb{Z}_2, Q_{16}, D_8, Q_8\}$ of all non-cyclic groups, which are n-cyclic, where $n \leq 9$ by [1, Theorem 1.2 and Lemma 2.1]. Then M is isomorphic to a member of S. Therefore, if $p = 2$, then $a = 3, 4$ or 5, if $p = 3$ then $a = 3, 4$, and if $p = 5, 7$ then $a = 3$. Moreover, $|G| \in \{8, 16, 27, 32, 81, 125, 343\}$. We now apply a simple GAP program to verify that no group of these orders is 11-cyclic. Hence, the only non-abelian 11-cyclic p-group is $\mathbb{Z}_{27} \rtimes \mathbb{Z}_3$.

The following conclusion can be made by using the above case. A group G is cyclic and 11-cyclic if and only if $G \cong \mathbb{Z}_{p^{10}}$. Furthermore, by using [13, Theorem 1], it is easy to check that if G is an abelian group of order n, where $n \in \{pq, p^2q, pqr, p^2q^2, p^3q, p^4q\}$, then G is not 11-cyclic. Hence, the only non-cyclic abelian 11-cyclic group is $\mathbb{Z}_{27} \times \mathbb{Z}_3$.

From now onwards, all the groups are supposed to be non-abelian.

(2) $|\mathbf{G}| = \mathbf{pq}$: If $p < q$, then by [7, Lemma 3.1], $c(G) = q + 2$. Therefore $q = 9$. Hence no group of order pq is 11-cyclic.

(3) $|\mathbf{G}| = \mathbf{p^2q}$: We prove that G is isomorphic to $Dic_7, \mathbb{Z}_7 \rtimes \mathbb{Z}_9$ and $\mathbb{Z}_3 \times S_3$. There are the following two sub-cases:

(a) If $p < q$, then according to [7, Proposition 3.2], $c(G) \in \{6, 2p + 4, pq + 4, q + 4, 2q + 2\}$. Hence G is 11-cyclic only if $p = 2, 3$ and $q = 7$. Moreover, by [4] one can notice that G is isomorphic to $Dic_7 =$*SmallGroup*$(28, 1)$ and $\mathbb{Z}_7 \rtimes \mathbb{Z}_9 =$*SmallGroup*$(63, 1)$.

(b) If $p > q$, then by [7, Proposition 3.2], $c(G) \in \{6, 2p + 4, p^2 + 3, p^2 + p + 2, 2p + 3, 3p + 2\}$. Thus $c(G) = 11$ is possible only if $p = 3$ and $q = 2$. Again by [4], we have $G \cong \mathbb{Z}_3 \times S_3 =$*SmallGroup*$(18, 3)$.

(4) $|\mathbf{G}| = \mathbf{pqr}$: We prove that there is no 11-cyclic group of order pqr. Since every group of square-free orders is solvable, then G has Hall subgroups of orders pq, pr and qr. These Hall subgroups are either cyclic or they are isomorphic to S_3, D_{14}, D_{10} or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ by [1, Theorem 1.2 and Lemma 2.1]. If all the Hall subgroups of G are cyclic, then G is a minimal non-cyclic group, which is not possible by [6, Proposition 2.8]. Let M be a non-abelian, maximal Hall subgroup of G. Then we have the following four sub-cases:

(a) If $M \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, then $|G| = 21p$. One can observe that G has a unique cyclic subgroup of orders 1 and 7, and at least 7 cyclic subgroups of order 3. Also, G has a unique cyclic subgroup of orders p and $7p$. Now by equation 2, $T(G) = 0$ has no solution. Hence no such group is 11-cyclic.

(b) If $M \cong D_{14}$, then $|G| = 14p$ and $c(M) = 9$ by [1, Lemma 2.1]. By Sylow theorem, there is a unique subgroup of orders 7 and p. Also, G has at least 7 subgroups of order 2 and a unique cyclic subgroup of order $7p$. By equation 2, $T(G) = 0$ has no solution. Therefore no such group is 11-cyclic.

(c) If $M \cong S_3$, then $|G| = 6p$, and G has at least three subgroups of order 2. If G has 6 cyclic subgroups of order p, then by Lemma 2.3, $p = 5$ and $|G| = 30$. We use GAP[4] to obtain the number of cyclic subgroups of groups of order 30, which shows that there is no 11-cyclic group of this type. Again by using Sylow theorems, Lemma 2.3 and $c(G) = 11$, all the possibilities for the number of cyclic subgroups of G along with the function $T(G)$ using equation 2 are recorded in Table 1.

Since p is prime and $p \geq 5$, then the only possible solution of equation $T(G) = 0$ is $p = 5$ and 7, and $|G|$ is 30 and 42. After checking the number of cyclic subgroups of groups of order 30 and 42 by GAP [4], we can say that no such group is 11-cyclic.

(d) If $M \cong D_{10}$, then $|G| = 10p$ and G has at least 5 subgroups of order 2. If G has 4 subgroups of order p, then by Lemma 2.3, $p = 3$ and $|G| = 30$. By GAP[4], no such group is 11-cyclic. Thus, by using Sylow theorems and Lemma 2.3, all the possibilities for the number of cyclic subgroups of G and the corresponding value of $T(G)$ using equation 2 are recorded in Table 2. Since $p \notin \{2, 5\}$ so it is easy to check that $T(G) = 0$ has no solution.

Consequently, there does not exist any 11-cyclic group of order pqr.

(5) $|\mathbf{G}| = \mathbf{p^2q^2}$: We show that no group of order p^2q^2 is 11-cyclic. In this case, Sylow p and q subgroups of G are Dedekind. Now, we prove the result by examining the following sub-cases.

Table 1.

$c(1)$	$c(2)$	$c(3)$	$c(p)$	$c(2p)$	$c(3p)$	$c(6)$	$T(G)$
1	3	4	1	1	1	0	$p-4$
1	3	4	1	1	0	1	$p-3$
1	3	4	1	0	1	1	$3p-11$
1	3	4	1	2	0	0	$p-3$
1	3	4	1	0	2	0	$p-7$
1	3	4	1	0	0	2	$p-3$
1	3	1	1	4	1	0	$p-1$
1	3	1	1	0	1	4	$3p-11$
1	3	1	1	2	1	2	$p-5$
1	5	1	1	2	1	0	$p-3$
1	5	1	1	0	1	2	$p-3$
1	5	1	1	1	1	1	$p-3$
1	5	4	1	0	0	0	$5p-13$
1	7	1	1	0	1	0	$3p-7$

Table 2.

$c(1)$	$c(2)$	$c(5)$	$c(p)$	$c(2p)$	$c(5p)$	$c(10)$	$T(G)$
1	5	1	1	1	1	1	$p-2$
1	5	1	1	2	1	0	$p-1$
1	5	1	1	0	1	2	$5p-13$
1	7	1	1	0	1	0	$5p-7$

(a) **G *has a unique subgroup of orders* p *and* q.** In this case Sylow p and q subgroups of G are cyclic. If $n_p(G) = n_q(G) = 1$, then G is cyclic, which is a contradiction. Also, by Sylow theorems, $n_p(G), n_q(G) > 1$ is not possible. Let us assume that $n_p(G) > 1$ and $n_q(G) = 1$. Then by Sylow theorem, $n_p(G) \geq 1 + p$ and $n_p(G) \in \{q, q^2\}$. Moreover, G has a unique cyclic subgroup of orders $1, p, pq, q$ and q^2.
If $n_p(G) = q$, then $p < q$ and $c(G) \geq q + 5$. Hence $p \in \{2, 3\}$ and $q \in \{2, 3, 5\}$. Also $|G| = 36, 100$ or 225. If $n_p(G) = q^2$, then either $|G| = 36$ or $c(G) > 11$. A simple calculation with GAP[4], shows that no such group of order $36, 100$ and 225 is 11-cyclic.

(b) **G *has a unique subgroup of order* q *and at least* p + 1 *subgroups of order* p**. This implies that Sylow q-subgroup of G is cyclic. Therefore $c(G) \geq p + 4$, which implies that $p \in \{2,3,5,7\}$. If Sylow p-subgroup of G is not normal, then by Lemma 2.2, $q < 10$ and $|G| \in \{36, 100, 196, 225, 441, 1225\}$. Again if Sylow p-subgroup of G is normal, then Sylow q-subgroup of G is not normal (being G non-abelian) that is $n_q(G) \geq 1+q$. Since G has at least 3 subgroups of order p, then $c(G) \geq 6+q$ and $q \leq 5$. Hence $|G| \in \{36, 100, 196, 225, 441, 1225\}$. By using a simple GAP[4] program, there is no 11-cyclic group of these orders.

(c) ***Subgroups of orders* $\mathbf{p}$ *and* $\mathbf{q}$ *are not unique.*** By lemma 2.3, G has at least $p+1$ and $q+1$ subgroups of order p and q, respectively. Therefore, $c(G) \geq p+q+3$. Accordingly, $p+q \in \{2,3,4,5,6,7,8\}$ and $|G| \in \{36, 100, 225\}$. By GAP[4], after analyzing all the groups of order $36, 100$ and 225 it can be seen that none of them is 11-cyclic. Hence no group of order p^2q^2 is 11-cyclic.

(6) $|\mathbf{G}| = \mathbf{p^3q}$: Specifically, we prove that $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_8$. If G is a non-CLT group of order p^3q, then by [10, Theorem 1.1], either G is isomorphic to $SL(2,3)$ $=$*Small Group*$(24,3)$ or $E(p^3) \rtimes \mathbb{Z}_q$, where $E(p^3)$ is the elementary abelian p-group of order p^3. We can check $c(SL(2,3)) = 13$, with the help of GAP[4]. Since every non-identity element of $E(p^3)$ has order p, then $c(E(p^3)) = p^2+p+2$. If $p > 2$, then $c(E(p^3)) \geq 14$. If $p = 2$, then $c(E(p^3)) = 8$ also $E(p^3) \rtimes \mathbb{Z}_q$ has at least $q+1$ subgroups of order q, where $q \geq 3$. Therefore, $c(E(p^3) \rtimes \mathbb{Z}_q) > 11$, which is a contradiction. Thus G is a CLT group, so it has a subgroup of order p^2q, let us call it M. By Proposition 2.4, we have $c(M) \leq 9$. Also by [1, Theorem 1.2 and Lemma 2.1], $M \in \{\mathbb{Z}_{p^2q}, \mathbb{Z}_3 \rtimes \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_{2p}, A_4, \mathbb{Z}_5 \rtimes \mathbb{Z}_4\}$. If $M \in \{A_4, \mathbb{Z}_3 \rtimes \mathbb{Z}_4, \mathbb{Z}_5 \rtimes \mathbb{Z}_4\}$, then $|G| \in \{24, 40\}$. It is easy to verify using GAP[4], that no such group of order 24 and 40 is 11-cyclic. Now, we discuss the remaining possibilities of M separately.

(a) Let $M \cong \mathbb{Z}_{p^2q}$ then there are following sub-cases:

(i) $\mathbf{G}$ ***has at least*** $\mathbf{p+1}$ ***subgroups of order*** $\mathbf{p}$***.*** Since $c(M) = 6$, then $c(G) \geq 6+p$ and $p \leq 5$, now we discuss these possibilities separately.

If $p = 5$, then G has 6 cyclic subgroups of order 5 and a unique cyclic subgroup of orders $1, 25, q, 5q$ and $25q$. By equation 2, $T(G) = 0$ has no solution.

If $p = 3$, then by Lemma 2.3, the number of subgroups of order 3 is $1+3k$, where $k \in \mathbb{N}$. Since $c(G) = 11$, then G has exactly 4 subgroups of order 3. Also, by [7, Table 1], Sylow 3-subgroup of G lies in $\{C_{27}, C_9 \times C_3, \langle a, b | a^9 = b^3 = 1, ba = a^4b\rangle\}$ as G has an element of order 9. If a Sylow 3-subgroup (P) of G is isomorphic to $C_9 \times C_3$, then it is easy to see that $c(P) = 8$. Additionally, G has a unique cyclic subgroup of orders $q, 3q$ and $9q$. After using equation 2, $T(G) = 0$ has no solution. If a Sylow 3-subgroup of G is isomorphic to C_{27}, then $n_3(G) \geq 4$ and $c(G) > 11$. Therefore, this case is not possible. Finally if a Sylow 3-subgroup of G is isomorphic to $\langle a, b | a^9 = b^3 = 1, ba = a^4 b\rangle$, then by GAP[4], $c(P) = 8$. Furthermore, G has a unique cyclic subgroup of orders $q, 3q$ and $9q$. Thus by equation 2, $T(G) = 0$ has no solution, which is a contradiction.

If $p = 2$, then by Lemma 2.3, number of subgroups of order 2 is $1+2k$, where $k \in \mathbb{N}$. Since $c(G) = 11$, then G has either 3 or 5 subgroups of order 2. Let us first assume that Sylow q-subgroup of G is not normal, then $n_q(G) \geq 1 + q$ and $q = 3$ by using the fact that $c(G) = 11$. This shows that $|G| = 24$, and we can verify by using GAP[4] that no such group is 11-cyclic. Therefore from now onwards, Sylow q-subgroup of G is normal. By [7, Table 1] Sylow 2-subgroups of G lie in the set $\{C_8, C_4 \times C_2, Q_8, D_8\}$ as G has a subgroup of order 4. Now, there are the following two situations. At first, assume that Sylow 2-subgroup of G is normal, then $G \cong Q_8 \times C_q$ or $D_8 \times C_q$, with some easy calculation, it can be seen that none of these groups is 11-cyclic. Finally, assume that Sylow 2-subgroup of G is not normal. This further gives the following two possibilities. If Sylow 2-subgroup of G is Dedekind, then by Lemma 2.2, $q < 10$. Thus $|G| \in \{24, 40, 56\}$, with the help of a simple program using GAP[4], we can check that no such group is 11-cyclic. At last, if Sylow 2-subgroup of G is not Dedekind, then it is isomorphic to D_8. Moreover, it is easy to see that G has at least 8 cyclic subgroups of orders $1, 2$ and 4 and a unique cyclic subgroup of orders $q, 2q$ and $4q$. By equation 2, there is no solution of the equation $T(G) = 0$. Hence no such group is 11-cyclic.

(ii) **G *has a unique subgroup of order* p.** In this case, Sylow p-subgroup of G is either cyclic or generalized quaternion. First, suppose that G has a unique cyclic Sylow p-subgroup, then $n_q(G) \geq 1 + q$. This shows that, $c(G) \geq 7 + q$ and $q = 2$ or 3. Since Sylow q-subgroup of G is Dedekind but not normal, then by Lemma 2.2, $p < 10$. Hence $|G| \in \{24, 54, 250, 375, 686, 1029\}$. Now it is easy to verify using GAP[4] that no group of above orders in which Sylow p-subgroup is cyclic and normal is 11-cyclic. Therefore, if Sylow p-subgroup of G is cyclic, then it is not normal and $n_p(G) = q$. Moreover, $c(G) \geq 6 + q$ and $q \in \{2, 3, 5\}$. Again by using Sylow theorem $n_p(G) \geq 1 + p$, which implies that $c(G) \geq 7 + p$, and $p = 2$ or 3. Thus $|G| \in \{24, 40, 54, 135\}$. Now with the help of a simple GAP program [4], after checking the number of cyclic subgroups of all the groups of the orders obtained above, $G \cong SmallGroup(40, 1) = \mathbb{Z}_5 \rtimes \mathbb{Z}_8$.

If Sylow p-subgroup of G is a generalized quaternion, then Sylow p and q subgroups of G are Dedekind. Let us first assume that Sylow q-subgroup of G is normal. If Sylow p-subgroup of G is also normal, then $G \cong Q_8 \times \mathbb{Z}_q$, and it is easy to see that

$c(G) = 10$. Hence this case is not possible. Therefore Sylow p-subgroup of G is not normal. Then by Lemma 2.2, $q < 10$, which implies that $|G| \in \{24, 40, 56\}$. After using a simple GAP[4] program, no such group of these orders is 11-cyclic. Now we are left with the case when Sylow q-subgroup of G is not normal, then by Sylow theorem $n_q(G) = 1 + kq$, where $k \in \mathbb{N}$ and $n_q(G) \geq 4$. As a consequence, either $|G| = 24$ or $c(G) > 11$. By GAP[4], no group of order 24 in which Sylow p-subgroup is generalized quaternion is 11-cyclic.

(b) If $M \cong \mathbb{Z}_2 \times \mathbb{Z}_{2q}$, then $c(M) = 8$, $|G| = 8q$ and G has no cyclic subgroup of order 8. Also, by Sylow theorem $n_2(G) \in \{1, q\}$ and $n_q(G) \in \{1, 4, 8\}$. If $n_q(G) = 8$, then $c(G) > 11$. Also, if $n_q(G) = 4$, then by equation 2, $T(G) = 0$ has no solution. Consequently, G has a normal Sylow q-subgroup. Moreover, Sylow 2-subgroup is neither isomorphic to $\mathbb{Z}_8$ nor Q_8 as G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Further, if Sylow 2-subgroup of G is normal, then $G \cong D_8 \times \mathbb{Z}_q$, and it is easy to check that $c(G) = 14$. Hence, from now onwards, assume that Sylow 2-subgroup is not normal. This gives two possibilities. First is Sylow 2-subgroup is Dedekind, then by Lemma 2.2, $q < 10$ and $|G| \in \{24, 40, 56\}$. With the help of some simple program using GAP[4], we can check that no group of these orders satisfying the above conditions is 11-cyclic. Finally, if Sylow 2-subgroup is not Dedekind, then it is isomorphic to D_8. Also, G has at least 3 cyclic subgroups of order $2q$, then by equation 2, $T(G) = 0$ has no solution, which is a contradiction.

(7) $|\mathbf{G}| = \mathbf{p^4q}$: We prove that, $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_{16}$. Let us first suppose that G is a CLT group. Then G has a subgroup of order p^3q, say M such that $c(M) \leq 9$ by Proposition 2.4. By Theorem 1.2 and Lemma 2.2 of [1], $M \in \{\mathbb{Z}_{p^3q}, \mathbb{Z}_3 \rtimes \mathbb{Z}_8\}$. If $M \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_8$ then $|G| = 48$. By GAP[4], no group of order 48 containing a maximal subgroup isomorphic to $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$ is 11-cyclic. Finally, we are left with the case when $M \cong \mathbb{Z}_{p^3q}$. Now, we proceed further by discussing the following sub-cases.

(a) **G** ***has a unique subgroup of order*** **p.** In this case, Sylow p-subgroup of G is either cyclic or generalized quaternion. First, suppose that G has a unique cyclic Sylow p-subgroup. Then $n_q(G) \geq 1 + q$, otherwise G will be cyclic. As a consequence, we get $q = 2$, also by equation 2, $T(G) = 0$ has no solution, which is not possible. Therefore, if Sylow p-subgroup of G is cyclic, then it is not normal that is $n_p(G) \geq 1 + p$. This implies that $c(G) \geq 9 + p$, and $p = 2$. By applying equation 2, we get $q = 3$ and $|G| = 48$. Now with the help of some simple program in GAP[4], we get $G \cong SmallGroup(48, 1) = \mathbb{Z}_3 \rtimes \mathbb{Z}_{16}$. If Sylow p-subgroup of G is generalized quaternion (Q_{16}), then $|G| = 16q$. Now it is easy to see that $c(G) > 11$ as $c(Q_{16}) = 8$.

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(b) **G *has at least* p + 1 *subgroups of order* p.** In this case, it is easy to see that either $p = 2$ or 3. Using equation 2, all the possibilities for the number of cyclic subgroups of G and the corresponding function $T(G)$ are recorded in Table 3. By Table 3, the equation $T(G) = 0$ has no solution. Hence there does not exist any CLT group of such type of order p^4q, which is 11-cyclic.

TABLE 3.

p	$c(1)$	$c(p)$	$c(p^2)$	$c(p^3)$	$c(p^4)$	$c(q)$	$c(pq)$	$c(p^2q)$	$c(p^3q)$	$T(G)$
2	1	3	2	1	0	1	1	1	1	$2q-1$
	1	3	1	2	0	1	1	1	1	$4q-3$
	1	3	1	1	0	1	2	1	1	$7q-1$
	1	3	1	1	0	1	1	2	1	q
	1	3	1	1	0	1	1	1	2	$2q+1$
3	1	4	1	1	0	1	1	1	1	$9q-1$

Let G be a non-CLT group of order p^4q. Then any Sylow p-subgroup P of G lies in the set $\{\mathbb{Z}_{p^4}, \mathbb{Z}_8 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4, G_7, G_{10}, G_{11}, G_{13}, G_{14}\}$, by [7, Table 2, 3]. Now, we discuss these cases separately.

If P is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4, G_7, G_{10}, G_{13}$, then by [7, Table 2] we get $c(P) = 10$. Also, G has a unique subgroup of order q. This implies that $|G| = p^4 + q - 1$, which is a contradiction.

If P is isomorphic to $G_{11} \cong M_4(2)$ or $G_{14} \cong Q_{16}$, then there are the following possibilities. If Sylow p and q subgroups are normal, then G is isomorphic to $M_4(2) \times \mathbb{Z}_q$ or $Q_{16} \times \mathbb{Z}_q$, we discard this case as these groups are CLT. If Sylow q-subgroup is not normal, then $n_q(G) \geq 4$ as $q \geq 3$. Also, $c(M_4(2)) = c(Q_{16}) = 8$ by [7, Table 2]. Therefore $c(G) > 11$. Finally, let us assume that Sylow p-subgroup is not normal and Sylow q-subgroup is normal. If $P \cong Q_{16}$ (Dedekind), then by Lemma 2.2, $q < 10$ and $|G| \in \{48, 80, 112\}$. By GAP[4], no such group of these orders is 11-cyclic. If $P \cong M_4(2)$, then G has at least 9 subgroups of orders 1, 2 and 4, and one subgroup of order q. Moreover, G has a subgroup of order $2q$. This further gives the following two situations. Let G has a cyclic subgroup of order $2q$, then by equation 2, $T(G) = 0$ has no solution. Otherwise, G contains a non-cyclic subgroup of order $2q$ isomorphic to $\mathbb{Z}_q \rtimes \mathbb{Z}_2$. Then either G has at least 5 subgroups of order 2 or $|G| = 24$. If G has at least 5 subgroups of order 2, then $c(G) > 11$. If $|G| = 24$, then there does not exist a non-CLT group of such type.

If P is isomorphic to $\mathbb{Z}_{p^4}$ or $\mathbb{Z}_8 \times \mathbb{Z}_2$, then both Sylow p-subgroup (P) and Sylow q-subgroup (Q) can not be normal. Then let us first suppose that P is normal, but Q is not normal. Then by Lemma 2.2, $p < 10$. If $P \cong \mathbb{Z}_{p^4}$, then $q = 2, 3$ or 5 by using $n_q(G) \geq 1 + q$. Since G is

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a non-CLT group, then by [8], $|G| \in \{2^4 \times 3, 2^4 \times 5, 3^4 \times 5, 5^4 \times 3, 7^4 \times 5\}$. If $|G| = 48$, then GAP[4], no such group is 11-cyclic, for other orders $c(G) > 11$ by Sylow's third theorem. If $P \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, then $c(G) \geq 9 + q$ and $q = 2$ which is a contradiction. Finally, let us assume that Q is normal, but P is not normal. Then by Lemma 2.2, $q < 10$. If $P \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, then $|G| \in \{48, 80, 112\}$. A simple calculation using GAP[4], no such group is 11-cyclic. If $P \cong \mathbb{Z}_{p^4}$, then $n_p(G) \geq 1 + p$. In this case, $p \leq 5$, if $p = 5$, then by equation 2, we can not find the value of q. If $p = 2$ or 3, then $|G| \in \{48, 80, 112, 162, 405, 567\}$. By GAP[4], no such group of these orders is 11-cyclic. Similarly, we can show that if P and Q are both not normal, then G is not 11-cyclic.

4. Conclusion

In this work, a classification of 11-cyclic groups is given and they are $\mathbb{Z}_{p^{10}}, \mathbb{Z}_{27} \times \mathbb{Z}_3, \mathbb{Z}_{27} \rtimes \mathbb{Z}_3, Dic_7, \mathbb{Z}_7 \rtimes \mathbb{Z}_9, \mathbb{Z}_3 \times S_3, \mathbb{Z}_5 \rtimes \mathbb{Z}_8$ and $\mathbb{Z}_3 \rtimes \mathbb{Z}_{16}$. Here, one can observe that these groups are supersolvable with abelian Sylow subgroups, and the centres of these groups are cyclic. Also, these groups are generated by at most two elements. Moreover, every p-cyclic group, where $p \leq 11$ is supersolvable. In the future, we want to explore the problem of the classification of 12-cyclic groups and the properties of these groups.

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