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CUBIC SEMISYMMETRIC GRAPHS OF ORDER 44p OR 44p^2

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ABSTRACT. A simple graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let p be an arbitrary prime. Folkman [J. Folkman, Regular line-symmetric graphs, J. Combinatorial Theory, **3** (1967) 215–232.] proved that there are no cubic semisymmetric graphs of order 2p or 2p^2. In this paper, an extension of his result in the case of cubic graphs of order 44p or 44p^2 is given. By using group theoretic methods, we prove that there are no connected cubic semisymmetric graphs of order 44p or 44p^2.

1. **Introduction**

In through of this paper, all graphs are finite, undirected and simple, i.e. without loops or multiple edge. For a graph X, we use V(X), E(X) and A := Aut(X) to denote its vertex set, edge set and the full automorphism group, respectively. The graph is said to be vertex-transitive and edge-transitive, if A acts transitively on V(X) and E(X), respectively. It is worth mentioning that a graph is said semisymmetric if it is regular and edge-transitive, but not vertex-transitive. The first person who studied semisymmetric graphs was Folkman. In 1967 he constructed several infinite families of such graphs and proposed eight open problems (see [13]). Afterwards, Bouwer, Klin, Iofinova, Ivanov and others did much work on semisymmetric graphs (see[4, 18, 19]). They gave new constructions of such graphs and nearly solved all of Folkman's open problems. In particular, Iofinova and Ivanov [18] in 1985 classified biprimitive semisymmetric cubic graphs using group-theoretical methods. This was the

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first classification theorem for such graphs. For prime p, cubic semisymmetric graphs of order $2p^3$ were investigated in [21], in which the authors proved that there is no connected cubic semisymmetric graph of order $2p^3$ for any prime $p \neq 3$ and that for $p = 3$ the only such graph is the Gray garph.

Conder et al.[6] classified all semisymmetric cube graphs of order maximum 768 and also showed that there are no semisymmetric cube graphs of order 16p except $p = 7$. An interesting research problem is to classify connected cubic semisymmetric graphs of various types of orders. Among these, graphs of orders kp^i for prime p and small k and i have been a target of much research. A simple observation then shows that there are finitely many connected cubic semisymmetric graphs of order $4p^2$ in [2]. In fact, one can show that there are no connected cubic semisymmetric graphs of order 4p, $4p^2$ or $4p^3$. Lu et al.[15] classified cubic semisymmetric graphs of order $6p^2$.

As an application of the classification, cubic symmetric graphs of order $10p^3, 34p^3, 6p^3, 8p^3$ and $18p^n (n \geq 1)$ were also classified in [3, 8, 11, 17, 29]. The present work has the following motivation. Du and Xu in [10] classified semisymmetric graphs of order 2pq but the classification of semisymmetric graphs of order 4pq is still open. Feng et al.[12] Classifying cubic symmetric graphs of order 8p or $8p^2$ and cubic symmetric graphs of order $28p^2, 20p^2$ and 20p classified in [1, 25, 26]. An extension of their result in the case of cubic graphs of order 44p or $44p^2$ is given. Hence it is worth studying semisymmetric graphs of such order. On the other hand, we hope that a complete classification of symmetric graphs of order 4pq or $4p^2q$ will be provided.

2. **Preliminaries**

In the present study, S_n, A_n, Z_n and D_{2n} represent the symmetric and the alternating groups of degree n, the cyclic groups of order n and the dihedral groups of order 2n respectively. In addition, we denote a projective special linear group by $L_n(q)$ and $U_n(q)$ refers to a projective special unitary group. For each prime p that dividing the order of finite group G, $O_p(G)$ denotes the largest normal p-subgroup of G and it is easy to verify that $O_p(G) \trianglelefteq^c G$. For a group G and a nonempty set Ω, an action of G on Ω is a function $(g, w) \to g.w$ from $G \times \Omega$ to Ω, where $1.w = w$ and $g.(h.w) = (g.h).w$ for $g, h \in G$ and every $w \in \Omega$, in addition for $w \in \Omega$, the stabilizer of w in G is defined as $G_w = \{g \in G | gw = w\}$. Consider G as a transitive on Ω, if given any two elements x and y from Ω there is an element $g \in G$ such that $x.g = y$. Now, if for each $x \in G$, $G_x = 1$, the action of G on Ω is called semiregular. Furthermore, if it is both semiregular and transitive, it can be called as regular. For any two groups H and K and any homomorphism $\varphi : H \to Aut(K)$ the external semidirect product $H \rtimes_\varphi K$ is defined as the group whose underlying set is the Cartesian product $H \times K$ and whose binary operation is defined as $(h_1, k_1)(h_2, k_2) = (h_1\varphi(k_1)h_2, k_1k_2)$. Let X be a simple graph. If two vertices u and v be adjacent, we write $u \sim v$. The set of all vertices adjacent to a vertex u is denoted by X(u) and the degree of u is $|X(u)|$. If all vertices have the same degree, the graph would be regular in addition, V(X), E(X), Arc(X) and Aut(X) denote the vertex set, the edge set, the arc set, and the set of all automorphism of X, respectively. Let N be a subgroup of Aut(X). The quotient graph X_N is a graph whose the vertex

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set is the orbits N in V(X) and two orbits is adjacent if and only if a element of ones as a element of the other in V(X) be adjacent.

Let X_c and Y be two graphs. Then X_c is said to be a covering graph for Y if there is a surjection $f : V(X_c) \to V(Y)$ which preserves adjacency and for each $u \in V(X_c)$, the restricted function $f|_{X_c(u)} : X_c(u) \to Y(f(u))$ is a one to one correspondence. Clearly, if X is bipartite, then so is X_c.

For each $u \in V(X)$, the fiber on u is defined as $fib_u = f^{-1}(u)$. The following important set is a subgroup of $Aut(X_c)$ and is called the group of covering transformations for f:

$$CT(f) = \{\sigma \in Aut(X_c) | \forall u \in V(X), \sigma(fib_u) = fib_u\}.$$

If this action is regular, then X_c is said to be a regular K-cover of X.

Let G be a subgroup of Aut(X). If action G on V(X), E(X) and Arc(X) be transitive, X is called respectively G-vetex transitive, G-edge transitive and G-Arc transitive. X is called G-semisymmetic if it is regular and G-edge transitive but not G-vertex transitive. Furtheremore X is called symmetric if both G-vertex transitive and G-arc transitive. For G = Aut(X), we usually remove G and say X is vertex-trasitive, edge-trasitive, semisymmetric or symmetric. An G-edge transitive but not G-vertex transitive graph is necessarily bipartite, where the two partites are the orbits of the action of G on V(X). If X is regular, then the two partite sets have equal cardinality, so an G-semisymmetric graph is bipartite such that G is transitive on each partite but G carries no vertex from one partite set to the other.

Following [14] the coset graph $C(G; H_0, H_1)$ of a group G with respect to finite subgroups H_0 and H_1 is a bipartite graph with $\{H_0g | g \in G\}$ and $\{H_1g | g \in G\}$ as its bipartition sets of vertices where H_0g is adjacent to H_1g' whenever $H_0g \cap H_1g' \neq \varnothing$.

In the following we discussed about some important findings that are used in the present study.

Theorem 2.1. [5, 16, 27, 32]

i) *A* K_3*-group is isomorphic to one of the following groups:*

$$A_5, A_6, L_2(7), L_2(2^3), L_2(17), L_3(3), U_3(3), U_4(2)$$

ii) *A* K_4*-group is isomorphic to one of the following groups:*
(1) $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, L_2(2^4), L_2(5^2), L_2(7^2), L_2(3^4), L_2(97), L_2(3^5), L_2(577), L_3(2^2), L_3(5), L_3(7), L_3(2^3), L_3(17), L_4(3), U_3(2^2), U_3(5), U_3(7), U_3(2^3), U_3(3^2), U_4(3), U_5(2), S_4(2^2), S_4(5), S_4(7), S_4(3^2), S_6(2), O_8^+(2), G_2(3), S_Z(2^3), S_Z(2^5), {}^3D_4(2), {}^2F_4(2)$;
(2) $L_2(r)$ *where* r *is a prime and* $r^2 - 1 = 2^a.3^b.s, s > 3$ *is a prime,* $a, b \in \mathbb{N}$;
(3) $L_2(2^m)$, *where* $m, 2^m - 1, \frac{2^m+1}{3}$ *are primes grater* 3;
(4)$L_2(3^m)$ where $m, \frac{3^m+1}{4}, \frac{3^m-1}{2}$ *are odd primes.*

It is important to note that only nonabelian simple groups of order less than 300 are A_5 and $L_2(7)$.

Theorem 2.2. [28] *If* H *is a subgroup of a group* G*, then* $C_G(H) \trianglelefteq N_G(H)$ *and* $\frac{N_G(H)}{C_G(H)}$ *is isomorphic to a subgroup of* Aut(H).

The theorem below is also well-known see [24].

Proposition 2.3. *Let* G *be a finite group and* $\mathrm{N} \trianglelefteq \mathrm{G}$. *If* $|\mathrm{N}|$ *and* $|\frac{\mathrm{G}}{\mathrm{N}}|$ *are relatively prime, then* $\mathrm{G} \cong \mathrm{H} \rtimes \mathrm{N}$.

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

Theorem 2.4. [24] *For any two distinct primes* p *and* q *and any two nonnegative integers* a *and* b, *every finite group of order* $\mathrm{p^a q^b}$ *is solvable.*

Lemma 2.5. [25, Theorem 3.2] *Let* G *be a finite group and* $\mathrm{H} \trianglelefteq \mathrm{G}$ *such that* $\frac{\mathrm{G}}{\mathrm{H}}$ *is nonabelian simple. If* $\mathrm{H} \simeq \mathrm{Z_p}$ *or* $\mathrm{Z_{2p}}$ *for an odd prime p, then* $\mathrm{H} = \mathrm{Z(G)}$.

In the next theorem, inverse of pair (a, b) is denoted by (b, a) and $\mathrm{A_i}, \mathrm{B_i}$ and $\mathrm{C_i}$ are just noncyclic groups of order i which we don't know their structures.

Theorem 2.6. [11] *Let* X *be a connected cubic* G*-semisymmetric graph, in this case the order of the stabilizer of each vertex* v *is* $2^{\mathrm{r}} \times 3$, *where* $0 \leq \mathrm{r} \leq 7$. *Furthermore for each edge* $\{\mathrm{u}, \mathrm{v}\}$, $(\mathrm{G_u}, \mathrm{G_v})$ *is one of the following pairs or their inverses:*
$(\mathrm{Z_3}, \mathrm{Z_3}), (\mathrm{S_3}, \mathrm{S_3}), (\mathrm{S_3}, \mathrm{Z_6}), (\mathrm{D_{12}}, \mathrm{D_{12}}), (\mathrm{D_{12}}, \mathrm{A_4}), (\mathrm{S_4}, \mathrm{D_{24}}), (\mathrm{S_4}, \mathrm{Z_3} \ltimes \mathrm{D_8}), (\mathrm{A_4} \times \mathrm{Z_2}, \mathrm{D_{12}} \times \mathrm{Z_2}), (\mathrm{S_4}, \mathrm{S_4}),$
$(\mathrm{S_4} \times \mathrm{Z_2}, \mathrm{D_8} \times \mathrm{S_3}), (\mathrm{S_4} \times \mathrm{Z_2}, \mathrm{S_4} \times \mathrm{Z_2}), (\mathrm{A_{96}}, \mathrm{B_{96}}), (\mathrm{A_{192}}, \mathrm{B_{192}}), (\mathrm{C_{192}}, \mathrm{D_{192}}), (\mathrm{A_{384}}, \mathrm{B_{384}})$.

Theorem 2.7. [31, Proposition 2.5] *Let* X *be a connected cubic* G*-semisymmetric graph for some* $\mathrm{G} \leq \mathrm{Aut(X)}$ *and* $\mathrm{N} \trianglelefteq \mathrm{G}$. *If* $\frac{|\mathrm{G}|}{|\mathrm{N}|}$ *is not divisible by* 3 *then* X *is* N*-semisymmetric graph.*

Theorem 2.8. [21, Proposition 2.7] *Let* X *be a connected cubic* G*-semisymmetric graph for some* $\mathrm{G} \leq \mathrm{Aut(X)}$, *then* $\mathrm{X} \cong \mathrm{K_{3,3}}$, *the complete bipartite graph with six vertex, or* G *acts faithfully on each of the bipartition sets of* X.

Theorem 2.9. [15, Theorem 3.1] *Let* X *be a connected cubic* G*-semisymmetric graph and* $\{\mathrm{U}, \mathrm{W}\}$ *be a bipartition of* X *furthermore,* $\mathrm{N} \trianglelefteq \mathrm{G}$. *If The actions of* N *on both* U *and* W *are intransitive, then* N *acts semiregularly on both* U *and* W, $\mathrm{X_N}$ *is* $\frac{\mathrm{G}}{\mathrm{N}}$*-semisymmetric and* X *is a regular* N*-covering of* $\mathrm{X_N}$.

This theorem has a nice result. For every normal subgroup $\mathrm{N} \trianglelefteq \mathrm{G}$ either N is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of N is divisible by $|\mathrm{U}| = |\mathrm{W}|$. In the latter case, according to Theorem 2.9, the induced action of N on both U and W is semiregular and hence the order of N divides $|\mathrm{U}| = |\mathrm{W}|$. Thus, we have the following handy corollary.

Corollary 2.10. *Let* X *be a connected cubic* G*-semisymmetric graph with* $\{\mathrm{U}, \mathrm{W}\}$ *as a bipartition and* $\mathrm{N} \trianglelefteq \mathrm{G}$, *then either* $|\mathrm{N}|$ *divided* $|\mathrm{U}|$ *or* $|\mathrm{U}|$ *divided* $|\mathrm{N}|$.

In the following, we present a key theorem about coset graphs.

Proposition 2.11. [11] *Let* G *be a finite group and* H, K *are subgroups of* G*. The coset graph* C(G; H, K) *has the following propeties:*

(i) C(G; H, K) *is regular if and only if* $\frac{|H|}{|H \cap K|} = \frac{|K|}{|H \cap K|} = d$;

(ii) C(G; H, K) *is connected if and only if* $G =< H, K >$;

(iii) G *acts on* C(G; H, K) *with multiple of right and this action faithfull if and only if* $Core_G(H \cap K) = 1$, *in this case* C(G; H, K) *is* G*-semisymmetric.*

Theorem 2.12. [18] *Let* X *be a regular graph and* $G \leq Aut(X)$*. If* X *be* G*-semisymmetric, then* $X \cong C(G; G_u, G_v)$ *where* u, v *are adjacent vertices.*

3. Main results

In this section, our goal is to prove the following important result:

Theorem 3.1. *Let* p *be a prime.*

i *If* $p \neq 23, 5$, *then there is no connected cubic semisymmetric graph of order* 44p.

ii) *There is no connected cubic semisymmetric graph of order* $44p^2$.

Let X be a cubic semisymmetric graph of order 44p. If $p < 17$, then by [6] there is only one cubic semisymmetric graph S220 of order 44p, in which $p = 5$. Inaddition, there is no connected cubic semisymmetric graph of order $44p^2$ for $p < 5$ [6]. Therefore, in the following, we will prove the first part of Theorem 3.1 for $p > 17$ and the second part of Theorem 3.1 for $p > 5$. We need some lemmas that now state and prove.

Lemma 3.2. [27] *For any odd prime number* p*, there are only two groups of order* 2p : Z_{2p}, D_{2p}. *The cardinalities of* $Z(D_{2p})$ *and* $Aut(D_{2p})$ *are* 2 *and* $p(p-1)$ *respectively.*

Lemma 3.3. *There are only four groups of order* 154 : Z_{154}, D_{154}, G_2 *and* G_3, *where*

$$G_2 = \langle a, b | a^{77} = b^2 = 1, b^{-1}ab = a^{34} \rangle$$
$$G_3 = \langle a, b | a^{77} = b^2 = 1, b^{-1}ab = a^{43} \rangle$$

Moreover,

$$Z(G_2) = \langle a^{35} \rangle, Z(G_3) = \langle a^{44} \rangle$$
$$|Aut(G_2)| = |Aut(G_3)| = 60 \times 2$$

Proof: Since 154 = 2×7×11, it follows that G *has a characteristic subgroup of order 77, say* N*. Also since 7 does not divide 10 it implies that* N *is cyclic group. Let* $N = \langle a \rangle$ *and take* $b \in G$ *to be an element of order* 2*. There is some* $1 \leq i < 77$ *for which* $b^{-1}ab = a^i$.

Therefore $a^{i^2} = b^{-1}(b^{-1}ab)b = a$ *and so* $i^2 \equiv 1$ *(mod 77). This congruence has only four solutions* $i = 1, 34, 43, 76$ *which respectively correspond to* $Z_{154}, G_1, G_2, D_{154}$. *Now consider* G_2. *Each element*

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equals a^i *or* a^ib *for some* i. *According to the relationship* $b^{-1}ab = a^{34}$, *It can be easily verified that no element of the form* a^ib *belongs to the center, and that* $a^i \in Z(G_2)$ *if and only if* $i \equiv 0$ *(mod 35). So* $Z(G_2) = \langle a^{35} \rangle$.

Every automorphism f *of* G_2 *is uniquely characterized by the two values* f(a) *and* f(b). *By an order argument, we find out that a group of order* 154 *has only one subgroup of order* 77. *So* f *takes* $\langle a \rangle$ *to* $\langle a \rangle$ *and hence* $f(a) = a^i$ *for some* i *coprime to* 77. *Therefore there are* $\varphi(77)$ *possibilities for* f(a), *where* φ *is the Euler function. Also* f(b) *must be an element of order* 2 *and has* 2 *possibilities, since the elements of order* 2 *in* G_2 *are of the form* a^ib *where* $5i \equiv 0 \pmod{11}$. *We conclude that* $|Aut(G_2)| = \varphi(77) \times 2 = 60 \times 2$. *The proof for* G_3 *is similar.* □

The following theorem is obtained by a simple check of the list of groups in Theorem 2.1.

Lemma 3.4. *There are only two groups, simple* K_4*-group whose orders of the form* $2^i \times 3 \times 11 \times p$ *for some prime* p, $p > 5$ *and* $i \in \mathbb{N}, 1 \leq i \leq 8 : L_2(11), L_2(23)$.

Lemma 3.5. *If* $|O_P(A)| = 1$, *then* A *does not have normal subgroup of orders* 22 *and* 44.

Proof: Let $\{U, V\}$ *be the bipartition of* X. *Then* $|U| = |W| = 22p$ *and by Theorem* 2.6, $|A_u| = 2^r \times 3$ *some* r, $0 \leq r \leq 7$ *and* $u \in U$. *Duo to transitivity of* G *on* U, $|A| = |A_u||U| = 2^{r+1} \times 3 \times 11 \times p$. *Let* M *be a normal subgroup of* A *of order* 22, *since* M *is intransitive on* U, *by Theorem* 2.9 X_M *is a* $\frac{A}{M}$*-semisymmetric with a bipartition* $\{U_M, W_M\}$ *and* $|\frac{A}{M}| = 2^r \times 3 \times p, |U_M| = |W_M| = p$. *Let* $\frac{K}{M}$ *be a minimal normal subgroup of* $\frac{A}{M}$. *If* $\frac{K}{M}$ *is unsolvable, it must be a simple group of order* $2^i \times 3 \times p$ *for some* $i, 1 \leq i \leq 7$. *But since* $p > 17$ *according to Theorem* 2.1 (i) *there is no* K_3*-group of such order. Now if* $\frac{K}{M}$ *is solvable and hence elementary abelian, then by Corollary* 2.10, *its order must be* p, *implying* $|K| = 10p$. *The Sylow* p*-subgroup of* K *is normal in* K *and since it is caracterisitic in* K, *hence normal in* A *that contradicting with assume* $|O_P(A)| = 1$. *Let* M *be a subgroup of* A *of order* 11, *since* M *be intransitive on* U *due to Theorem* 2.6 X_M *is* $\frac{A}{M}$*-semisymmetric with a bipartition* $\{U_M, W_M\}$ *and* $|\frac{A}{M}| = 2^{r+1} \times 3 \times p$, $|U_M| = |W_M| = 2p$. *Let* $\frac{K}{M}$ *a minimal normal subgroup of* $\frac{A}{M}$. *If* $\frac{K}{M}$ *is unsolvable, it must be a simple group of order* $2^{i+1} \times 3 \times p$ *for some* $i, 1 \leq i \leq 8$ *but according to Theorem* 2.1 (i) *there is no* K_3*-group of such order (for* $p > 17$*). So* $\frac{K}{M}$ *is solvable and therefore elementary abelian, whether it is transitive or intransitive on the partition sets,* $|K| = 22$ *or* 11p. *The first case could not be possible, the second case The Sylow* p*-subgroup of* K *is normal in* K *and since it is caracterisitic in* K, *hence normal in* A *that contradicting with assumption* $|O_P(A)| = 1$

Lemma 3.6. *Let* X *is a connected cubic semisymmetric graph of order* 44p. *Then* $|O_p(A)| = p$ *and for* $p = 23$ *we have* $X \simeq C(L_2(23), D_{12}, D_{12})$.

Proof: Take $\{U, W\}$ *to be a bipartition for* X. *Then* $|U| = |W| = 22p$ *and* $|A| = 2^{r+1} \times 3 \times 11 \times p$ *for some* $0 \leq r \leq 7$. *Let* N *be a minimal normal subgroup of* A *then* $N \cong T^k$, *where* T *be a simple group. If* T *is nonabelian and since the powers* 3, 11 *in* |A| *equal* 1, *then* $k = 1$ *and* $N \cong T$. *According to Corollary* 2.10, *eithere* |N| *divides* $|U| = 22p$ *or* |U| *divides* |N|. *In first case, since* |N| *is divisible by at least three distinct primes so* $|N| = 2^i \times 11 \times p$ *and hence* N *is a simple* K_3*-group but the order of such groups, listed in Theorem 2.1 is divisible by* 3. *Therefore* |N| *is divisible by* 22p. *Again*

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since the order of every simple K_3*-group is divisible by 3,* N *must be a simple* K_4*-group of order* $2^i \times 3 \times 11 \times p$*. By a simple check the simple groups in Theorem 2.6,* $N \cong L_2(11), L_2(23)$*, these groups correspond to* $p = 5, p = 23$*. Since* $p > 17, N \cong L_2(23)$ *and the order of* $\frac{A}{N}$ *does not divisible by* 3*. According to Theorem2.7* X *is* N*-semisymmetric graph. For each* $u \in U$ *and* $v \in W$ *by Theorem 2.6,* $|N_u| = |N_v| = 12$ *and knowing that* N_u*,* N_v *are subgroups of* N*, by [3] we have* $|N_u| = |N_v| = D_{12}$ *and due to Theorem 2.12,* $X \cong C(N; N_u, N_v)$*. We note that by [23, Theorem 1.3],* $C(L_2(23); D_{12}, D_{12})$ *is a connected cubic semisymmetric graph.*

Now assume T *is abelian, So* N *is solvable and hence elementary abelian. According to Theorem 2.9, the quotient graph* X_N *is* $\frac{A}{N}$*-semisymmetric graph of order* $\frac{44p}{|N|}$*. By Corollary 2.10* $|N|$ *divides* $22p$ *and so* $|N| = 2, 11$ *or* p*. Assume* $|O_p(A)| = 1$*, then* $N \cong Z_2, Z_{11}$ *and* X_N *is* $\dfrac{A}{N}$*-semisymmetric graph of order* $22p$ *or* $4p$ *respectively. Take* $\{U_N, W_N\}$ *to be the bipartition of* X_N *and* $\frac{M}{N}$ *be a minimal normal subgroup of* $\frac{A}{N}$*. Assume that* $N \cong \mathbb{Z}_2$*, then* $|\frac{A}{N}| = 2^r \times 3 \times 11 \times p$ *and* $|U_N| = |W_N| = 11p$*. If* $\frac{M}{N}$ *is unsolvable, then it must be simple* K_4*-group of order* $2^i \times 3 \times 11 \times p$*. It follows from Lemma 3.5 that* $p = 5$ *or* 23*, which are ruled out by our assumption on* p*. On the other hand, If* $\frac{M}{N}$ *is solvable, then its order should divide* $|U_N| = 11p$ *and hense* $|\frac{M}{N}| = 11$ *or* p*. If* $|\frac{M}{N}| = 11$*, then* $|M| = 22$ *according to Lemma 3.6, it could not possible.*

If $|\frac{M}{N}| = p$*, then* $|M| = 2p$*, which contradicts our assumption on* $O_p(A)$ *since a Sylow* p*-subgroup of* M *would be characteristic in* M *and so would be normal in* A*.*

If $N \cong \mathbb{Z}_{11}$*, then* $|\frac{A}{N}| = 2^{r+1} \times 3 \times p$ *and* $|U_N| = |W_N| = 2p$*. In this case, if* $\frac{M}{N}$ *is unsolvable, it would be simple group of order* $2^i \times 3 \times p$ *for some* i*, and hence according to Theorem 2.1* $\frac{M}{N} \simeq Z_5$ *or* $L_2(7)$ *implying* $p = 5$ *or* 7*. This is in contradiction to our assumption on* p*. on the other hand, if* $\frac{M}{N}$ *is solvable then like before, we conclude that* $|\frac{M}{N}| = 2$ *or* p*. Therefore* $|M| = 22$ *or* $11p$ *and each of them is impossible. Therefore,* $|O_p(A)| = p$*.* □

proof of Theorem 3.1 (i) Let X be a cubic semisymmetric graph of order $44p$. According to [4] and we suppose that $p > 17$. Let M be a Sylow p -subgroup of A. We have $|A| = 2^{r+1} \times 3 \times 11 \times p$ and by Theorem 2.6, $M \trianglerighteq A$. Let $\{U, W\}$ be a bipartition of X. Then we have $|U| = |W| = 22p$ and M is on both U and W intransitive. Now by Theorem 2.9, X_M is a connected cubic $\frac{A}{M}$ -semisymmetric graph of order 44 but according to [4], there is no connected cubic semisummetric graph of order 44. Set $G = \frac{A}{M}$, since X_M is G-semisymmetric, we get by [4] that it is G-vertex transitive and hence it is symmetric but the smallest symmetric graph is of order 54. □

Remark 3.7. *In fact, we prove that if* X *is a connected cubic semisymmetric graph of order* $44p$*,* p *prime, then either* $p = 5$ *and* X *is isomorphic to the cubic semisymmetric graph of order* 220 *in [4] or* $p = 23$ *and by Lemma 3.6,* $X \simeq C(L_2(23), D_{12}, D_{12})$*.*

Lemma 3.8. *Let* X *be a cubic semisymmetric graph of order* $44p^2$ *and* $A = Aut(X)$*, if* $|O_p(A)| = 1$ *or* p*, then* A *has no normal subgroup of order* 22 *and* $22p$ *respectively. Proof: We show first if* $|O_p(A)| = 1$*, then* A *has no normal subgroup of order* 22*.*

Let $\{U, W\}$ *be the bipartition for* X*.*

Clearly according to Theorem 2.6 we have $|\mathrm{A}| = 2^{r+1} \times 3 \times 11 \times p^2$*. Let* M *be a normal subgroup of order* 22*, then* $|\mathrm{U_M}| = |\mathrm{W_M}| = p^2$ *and* $|\frac{A}{M}| = 2^r \times 3 \times p^2$*. Let* $\frac{K}{M}$ *be a minimal normal subgroup of* $\frac{A}{M}$*. If* $\frac{K}{M}$ *is unsolvable, it must be a simple group of order* $2^i \times 3 \times p^2$ *for some* i*. But there is no simple* K_3*-group of such order. So* $\frac{K}{M}$ *is solvable and hence elementary abelian. Whether it is intransitive or transitive on the partite sets, its order must be* p *or* p^2*. Therefore* $|K| = 22p^i$ *for* $i = 1$ *or* 2*. The Sylow* p*-subgroup of* K *is normal in* K*. So it is characteristic in* K *and hence normal in* A*, contradicting the assumption that* $|O_p(A)| = 1$*. Now suppose* $|O_p(A)| = p$ *and* $|M| = 22p$*. In this case* $|\mathrm{U_M}| = |\mathrm{W_M}| = p$ *and* $|\frac{A}{M}| = 2^r \times 3 \times p$*. Once more let* $\frac{K}{M}$ *be a minimal normal subgroup of* $\frac{A}{M}$*. If* $\frac{K}{M}$ *is unsolvable, it must be a simple group of order* $2^i \times 3 \times p^2$ *for some* i *and* $p > 5$*. So* $\frac{K}{M} \simeq L_2(7)$ *and* $p = 7$*, since* 3 *does not divided the order of* $\frac{A}{K} = \frac{\frac{A}{M}}{\frac{K}{M}}$*, by Proposition 2.7 we conclude that* X *is* K*-semisymmetric. From the group theory we have either* $MC_K(M) = M$ *or* $MC_K(M) = K$*. In the first case if* $MC_K(M) = M$*, then* $C_K(M) \leq M$ *and so* $C_K(M) = C_K(M) \cap M = Z(M)$*. Now according to Theorem 2.2, the order of* $\frac{K}{Z(M)}$ *must divide* $|\mathrm{Aut}(M)|$*. The order of* M *is* 154 *and so according to Lemma 3.3,* $M \simeq Z_{154}, D_{154}, G_2, G_3$*. We have* $|K| = |L_2(7)| \times |M| = 2^4 \times 7^2 \times 3 \times 11$*. If* $M \simeq Z_{154}$*, then* $|\frac{K}{Z(M)}| = 2^3 \times 7 \times 3$ *does not divided* $|\mathrm{Aut}(M)| = \varphi(77) = 66$*. In the same way, we examine* M *other states. So the equality* $MC_K(M) = M$ *could not be possible. So* $\frac{K}{M}$ *cannot be unsolvable. If* $\frac{K}{M}$ *is solvable, it is elementary abelian. Whether it is intransitive or transitive on the partite sets, its order must equal* p*. Therefore* $|K| = 22p^2$*. The Sylow* p*-subgroup of* K *is normal in* K*. So it is characteristic in* K *and hence normal in* A*, contradicting the assumption that* $|O_p(A)| = p$*.* □

Lemma 3.9. *If* $p > 5$ *is a prime and* X *is a connected cubic semisymmetric graph of order* $44p^2$*, then* Aut(X) *has a normal Sylow* p*-subgroup.*

Proof: Let $\{U, W\}$ to be a bipartition for X and $A = \mathrm{Aut}(X)$. Then $|U| = |W| = 22p^2$ and by Theorem 2.6, $|A| = 2^{r+1} \times 3 \times 11 \times p^2$ for some $0 \leq r \leq 7$. Let $N \simeq T^k$ be a minimal normal subgroup of A, where T is simple. If T is nonabelian, then $k = 1$ and $N = T$ since the powers of 3 and 11 in $|A|$ equal 1. According to Corollary 2.10, either $|N|$ divides $|U| = 22p^2$ or $22p^2$ divides $|N|$. If $|N|$ divides $22p^2$, then since $|N|$ is divisible by at least three distinct primes (Theorem 2.4), we must have $|N| = 2 \times 11 \times p^i$ for $i = 1$ or 2 and so N is a simple K_3-group. But the order of every simple K_3-group, listed in Theorem 2.1, is divisible by 3, a contradiction. Therefore $22p^2$ divides $|N|$. Again since the order of every simple K_3-group is divisible by 3, N must be a simple K_4-group whose order is of the form $2 \times 3 \times 11 \times p^2$. But no such simple K_4-group exists. We conclude that N is elementary abelian and hence it follows from Corollary 2.10 that $|N|$ divides $22p^2$. Therefore $N \simeq Z_2, Z_{11}, Z_p$ or Z_p^2. In each case X_N would itself be a connected cubic $\frac{A}{N}$-semisymmetric graph of order $\frac{44p^2}{|N|}$. Next we prove that $|O_p(A)| = p^2$:

Case 1. $|O_p(A)| = 1$. If this case happens, then the minimal normal subgroup of A is $N \simeq Z_2$ or Z_{11} and X_N is $\frac{A}{N}$-semisymmetric of order $22p^2$ or $4p^2$ respectively. Let $\{U_N, W_N\}$ to be the bipartition for X_N. Also let $\frac{M}{N}$ be a minimal normal subgroup of $\frac{A}{N}$. If $N \simeq Z_2$, then $|\frac{A}{N}| = 2^r \times 3 \times 11 \times p^2$ and $|U_N| = |W_N| = 11p^2$. If $\frac{M}{N}$ is unsolvable, then it must be a simple K_4-group whose order is of the form $2^i \times 3 \times 11 \times p^2$. But there is no such simple K_4-group. If $\frac{M}{N}$ is solvable, it is elementary abelian and hence according to Corollary 2.10, its order divides $11p^2$ which yields $\frac{M}{N} \simeq Z_{11}$ or Z_p^i for $i = 1$ or 2. But by Lemma 3.6, $\frac{M}{N}$ cannot be isomorphic to Z_{11}. Accordingly, $\frac{M}{N} \simeq Z_p^i$ and so $|M| = 2p^i$ for $i = 1$ or 2. If P is a Sylow p-subgroup of M, then $P \trianglelefteq^c M \trianglelefteq A$ which implies $P \trianglelefteq A$, contradictin our assumption.

Case 2: $|O_p(A)| = p$. Let $Q = Op(A)$. X_Q is connected cubic $\frac{A}{Q}$-semisymmetric with the bipartition $\{U_Q, W_Q\}$, where $|U_Q| = |W_Q| = 22p$ and $|\frac{A}{Q}| = 2^{r+1} \times 3 \times 11 \times p$. Let $\frac{B}{Q}$ be a minimal normal subgroup of $\frac{A}{Q}$ and $\frac{B}{Q}$ is unsolvable. In this case $\frac{B}{Q}$ should be of an order divisible by $|U_Q| = 22p$ and so it is a simple K_4-group whose order equals $2^i \times 3 \times 11 \times p$ for some $1 \leq i \leq 8$. According Theorem 3.4, $\frac{B}{Q}$ is isomorphic to either $L_2(11)$ and $L_2(23)$ These two groups correspond to $p = 11$ and 23 respectively. In each case the order of $\frac{A}{Q} = \frac{\frac{A}{B}}{\frac{B}{Q}}$ is not divisible by 3 so by Theorem 2.7, X would also be B-semisymmetric. By examining the assumptions of Lemma 3.5, $Z(B) = Q$ and $B' \cap Z(B) \leq Z(B)$. Therefore $B'Z(B) = Z(B)$ or $B'Z(B) = B$. In the first case $B' = B' \cap Z(B)$ which is not possible as B is not abalian. On the other hand if $B'Z(B) = B$ then $|B| = |B'||Z(B)|$ and so $B' = 2^i \times 3 \times 5 \times p$, where i depends on p (e.g. $i = 2$ for $p = 11$). Since X is B-semisymmetric, according to Corollary 2.10, B' either divides $|U| = 22p^2$ or is divisible by $22p^2$. With the order that we just obtained for B', none of these divisibilities hold.

If $\frac{B}{Q}$ is solvable. In this case, $\frac{B}{Q}$ is elementary abelian and hence intransitive on both U_Q and W_Q. So $\frac{B}{Q} \simeq Z_2, Z_{11}$ or Z_p. The isomorphism $\frac{B}{Q} \simeq Z_p$ could not hold as it would lead to $|B| = p^2$ which contradicts the assumption that $|O_p(A)| = p$. In the following we discuss the two remaining cases. Let $\frac{B}{Q} \simeq Z_2$. This yields $|B| = 2p$. Consider the graph X_B which is connected cubic $\frac{A}{B}$ -semisymmetric with the bipartition $\{U_B, W_B\}$, where $|U_B| = |W_B| = 11p$. Let $\frac{T}{B}$ be a minimal normal subgroup of $\frac{A}{B}$. It is either solvable or unsolvable. In the following we discuss that both cases lead to contradictions. Suppose $\frac{T}{B}$ is solvable, then $\frac{T}{B} \simeq Z_{11}$ or Z_P. The case $\frac{T}{B} \simeq Z_{11}$ by Lemma 3.8, is not possible. Also the case $\frac{T}{B} \simeq Z_p$ yields $|T| = 2p^2$ and a Sylow p-subgroup of T would be normal in A, contradicting our assumption that $|O_p(A)| = p$. Now suppose $\frac{T}{B}$ is unsolvable. So it should be a simple group of order $2^i.3.11.p$ for some $1 \leq i \leq 7$. Again according to Theorem 3.3, $\frac{T}{B} = L_2(11)$ or $L_2(23)$ which respectively correspond to $p = 11, 23$. In each case X is T- semisymmetric according to Corollary 2.10, since the order of $\frac{A}{T} = \frac{\frac{A}{B}}{\frac{T}{B}}$ is not divisible by 3. As $|B| = 2p$ and p is odd prime, according to Lemma 3.5 we have $B \simeq Z_{2p}$ or D_{2p}. First suppose $B \simeq Z_{2p}$, then $Z(T) = B$. Now $T' \cap Z(T) \leq Z(T) \simeq Z_{2p}$ and p does not divide the order of $T' \cap Z(T)$. Therefore $|T' \cap B| \simeq |T' \cap Z(T)| = 1$ or 2. Since $\frac{T}{B}$ is nonabelian simple, $\frac{T'B}{B} \simeq (\frac{T}{B})' \simeq \frac{T}{B}$ and so $T'.B = T$. So $|T| = |T'|.\frac{|B|}{|T' \cap B|}$. Because $|T| = 2^i \times 3 \times 11 \times p^2$ for some $i > 1$ and $\frac{|B|}{T' \cap B} = p$ or $2p$, we obtain $|T'| = 2^j \times 3 \times 11 \times p$ for some j. The graph X is T-semisymmetric and either $|T'|$ divides $|U| = 22p^2$ or $|T'|$ is divisible by $22p^2$. Both these cases are inconsistent with the order that we just obtained for T'. Next assume $B \simeq D_{2p}$. The relations $B \leq BC_T(L) \trianglelefteq T$ imply $LC_T(L) = L$ or $BC_T(B) = T$. If $BC_T(B) = T$, then $|T| = \frac{|B||C_T(L)|}{|B \cap C_T(L)|}$.

Since $|B\cap C_T(B)| = |Z(B)| = |Z(D_{2p})| = 2$, we will have $|T| = \frac{|B||C_T(L)|}{2}$ and so $|C_T(B)| = 2^j \times 3 \times 11 \times p$ for some j. With this order, the normal subgroup $C_T(B) \trianglelefteq T$ does not satisfy Corollary 2.10. If $BC_T(B) = B$, then $C_T(B \leq B$ and hence $C_T(B) = C_T(B) \cap B = Z(B) \simeq Z(D_{2p}) \simeq Z_2$. According to Theorem 2.2, $\frac{T}{C_T(B)} \leq Aut(B) \simeq Aut(D_{2p})$ and so $\frac{T}{C_T(B)} = \frac{2^i \times 3 \times 11 \times p^2}{2}$ divides $|Aut(D_{2p})| = p(p-1)$ which is impossible.

suppose $\frac{B}{Q} \simeq Z_{11}$. In this case $|L| = 11p$. The graph X_B is connected cubic $\frac{A}{B}$ semisymmetric with the bipartition $\{U_B, W_B\}$, where $|U_B| = |W_B| = 2p$. Let $\frac{T}{B}$ be a minimal normal subgroup of $\frac{A}{B}$. If $\frac{T}{B}$ is solvable, then $\frac{T}{B} \simeq Z_2$ or Z_p. The case $\frac{T}{B} \simeq Z_2$ is ruled out by Lemma 3.8. Also the case $\frac{T}{B} \simeq Z_p$ yields $|T| = 11p^2$ and a Sylow p-subgroup of T would be normal in A, contradicting our assumption that $|O_p(A)| = p$. If $\frac{T}{B}$ is unsolvable, then it is a simple group of order $2^i \times 3 \times p$ for some $1 \leq i \leq 8$. So $\frac{T}{B} \simeq A_5, L_2(7)$. But since $p > 5$, we may only have $\frac{T}{B} \simeq L_2(7)$ and $p = 7$. The order of $\frac{\frac{A}{B}}{\frac{T}{B}}$ is not divisible by 3. So X_B is also G- semisymmetric where $G \simeq \frac{T}{B}$. Therefore G is transitive on both U_B and W_B, each with $2p = 14$ points. So for any pair of vertices $u \in U_B$ and $w \in W_B$, the stabilizers G_u and G_w are of order 12. For any prime power q all subgroups of the group $L_2(q)$ have been characterized. The only subgroup of $L_2(7)$ of order 12 is A_4. So $G_u \simeq G_w \simeq A_4$. But the pair $(G_u, G_w) = (A_4, A_4)$ is not possible for an edge $\{u, w\}$ of a cubic G-semisymmetric graph according to Theorem 2.6. □

proof of Theorem 3.1 (ii) According to [4], for $p = 2, 3$, there is no connected cubic semisymmetric graph of order $44p^2$. So we continue the proof for $p > 5$. Let $\{U, W\}$ be a bipartition for X and $A = Aut(X)$. Then $|U| = |W| = 22p^2$ and $|A| = 2^{r+1} \times 3 \times 11 \times p^2$ for some $0 \leq r \leq 7$. By Lemma 3.9, there is Sylow p-subgroup M of A and $M \trianglelefteq A$. Due to its order, M is intransitive on both U and W and so according to Theorem 2.9, X_M is a connected cubic G-semisymmetric graph of order 44 with the bipartition $\{U_M, W_M\}$, where $G = \dfrac{A}{M}$ and $|U_M| = |W_M| = 22$ and X_M is G-edge transitive and hence edge transitive. Since by [4] there is no semisymmetric cubic graph of order 22, so X_M should be vertex-transitive and hence symmetric since a cubic vertex and edge-transitive graph is necessarily symmetric. According to [7], there are no symmetric cubic graphs of order 22. □

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