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A CLOSED FORMULA FOR THE NUMBER OF INEQUIVALENT ORDERED INTEGER QUADRILATERALS WITH FIXED PERIMETER

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Abstract. Given an integer $n \geq 4$, how many inequivalent quadrilaterals with ordered integer sides and perimeter n are there? Denoting such number by $Q(n)$, the answer is given by the following closed formula:

$$Q(n) = \left\{ \frac{1}{576} n\,(n+3)\,(2n+3) - \frac{(-1)^n}{192} n\,(n-5) \right\}.$$

1. Introduction

In [5], Jordan, Walch, and Wisner characterized the number $T(n)$ of incongruent triangles with integer sides that have a fixed perimeter n by proving that $T(2n+12) = T(2n-3) + n + 3$, for $n \geq 1$. However, in [1], George E. Andrews noted that $T(n)$ can simply be handled by relating it to $p(n,2)$ and $p(n,3)$, the number of partitions of n into 2 and 3 parts, respectively, and proved the following elegant analytical formula

$$T(n) = \left\{ \frac{n^2}{12} \right\} - \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n+2}{4} \right\rfloor,$$

where $\{x\}$ is the nearest integer function and $\lfloor x \rfloor$ the greatest integer function.

Keywords: Integer quadrilaterals, Ordered quadrilaterals, Integer partitions, Generating function.

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Reference [4] introduces an enhanced formula for $T(n)$, which is notably more user-friendly, providing greater ease of use and improved accessibility for users. Moreover, Honsberger showed that

$$T(n) = \begin{cases} \left\{ \dfrac{n^2}{48} \right\} & \text{if } n \text{ is even} \\ \left\{ \dfrac{(n+3)^2}{48} \right\} & \text{if } n \text{ is odd} \end{cases}$$

In [3], James East and Ron Niles considered analogous problem, but concerning the number of inequivalent integer m-gons having perimeter n. The authors considered two m-gons P and Q for some $m \geq 3$, with side lengths $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$, respectively, beginning from any side and reading clockwise or anti-clockwise as equivalent, if we may obtain the m-tuple $(b_1, \ldots, b_m)$ from $(a_1, \ldots, a_m)$ by cyclically re-ordering and/or reversing the entries. Thus, in the figure bellow, the first and second quadrilaterals are equivalent, the third and fourth are equivalent but the first and third are inequivalent.

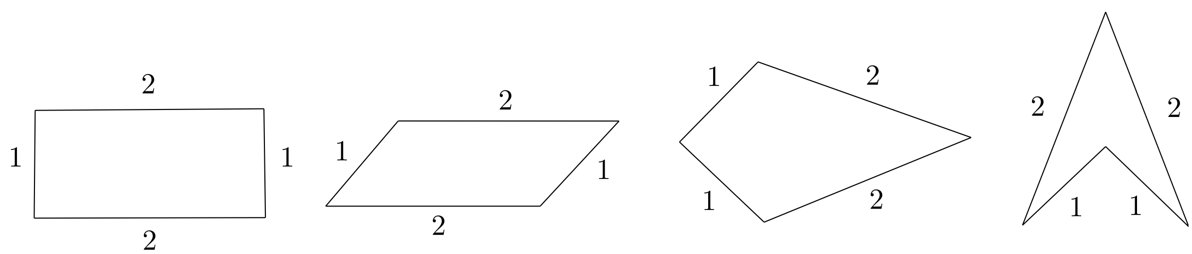

FIGURE 1. Several quadrilaterals with edge lengths 1, 1, 2, 2.

Specializing to the case $m = 4$, James East and Ron Niles [3], proved the following result on quadrilaterals.

Theorem 1.1. *The number $p_{4,n}$ of inequivalent integer quadrilaterals with perimeter n is given by*

$$p_{4,n} = \begin{cases} \dfrac{n^3 - 3n^2 + 20n}{96} & \text{if } n \equiv 0 \pmod 4 \\ \dfrac{n^3 - 7n + 6}{96} & \text{if } n \equiv 1 \pmod 4 \\ \dfrac{n^3 - 3n^2 + 20n - 36}{96} & \text{if } n \equiv 2 \pmod 4 \\ \dfrac{n^3 - 7n - 6}{96} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Table 1 gives calculated values of $p_{4,n}$, for $4 \leq n \leq 20$.

In the following, we will deal with the same problem, but by regarding the number $Q(n)$ of inequivalent ordered quadrilaterals with integer sides and perimeter n,

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TABLE 1. The number $p_{4,n}$, for $4 \le n \le 20$.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$p_{4,n}$	1	1	2	3	5	7	9	13	16	22	25	34	38	50	54	70	75

Definition 1.2. *Let P be a quadrilateral, with side lengths a_1, a_2, a_3, a_4 beginning from any side and reading clockwise or anti-clockwise. We say that P is ordered if we may obtain an ordered sequence by cyclically re-ordering and/or reversing the entries.*

In Figure 1, for example, the third and fourth quadrilaterals are ordered but the first and second are unordered.

2. Preliminary results

The partition of $n \in \mathbb{N}$ into k parts is a tuple $\pi = (\pi_1, \ldots, \pi_k) \in \mathbb{N}^k, k \in \mathbb{N}$, such that

$$n = \pi_1 + \cdots + \pi_k,\ 1 \le \pi_1 \le \cdots \le \pi_k,$$

where the positive integers π_i are called parts. We denote the number of partitions of n into k parts by $p(n, k)$.

Lemma 2.1. *For $n \ge 4$, we have*

$$Q\,(n) = p\,(n, 4) - \sum_{m=3}^{\left\lfloor \frac{n}{2} \right\rfloor} p\,(m, 3) \cdot$$

Proof. At first sight, it should be noted that any partition of n into four parts generates an ordered integer quadrilateral and vice versa, except the partitions for which the sum of its three small parts does not exceed the largest part, due to the triangle inequality, the such partitions verify

$$n = a + b + c + d,\quad 1 \le a \le b \le c \le d \ \text{ and } \ a + b + c \le d.$$

Equivalently

$$n - d = a + b + c,\quad 1 \le a \le b \le c \le d \le n.$$

But

$$n - d \le d \Longleftrightarrow n - d \le \frac{n}{2} \cdot$$

Hence

$$Q(n) = p\,(n, 4) - \sum_{m=3}^{\left\lfloor \frac{n}{2} \right\rfloor} p\,(m, 3) \cdot$$

□

If we consider, for example, the perimeter $n = 10$, then the number of partitions of n is 9, which are: 7111, 6211, 5311, 4411, 5221, 4321, 3331, 4222 and 3322, they form a quadrilateral only those we have underlined as shown below

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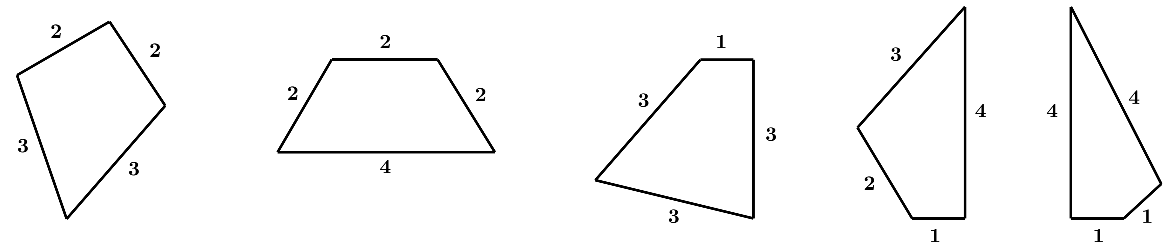

As we can check

$$Q(10) = p\,(10, 4) - \sum_{m=3}^{5} p(m, 3) = 9 - (1 + 1 + 2) = 5.$$

It should be noted that each quadrilaterals in the figure above represents an equivalence class of quadrilaterals that share the same partition. So, the number $Q(n)$ counts only the inequivalent ordered integer quadrilaterals representing the equivalence classes modulo the same partition.

Lemma 2.2. *For $n \geq 3$, we get*

$$\sum_{m=3}^{n} p\,(m, 3) = \frac{n\,(n-2)\,(2n+7)}{72} + \frac{1}{3}\left\lfloor \frac{n}{3} \right\rfloor + \frac{1-(-1)^n}{16}.$$

Proof. Let $f(z)$ be the known generating function of $p(m, 3)$ [1]:

$$f\,(z) = \frac{z^3}{(1-z)\,(1-z^2)\,(1-z^3)}.$$

Then

$$\sum_{m=0}^{n} p(m, 3) = [z^n] \left(\frac{f\,(z)}{1-z} \right).$$

From expanding $\frac{f\,(z)}{1-z}$ in partial fractions, we obtain

$$\frac{f\,(z)}{1-z} = \frac{7}{144\,(1-z)} - \frac{1}{72\,(1-z)^2} - \frac{1}{4\,(1-z)^3} + \frac{1}{6\,(1-z)^4} - \frac{1}{16\,(1+z)} + \frac{1+z}{9\,(1+z+z^2)}$$

Since

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n \geq 0} z^n, \\ \frac{1}{1+z} &= \sum_{n \geq 0} (-1)^n\, z^n, \\ \frac{1}{(1-z)^2} &= \sum_{n \geq 0} (n+1) z^n, \\ \frac{1}{(1-z)^3} &= \sum_{n \geq 0} \frac{(n+1)(n+2)}{2} z^n, \\ \frac{1}{(1-z)^4} &= \sum_{n \geq 0} \frac{(n+1)(n+2)(n+3)}{6} z^n, \end{aligned}$$

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and

$$\begin{aligned}\frac{1+z}{1+z+z^2} &= \frac{1-z^2}{1-z^3},\\ &= \frac{1}{1-z^3}-\frac{z^2}{1-z^3},\\ &= \sum_{n\geq 0} z^{3n}-\sum_{n\geq 0} z^{3n+2},\\ &= \sum_{n\geq 0} a_n z^n,\end{aligned}$$

where

$$a_n = \begin{cases} 1, & n\equiv 0 \pmod 3,\\ 0, & n\equiv 1 \pmod 3,\\ -1, & n\equiv 2 \pmod 3.\end{cases}$$

In a simplified way,

$$a_n = 1-n+3\left\lfloor\frac{n}{3}\right\rfloor\cdot$$

Summing all coefficients of z^n, the result yields. □

Corollary 2.3. *For $n\geq 6$, we have*

$$\sum_{m=3}^{\lfloor\frac{n}{2}\rfloor} p(m,3) = \frac{1}{576}\left(2n^3+3n^2-59n+30\right)+\frac{(-1)^n}{192}\left(n^2+n-10\right)+\frac{1}{3}\left\lfloor\frac{n}{6}\right\rfloor+\frac{1-(-1)^{\lfloor\frac{n}{2}\rfloor}}{16}\cdot$$

Proof. While observing that

$$\left\lfloor\frac{n}{2}\right\rfloor = \frac{n}{2}-\frac{1-(-1)^n}{4},$$

we get,

$$\frac{1}{72}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-4}{2}\right\rfloor\left(2\left\lfloor\frac{n}{2}\right\rfloor+7\right) = \frac{1}{576}\left(2n^3+3n^2-59n+30\right)+\frac{(-1)^n}{192}\left(n^2+n-10\right)\cdot$$

Hence, from Lemma 2.2 the result follows. □

3. Main result

Theorem 3.1. *For $n\geq 4$, we have*

$$Q(n) = \left\{\frac{1}{576}n(n+3)(2n+3)-\frac{(-1)^n}{192}n(n-5)\right\}\cdot$$

Proof. The generating function of $p(n,4)$ is as follows [2]:

$$g(z) = \frac{z^4}{(1-z)(1-z^2)(1-z^3)(1-z^4)}\cdot$$

Via straightforward calculations, it can be proved that

$$p(n,4) = \frac{n^3}{144}+\frac{n^2}{48}-\frac{(1-(-1)^n)\,n}{32}+\frac{(-1)^n}{32}-\frac{13}{288}+\frac{\alpha_n}{72},$$

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where

$$\alpha_n \in \{-17, -9, -8, -1, 0, 1, 8, 9, 17\} \cdot$$

Then, from Lemma 2.1 and Corollary 2.3, we get

$$Q_n = \frac{1}{576} n (n+3)(2n+3) - \frac{(-1)^n}{192} n (n-5) + \beta_n,$$

where

$$\beta_n = -\frac{23}{144} + \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{16} + \frac{1}{3}\left(\frac{n}{6} - \left\lfloor \frac{n}{6} \right\rfloor\right) + \frac{(-1)^n}{12} - \frac{\alpha_n}{72},$$

with

$$\beta_n \in \left\{-\frac{3}{8}, -\frac{1}{4}, -\frac{11}{72}, -\frac{5}{36}, -\frac{1}{8}, -\frac{1}{36}, -\frac{1}{72}, 0, \frac{5}{72}, \frac{7}{72}, \frac{2}{9}, \frac{4}{9}\right\}.$$

Since $Q(n)$ is an integer and $|\beta(n)| < 1/2$, we finally get

$$Q(n) = \left\{\frac{1}{576} n (n+3)(2n+3) - \frac{(-1)^n}{192} n (n-5)\right\}.$$

□

Table 2 gives calculated a few values of $Q(n)$, for $4 \leq n \leq 20$.

TABLE 2. The number $Q(n)$, for $4 \leq n \leq 20$.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Q(n)$	1	1	1	2	3	4	5	7	8	11	12	16	18	23	24	31	33

It is easy to see, from Theorem 3.1, that for $k \geq 1$, we get

$$Q(24^2 k) = 663552k^3 + 3456k^2 + 24k.$$

Thus, Table 3 gives calculated values of $Q(n)$, for a few large values of n.

k	1	2	3	4	5	6	7
$n = 24^2 k$	576	1152	1728	2304	2880	3456	4032
$Q(n)$	667032	5322288	17947080	42522720	83030520	143451792	227767848

TABLE 3. The number $Q(24^2 k)$, for $1 \leq k \leq 7$.

Remark 3.2. *There is an apparent adaptation of Lemma 2.1 for ordered integer k-gons with a perimeter of n. Let $Q(n,k)$ represent the number of such k-gons, then*

$$Q(n,k) = p(n,k) - \sum_{m=k-1}^{\lfloor \frac{n}{2} \rfloor} p(m, k-1) \cdot$$

So $Q(n) = Q(n,4)$ and $T(n) = Q(n,3)$.

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By summing over k, we get

$$P(n) = \sum_{k=3}^{n} Q(n,k),$$

the number of ordered integer polygons of perimeter n. A table with values of $Q(n,k)$ for $3 \leq n, k \leq 20$, and $P(n)$ for $3 \leq n \leq 20$ has been generated using a basic code. Nevertheless, it remains an intriguing future endeavor to derive formulas for $Q(n,k)$ applicable to larger values of k, as well as to investigate the behavior of the sequence $P(n)$.

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$P(n)$
3	1																		1
4	0	1																	1
5	1	1	1																3
6	1	1	1	1															4
7	2	2	2	1	1														8
8	1	3	2	2	1	1													10
9	3	4	4	3	2	1	1												18
10	2	5	5	4	3	2	1	1											23
11	4	7	8	6	5	3	2	1	1										37
12	3	8	9	9	6	5	3	2	1	1									47
13	5	11	14	12	10	7	5	3	2	1	1								71
14	4	12	16	16	13	10	7	5	3	2	1	1							90
15	7	16	23	22	19	14	11	7	5	3	2	1	1						131
16	5	18	25	28	24	20	14	11	7	5	3	2	1	1					164
17	8	23	35	37	34	27	21	15	11	7	5	3	2	1	1				230
18	7	24	39	46	42	36	28	21	15	11	7	5	3	2	1	1			288
19	10	31	52	59	58	48	39	29	22	15	11	7	5	3	2	1	1		393
20	8	33	57	71	70	63	50	40	29	22	15	11	7	5	3	2	1	1	488

TABLE 4. $P(n)$, $3 \leq n \leq 20$.

4. **Conclusion**

The values $Q(n)$ in Table 2 are sequence A062890 in the Online Encyclopedia of Integer Sequences [6], but no explicit formula has been given for this sequence.

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