https://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 13 No. 3 (2024), pp. 213-223.

www.ui.ac.ir

TOTAL ROMAN DOMINATION AND 2-INDEPENDENCE IN TREES

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A *total Roman dominating function* on a graph G is a function $f : V \to \{0, 1, 2\}$ satisfying the following conditions: (i) every vertex u such that $f(u) = 0$ is adjacent to at least one vertex v such that $f(v) = 2$ and (ii) the subgraph of G induced by the set of all vertices of positive weight has no isolated vertex. The weight of a total Roman dominating function f is the value, $f(V) = \Sigma_{u\in V(G)} f(u)$. The *total Roman domination number* $\gamma_{tR}(G)$ of G is the minimum weight of a total Roman dominating function of G. A subset S of V is a 2-independent set of G if every vertex of S has at most one neighbor in S. The maximum cardinality of a 2-independent set of G is the 2-independence number $\beta_2(G)$. These two parameters are incomparable in general, however, we show that if T is a tree, then $\gamma_{tR}(T) \le \frac{3}{2}\beta_2(T)$ and we characterize all trees attaining the equality.

1. Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [18, 19]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A *leaf* is a vertex of degree 1, a *support vertex* is

Keywords: total Roman dominating function, total Roman domination number, 2-independent set, 2-independence number.

MSC(2010): Primary 05C69, 05C05.

Communicated by Alireza Abdollahi.

Article Type: Research Paper.

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Received: 21 July 2022, Accepted: 08 June 2023.

Cite this article: H. Abdollahzadeh Ahangar, M. Soroudi, J. Amjadi and S. M. Sheikholeslami, Total Roman domination and 2-independence in trees, Trans. Comb., **13** no. 2 (2024) 213–223. http://dx.doi.org/10.22108/toc.2023.134483.2005 .

a vertex adjacent to a leaf, and a *strong support vertex* is a support vertex adjacent to at least two leaves. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. Also, the *depth of* v, depth(v), is the largest distance from v to a vertex in $D[v]$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We denote the set of leaves adjacent to a vertex v by L_v. A *pendant path* P of a graph G is an induced path such that one of its end-vertex has degree one in G, and its other end-vertex has degree at least 3. The *diameter* of a graph G, denoted by diam(G), is the greatest distance between two vertices of G. We write P_n for the *path* of order n and $K_{1,n-1}$ for the *star* of order n. A *double star* $DS_{p,q}$ is a tree obtained from $K_{1,p}$ and $K_{1,q}$ by connecting the center of $K_{1,p}$ with that of $K_{1,q}$. For a subset $S \subseteq V(G)$ and a function $f:\ V(G) \longrightarrow \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$.

A subset S of vertices of G is a *dominating set* if $N[S] = V$ and is a *total dominating set* if the induced subgraph $G[S]$ has no isolated verte). The *(total) domination number* $\gamma(G)$ $(\gamma_t(G))$ is the minimum cardinality of a (total) dominating set of G, and a (total) dominating set of minimum cardinality is called a γ-set (γ_t-set). The subset S is k*-dominating* if every vertex of $V - S$ has at least k neighbors in S. The k*-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. The literature on the subject of domination parameters in graphs has been surveyed in the three main books [18, 19, 20, 21].

In [17], Fink and Jacobson generalized the concept of independent sets as follows. Let k be a positive integer, a subset X of V is k*-independent* if the maximum degree of the subgraph induced by the vertices of X is at most $k-1$. The k*-independence number* $\beta_k(G)$ is the maximum cardinality among all k-independent sets of G. A k-independent set with maximum cardinality of a graph G is called a $\beta_k(G)$-set. For additional information on k-independence see the survey by Chellali et al. [12]. Relations between domination parameters and independence parameters have been studied by several authors [4, 11, 14, 15, 22, 24].

A *Roman dominating function* on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v such that $f(v) = 0$ is adjacent to at least one vertex u such that $f(u) = 2$. The *weight*, $\omega(f)$, of f is defined as $f(V(G))$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight over all Roman dominating function of G. That is $\gamma_R(G) = \min\{\omega(f)|\ f$ is a Roman dominating function in $G\}$. A Roman dominating function with minimum weight $\gamma_R(G)$ in G is called a $\gamma_R(G)$-function. For a Roman dominating function f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$. We observe that $\omega(G) = |V_1| + 2|V_2|$. The concept of Roman dominating function was first defined by Cockayne, Dreyer, Hedetniemi and Hedetniemi [13] and was motivated by the works [25, 26]. Roman domination in graphs is now very well studied (see for example [10, 16, 27]).

Liu and Chang [23] introduced the concept of total Roman domination in graphs albeit in a more general setting. A *total Roman dominating function* of a graph G with no isolated vertex, abbreviated

http://dx.doi.org/10.22108/toc.2023.134483.2005

TRD-*function*, is a Roman dominating function $f = (V_0, V_1, V_2)$ on G with the additional property that the subgraph of G induced by the set of all vertices $V_1 \cup V_2$ of positive weight under f has no isolated vertices. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a total dominating function on G. The total Roman domination in graphs is less studied until now and only a couple of works are already known. For instance, Liu and Chang [23] investigated algorithmic aspects of total Roman domination in graphs and several other combinatorial results were published in [1, 2, 3, 5, 6, 7, 8].

Abdollahzadeh Ahangar et al. [2], show that $\gamma_{tR}(G) \leq 2\gamma_t(G)$ for every graph G with no isolated vertex. In [9], the authors prove that, for every graph G without isolated vertices, $\gamma_t(G) \leq \frac{3}{2}\gamma_2(G) - \frac{1}{2}$. Favaron [15] proved that for any graph G and positive integer k, $\gamma_k(G) \leq \beta_k(G)$. Combining these results, we obtain the next results.

Proposition 1.1. For any graph G without isolated vertices, $\gamma_{tR}(G) \leq 3\beta_2(G) - 1$.

It seems that that aforementioned bound is not sharp and it would be interesting to find a sharp relation between these parameters. In this paper we tackle the problem of 2-independent set with total Roman domination in trees and improves the above bound considerably. In fact, we prove that for any tree T, $\gamma_{tR}(T) \leq \frac{3}{2}\beta_2(T)$, and we provide a constructive characterization of the trees T with $\gamma_{tR}(T) = \frac{3}{2}\beta_2(T)$.

We make use of the following results and notation in this paper.

Observation 1.2. [3] *If v is a strong support vertex in a graph G, then there exists a $\gamma_{tR}(G)$-function f such that $f(v) = 2$.*

Observation 1.3. *If v is a strong support vertex in a graph G, then there exists a $\beta_2(G)$-set S such that $v \notin S$.*

By Observations 1.2 and 1.3, in what follows, when we consider a $\gamma_{tR}(G)$-function of a graph G, we assume each strong support vertex is assigned the value 2, and when we consider a $\beta_2(G)$-set of a graph G, we assume no strong support vertex belongs to it.

Let v be a vertex of a graph G. A function $f : V(G) \to \{0, 1, 2\}$ is said to be a *nearly total Roman dominating function* (nearly TRDF) with respect to v, if the following three conditions hold: (i) every vertex $x \in V(G) - \{v\}$ such that $f(x) = 0$ is adjacent to at least one vertex $y \in V(G)$ such that $f(y) = 2$, (ii) every vertex $x \in V(G) - \{v\}$ such that $f(x) \geq 1$ is adjacent to at least one vertex $y \in V(G)$ such that $f(y) \geq 1$ and (iii) $f(v) \geq 1$ or $f(v) + f(u) \geq 2$ for some $u \in N(v)$. Let

$$\gamma_{tR}(G; v) = \min\{\omega(f) \mid f \text{ is a nearly TRDF with respect to } v\}.$$

Clearly, any total Roman dominating function on G is a nearly TRDF with respect to each vertex of G. Hence $\gamma_{tR}(G; v) \leq \gamma_{tR}(G)$ for each $v \in V(G)$.

http://dx.doi.org/10.22108/toc.2023.134483.2005

For a graph G and $v \in V(G)$, we define W_G^i for $i = 1, 2$, as follows:

$$W_G^1 = \{v \in V(G) | \gamma_{tR}(G; v) = \gamma_{tR}(G)\}$$

and

$$W_G^2 = \{v \in V(G) | \text{for any } \gamma_{tR}(G) - \text{ function } f,\ f(v) \leq 1\}.$$

Lemma 1.4. *Let T' be a tree, and let $u \in V(T')$. If T is a tree obtained from T' by adding a path $P_3 = x_3x_2x_1$, and joining u to x_3, then $\beta_2(T) = \beta_2(T') + 2$ and $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$.*

Proof. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding x_1, x_2, and so $\beta_2(T) \geq \beta_2(T') + 2$. Also if S is a $\beta_2(T)$-set, then obviously $|S \cap \{x_1, x_2, x_3\}| = 2$ and $S \cap V(T')$ is a 2-independent set of T' and hence $\beta_2(T) \leq \beta_2(T')+2$. Thus $\beta_2(T) = \beta_2(T')+2$. Assume now that f is a $\gamma_{tR}(T')$-function. Then the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(x_2) = 2, g(x_1) = 1, g(x_3) = 0$, and $g(x) = f(x)$ otherwise, is a TRDF of T implying that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. □

2. Main Result

In this section, we show that if T is a tree, then $\gamma_{tR}(T) \leq \frac{3}{2}\beta_2(T)$ and we provide a constructive characterization of all trees T with $\gamma_{tR}(T) = \frac{3}{2}\beta_2(T)$. In order to do this, let $\mathcal{T}$ be the family of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ $(m \geq 1)$ of trees such that $T_1 = P_3$, and, if $m \geq 2$, T_{i+1} can be obtained recursively from T_i by the following operation for $1 \leq i \leq m - 1$.

Operation $\mathcal{O}$. If $u \in W_{T_i}^1 \cap W_{T_i}^2$, then Operation $\mathcal{O}$ adds a path $P_3 = x_3x_2x_1$, and the edge ux_3 to obtain T_{i+1}

Lemma 2.1. *If T_i is a tree with $\gamma_{tR}(T_i) = \frac{3}{2}\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}$, then $\gamma_{tR}(T_{i+1}) = \frac{3}{2}\beta_2(T_{i+1})$.*

Proof. By Lemma 1.4, $\beta_2(T_{i+1}) = \beta_2(T_i) + 2$, and $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 3$. Assume now, that f is a $\gamma_{tR}(T_{i+1})$-function such that $f(x_3) \leq 1$. Clearly $f(x_2) + f(x_1) \geq 2$. If $f(x_3) + f(x_2) + f(x_1) \geq 3$, then we may assume that $f(x_2) = 2$ and $f(x_3) = 1$. Hence, the function f restricted to T_i is a nearly TRDF of T_i with respect to u and we deduce from the assumption $u \in W_{T_i}^1$ that $\gamma_{tR}(T_{i+1}) \geq \gamma_{tR}(T_i; u) + 3 = \gamma_{tR}(T_i) + 3$.

Now let $f(x_3) + f(x_2) + f(x_1) = 2$. Then we must have $f(x_2) = f(x_1) = 1$ and $f(x_3) = 0$. Then the function f, restricted to T_i is an TRDF of T_i of weight $\gamma_{tR}(T_{i+1}) - 2$ with $f(u) = 2$. Since $u \in W_{T_i}^2$, we obtain $\gamma_{tR}(T_{i+1}) - 2 = \omega(f|_{T_i}) \geq \gamma_{tR}(T_i) + 1$ and so $\gamma_{tR}(T_{i+1}) \geq \gamma_{tR}(T_i) + 3$. By the assumption $\gamma_{tR}(T_i) = \frac{3}{2}\beta_2(T_i)$, we obtain $\frac{3}{2}\beta_2(T_{i+1}) = \frac{3}{2}\beta_2(T_i) + 3 = \gamma_{tR}(T_i) + 3 = \gamma_{tR}(T_{i+1})$. □

Theorem 2.2. *If $T \in \mathcal{T}$, then $\gamma_{tR}(T) = \frac{3}{2}\beta_2(T)$.*

http://dx.doi.org/10.22108/toc.2023.134483.2005

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that T_1 is P_3, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by the above Operation for $i = 1, 2, \ldots, k-1$.

We proceed by induction on the number of operations applied to construct T. If $k = 1$, then $T = P_3 \in \mathcal{T}$. Suppose that the result is true for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{tR}(T') = \frac{3}{2}\beta_2(T')$. Since $T = T_k$ is obtained by Operation $\mathcal{O}$ from T', we conclude from Lemma 2.1 that $\gamma_{tR}(T) = \frac{3}{2}\beta_2(T)$. □

Next, we prove the main result of this paper.

Theorem 2.3. *For every tree T of order $n \geq 2$,*

$$\gamma_{tR}(T) \leq \frac{3}{2}\beta_2(T),$$

with equality if and only if $T \in \mathcal{T}$.

Proof. Let T be a tree of order $n \geq 2$. The proof is by induction on n. If $n = 2$, then $T = P_2$ and we have $\gamma_{tR}(T) = 2 < 3 = \frac{3}{2}\beta_2(T)$. Let $n \geq 3$ and let the statement hold for any tree of order less than n. If $\mathrm{diam}(T) = 2$, then T is the star $K_{1,n-1}$ and we have $\gamma_{tR}(T) = 3 \leq \frac{3(n-1)}{2} = \frac{3}{2}\beta_2(T)$ with equality if and only if $n = 3$, that is $T = P_3 \in \mathcal{T}$. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{p,q}$ $(q \geq p \geq 1)$ and we have $\gamma_{tR}(T) = 4 < \frac{3(p+q)}{2} = \frac{3}{2}\beta_2(T)$, if $p \geq 2$ and $\gamma_{tR}(T) = 4 < \frac{3\times(p+q+1)}{2} = \frac{3}{2}\beta_2(T)$ if $p = 1$. Assume that $\mathrm{diam}(T) \geq 4$ and let $v_1 v_2 \cdots v_{d+1}$ be a diametrical path in T such that $\deg(v_2)$ is as large as possible. Let v_{d+1} be a root of T. We consider the following cases.

Case 1. $\deg(v_2) \geq 4$.

Let $T' = T - T_{v_1}$. Then clearly, every $\beta_2(T')$-set not containing v_2, can be extended to a 2-independent set of T by adding v_1 which implies that $\beta_2(T) \geq \beta_2(T') + 1$. On the other hand, any $\gamma_{tR}(T')$-function f assigning 2 to v_2 and can be extended to a TRDF of T by assigning 0 to v_1 yielding $\gamma_{tR}(T) \leq \gamma_{tR}(T')$. It follows from the induction hypothesis that,

$$\begin{aligned}
\tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + \tfrac{3}{2} \\
&\geq \gamma_{tR}(T') + \tfrac{3}{2} \\
&\geq \gamma_{tR}(T) + \tfrac{3}{2} \\
&> \gamma_{tR}(T).
\end{aligned}$$

Case 2. $\deg(v_2) = 2$.

By the choice of diametrical path, all children of v_3 with depth 1, have degree 2. Assume that v_3 has t children with depth 1. Clearly, $t \geq 1$. We distinguish the following subcases.

http://dx.doi.org/10.22108/toc.2023.134483.2005

Subcase 2.1. $\deg(v_3) = 2$.

Let $T' = T - T_{v_3}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding v_1, v_2 yielding $\beta_2(T) \geq \beta_2(T') + 2$. On the other hand, any $\gamma_{tR}(T')$-function g can be extended to a TRDF of T by assigning 2 to v_2, 1 to v_3 and 0 to v_1, implying that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. It follows from the induction hypothesis that

$$\begin{array}{rcl} \frac{3}{2}\beta_2(T) & \geq & \frac{3}{2}\beta_2(T') + 3 \\ & \geq & \gamma_{tR}(T') + 3 \\ & \geq & \gamma_{tR}(T) - 3 + 3 \\ & = & \gamma_{tR}(T). \end{array}$$

Let the equality hold. Then all inequalities occurring in above inequality chain, become equality and so $\beta_2(T) = \beta_2(T') + 2$, $\gamma_{tR}(T) = \gamma_{tR}(T') + 3$ and $\gamma_{tR}(T') = \frac{3}{2}\beta_2(T')$. We conclude from the induction hypothesis that $T' \in \mathcal{T}$. We prove now that $v_4 \in W^1_{T'} \cap W^2_{T'}$. First we show that $v_4 \in W^1_{T'}$. Suppose, to the contrary, that $v_4 \notin W^1_{T'}$ and let g be a nearly TRDF of T' with respect to v_4 of weight at most $\gamma_{tR}(T') - 1$. Define $h : V(T) \to \{0, 1, 2\}$ by $h(u) = g(u)$ for $u \in V(T')$, $h(v_2) = 2$, $h(v_3) = 1$ and $h(v_1) = 0$. Clearly, h is a TRDF of T of weight $\gamma_{tR}(T') + 2$ which leads to a contradiction. Hence, $v_4 \in W^1_{T'}$. Next, we show that $v_4 \in W^2_{T'}$. Suppose, to the contrary, that $v_4 \notin W^2_{T'}$ and let g be a $\gamma_{tR}(T')$-function with $g(v_4) = 2$. Define $h : V(T) \to \{0, 1, 2\}$ by $h(u) = g(u)$ for $u \in V(T')$, $h(v_1) = h(v_2) = 1$ and $h(v_3) = 0$. Obviously h is a TRDF of T of weight $\gamma_{tR}(T') + 2$ which leads to a contradiction. Hence $v_4 \in W^2_{T'}$. Now, T can be obtained from T' by Operation $\mathcal{O}$ and so $T \in \mathcal{T}$.

Subcase 2.2. v_3 has $\ell \geq 1$ children with depth 0.

Let $T' = T - T_{v_3}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $D(v_3)$ and so $\beta_2(T) \geq \beta_2(T') + 2t + \ell$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning 2 to v_3 and all children of v_3 with depth 1 and 0 to all leaves of T_{v_3} and hence $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2 + 2t$. It follows from the induction hypothesis that,

$$\begin{array}{rcl} \frac{3}{2}\beta_2(T) & \geq & \frac{3}{2}\beta_2(T') + 3t + \frac{3\ell}{2} \\ & \geq & \gamma_{tR}(T') + 3t + \frac{3\ell}{2} \\ & \geq & \gamma_{tR}(T) - 2 - 2t + 3t + \frac{3\ell}{2} \\ & = & \gamma_{tR}(T) + \frac{3\ell}{2} - 2 + t \\ & > & \gamma_{tR}(T). \end{array}$$

Subcase 2.3. $\deg(v_3) = t + 1 \geq 3$ and v_3 is not a support vertex.

Let $T' = T - T_{v_3}$. Obviously, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding all vertices of T_{v_3} except v_3. It implies that $\beta_2(T) \geq \beta_2(T') + 2t$. On the other hand, any $\gamma_{tR}(T')$-function, can be extended to a TRDF of T by assigning 1 to all vertices of T_{v_3} and so $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 1 + 2t$.

It follows from the induction hypothesis that,

$$\begin{aligned}\tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + 3t\\ &\geq \gamma_{tR}(T') + 3t\\ &\geq \gamma_{tR}(T) - 1 - 2t + 3t\\ &= \gamma_{tR}(T) + t - 1.\\ &> \gamma_{tR}(T).\end{aligned}$$

Case 3. $\deg(v_2) = 3$ and $\deg(v_3) \geq 3$.

We distinguish the following subcases.

Subcase 3.1. v_3 has $\ell \geq 1$ children with depth 0.
Let $T' = T - T_{v_3}$. Assume that v_3 has $t \geq 1$ children with depth 1. Clearly, every $\beta_2(T')$-set can be extended to a 2-independent set of T by adding all descendents of v_3 with degree at most two. This implies that $\beta_2(T) \geq \beta_2(T') + 2t + \ell$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning 2 to v_3 and all of its children with depth 1 and assigning 0 to other vertices of T_{v_3} and so $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2t + 2$. It follows from the induction hypothesis that,

$$\begin{aligned}\tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + 3t + \tfrac{3\ell}{2}\\ &\geq \gamma_{tR}(T') + 3t + \tfrac{3\ell}{2}\\ &\geq \gamma_{tR}(T) - 2t - 2 + 3t + \tfrac{3\ell}{2}\\ &= \gamma_{tR}(T) + t - 2 + \tfrac{3\ell}{2}\\ &> \gamma_{tR}(T).\end{aligned}$$

Subcase 3.2. v_3 is not a support vertex.
Let $T' = T - T_{v_3}$. As in Subcase 3.1, we have $\beta_2(T) \geq \beta_2(T') + 2|C(v_3)|$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning 1 to v_3, 2 to all children of v_3 and 0 to other vertices of T_{v_3} and hence $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2|C(v_3)| + 1$. By considering $|C(v_3)| = t$ and by the induction hypothesis that,

$$\begin{aligned}\tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + 3t\\ &\geq \gamma_{tR}(T') + 3t\\ &\geq \gamma_{tR}(T) - 2t - 1 + 3t\\ &= \gamma_{tR}(T) + t - 1\\ &> \gamma_{tR}(T).\end{aligned}$$

Case 4. $\deg(v_2) = 3$ and $\deg(v_3) = 2$.

We consider following subcases.

Subcase 4.1. T has a pendant path $v_4y_3y_2y_1$ such that $y_3 \notin \{v_3, v_5\}$.
Considering above cases, we may assume that $\deg(y_2) = 3$ and $\deg(y_3) = 2$. Let $T' = T - (V(T_{y_3}) \cup V(T_{v_3}))$. Clearly, for any $\beta_2(T')$-set S, the set $(S - \{v_4\}) \cup (L_{y_2} \cup L_{v_2} \cup \{v_3, y_3\})$ is a 2-independent set of T and so $\beta_2(T) \geq \beta_2(T') + 5$. On the other hand, any $\gamma_{tR}(T')$-function g can be extended to a TRDF of T by assigning 2 to v_2, y_2, 1 to v_3, y_3 and 0 to the vertices in $L_{y_2} \cup L_{v_2}$, implying that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 6$. It follows from the induction hypothesis that

$$\begin{aligned}
\tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + \tfrac{15}{2}\\
&\geq \gamma_{tR}(T') + \tfrac{15}{2}\\
&\geq \gamma_{tR}(T) - 6 + \tfrac{15}{2}\\
&> \gamma_{tR}(T).
\end{aligned}$$

Subcase 4.2. v_4 has $\ell \geq 2$ children with depth 0 or v_4 has a child with depth 1 and degree 2.
Let $T' = T - T_{v_3}$. Clearly, T' has a $\beta_2(T')$-set S not containing v_4. Then $S \cup \{v_3\} \cup L_{v_2}$ is a 2-independent set of T and so $\beta_2(T) \geq \beta_2(T') + 3$. On the other hand, any $\gamma_{tR}(T)$-function can be extended to a TRDF of T by assigning 2 to v_2, 1 to v_3 and 0 to the vertices in L_{v_2}, and hence $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. It follows from the induction hypothesis that

$$\begin{aligned}
\tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + \tfrac{9}{2}\\
&\geq \gamma_{tR}(T') + \tfrac{9}{2}\\
&\geq \gamma_{tR}(T) - 3 + \tfrac{9}{2}\\
&> \gamma_{tR}(T).
\end{aligned}$$

Subcase 4.3. v_4 has two children w_1, z_1 with depth 1 and degrees at least 3.
Let $T' = T - (V(T_{w_1}) \cup V(T_{z_1}))$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $L_{w_1} \cup L_{z_1}$ yielding $\beta_2(T) \geq \beta_2(T') + 4$. On the other hand, any $\gamma_{tR}(T')$-function g can be extended to a TRDF of T by assigning 2 to w_1, z_1, $\max\{1, g(v_4)\}$ to v_4 and 0 to the vertices in $L_{w_1} \cup L_{z_1}$, and so $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 5$. It follows from the induction hypothesis that

$$\begin{aligned}
\tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + 6\\
&\geq \gamma_{tR}(T') + 6\\
&\geq \gamma_{tR}(T) - 5 + 6\\
&> \gamma_{tR}(T).
\end{aligned}$$

Subcase 4.4. v_4 is a support vertex and it has a child w_1 with depth 1 and degree at least 3.
Let $T' = T - T_{w_1}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding L_{w_1} yielding $\beta_2(T) \geq \beta_2(T') + 2$. On the other hand, for any $\gamma_{tR}(T')$-function g we have $g(v_4) \geq 1$ and it can be extended to a TRDF of T by assigning 2 to w_1 and 0 to the vertices in L_{w_1}, yielding

$\gamma_{tR}(T) \le \gamma_{tR}(T') + 2$. It follows from the induction hypothesis that

$$\begin{aligned} \tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + 3 \\ &\geq \gamma_{tR}(T') + 3 \\ &\geq \gamma_{tR}(T) - 2 + 3 \\ &> \gamma_{tR}(T). \end{aligned}$$

Subcase 4.5. $\deg(v_4) = 3$ and v_4 has a child w_1 with depth 1 and degree at least 3.
Let $T' = T - T_{v_4}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $L_{w_1} \cup L_{v_2} \cup \{v_3\}$ yielding $\beta_2(T) \ge \beta_2(T') + 5$. On the other hand, any $\gamma_{tR}(T')$-function g can be extended to a TRDF of T by assigning 2 to w_1, v_2, 1 to v_4, v_3 and 0 to the vertices in $L_{w_1} \cup L_{v_2}$, yielding $\gamma_{tR}(T) \le \gamma_{tR}(T') + 6$. By the induction hypothesis, we have

$$\begin{aligned} \tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + \tfrac{15}{2} \\ &\geq \gamma_{tR}(T') + \tfrac{15}{2} \\ &\geq \gamma_{tR}(T) - 6 + \tfrac{15}{2} \\ &> \gamma_{tR}(T). \end{aligned}$$

Subcase 4.6. $\deg(v_4) = 3$ and v_4 is a support vertex.
If $\operatorname{diam}(T) = 4$, then clearly $\frac{3}{2}\beta_2(T) > \gamma_{tR}(T)$. Assume that $\operatorname{diam}(T) \ge 5$. Let w be the leaf adjacent to v_4 and let $T' = T - T_{v_4}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $L_{v_2} \cup \{v_3, w\}$ yielding $\beta_2(T) \ge \beta_2(T') + 4$. Also, any $\gamma_{tR}(T')$-function g can be extended to a TRDF of T by assigning 2 to v_4, v_2, 1 to v_3 and 0 to the vertices in $\{w\} \cup L_{v_2}$, yielding $\gamma_{tR}(T) \le \gamma_{tR}(T') + 5$. By the induction hypothesis, we have

$$\begin{aligned} \tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + 6 \\ &\geq \gamma_{tR}(T') + 6 \\ &\geq \gamma_{tR}(T) - 5 + 6 \\ &> \gamma_{tR}(T). \end{aligned}$$

Subcase 4.7. $\deg(v_4) = 2$.
If $\operatorname{diam}(T) = 4$, then obviously $\frac{3}{2}\beta_2(T) > \gamma_{tR}(T)$. Suppose $\operatorname{diam}(T) \ge 5$. Let $T' = T - T_{v_4}$. Clearly, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning 2 to v_2, v_3 and 0 to the vertices in $\{v_4\} \cup L_{v_2}$, and this implies that $\gamma_{tR}(T) \le \gamma_{tR}(T') + 4$. On the other hand, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $\{v_3\} \cup L_{v_2}$, implying that $\beta_2(T) \ge \beta_2(T') + 3$. By the induction hypothesis we have

$$\begin{aligned} \tfrac{3}{2}\beta_2(T) &\geq \tfrac{3}{2}\beta_2(T') + \tfrac{9}{2} \\ &\geq \gamma_{tR}(T') + \tfrac{9}{2} \\ &\geq \gamma_{tR}(T) - 4 + \tfrac{9}{2} \\ &> \gamma_{tR}(T). \end{aligned}$$

This completes the proof. □

http://dx.doi.org/10.22108/toc.2023.134483.2005

Acknowledgments

H. Abdollahzadeh Ahangar was supported by the Babol Noshirvani University of Technology under research Grant Number BNUT/385001/1401.

References

[1] H. Abdollahzadeh Ahangar,Trees with total Roman domination number equal to Roman domination number plus its domination number: complexity and structural properties, *AKCE Int. J. Graphs and Combin.*, **19** (2022) 74–78.

[2] H. Abdollahzadeh Ahangar, M. A. Henning, V. Samodivkin and I. G. Yero, Total Roman domination in graphs, *Appl. Anal. Discrete Math.*, **10** (2016) 501–517.

[3] H. Abdollahzadeh Ahangar, J. Amjadi, S. M. Sheikholeslami and M. Soroudi, On the total Roman domination number of graphs, *Ars Combin.*, **150** (2020) 225–240.

[4] J. Amjadi, N. Dehgardi, S. M. Sheikholeslami and M. Valinavaz, Independent Roman domination and 2-independence in trees, *Discrete Math. Algorithms Appl.*, **10** (2018) 10 pp.

[5] J. Amjadi, S. Nazari-Moghaddam, S. M. Sheikholeslami and L. Volkmann, Total Roman domination number of trees, *Australas. J. Combin.*, **69** (2017) 271–285.

[6] J. Amjadi, S. M. Sheikholeslami and M. Soroudi, Nordhaus–Gaddum bounds for total Roman domination, *J. Comb. Optim.*, **35** (2018) 126–133.

[7] J. Amjadi, S.M. Sheikholeslami and M. Soroudi, On the total Roman domination in trees, *Discuss. Math. Graph Theory*, **39** (2019) 519–532.

[8] J. Amjadi and M. Soroudi, Twin signed total Roman domination numbers in digraphs, *Asian-European J. Math.*, **11** (2018) 22 pp.

[9] F. Bonomo, B. Brešar, L. Grippo, M. Milanič and M. Safe, Domination parameters with number 2: interrelations and algorithmic consequences, *Discrete Appl. Math.*, **235** (2018) 23–50.

[10] E. W. Chambers, B. Kinnersley, N. Prince and D. B. West, Extremal problems for Roman domination, *SIAM J. Discrete Math.*, **23** (2009) 1575–1586.

[11] M. Chellali and N. Meddah, Trees with equal 2-domination and 2-independence numbers, *Discuss. Math. Graph Theory.*, **32** (2012) 263–270.

[12] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, k-domination and k-independence in graphs: a survey, *Graphs Combin.*, **28** (2012) 1–55.

[13] E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S.T. Hedetniemi, Roman domination in graphs, *Discrete Math.*, **278** (2004) 11–22.

[14] N. Dehgardi, Mixed Roman domination and 2-independence in trees, *Commun. Comb. Optim.*, **3** (2018) 79–91.

[15] O. Favaron, On a conjecture of Fink and Jacobson concerning k-domination and k-dependence, *J. Combin. Theory Ser.*, **39** (1985) 101–102.

[16] O. Favaron, H. Karami, R. Khoeilar and S. M. Sheikholeslami, On the Roman domination number of a graph, *Discrete Math.*, **309** (2009) 3447–3451.

[17] J. F. Fink and M. S. Jacobson, *On n-domination, n-dependence and forbidden subgraphs*, Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984), Wiley-Intersci. Publ., Wiley, New York, 1985 301–311.

[18] T. W. Haynes, S.T. Hedetniemi and P. J. Slater (Eds.), *Fundamentals of domination in graphs*, Monographs and Textbooks in Pure and Applied Mathematics, **208**, Marcel Dekker, Inc., New York, 1998.

[19] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (Eds.), *Domination in graphs: advanced topics*, Marcel Dekker, Inc. New York, 1998.

[20] M. A. Henning, Recent results on total domination in graphs: A survey, *Discrete Math.*, **309** (2009) 32–63.

[21] M. A. Henning and A. Yeo, *Total domination in graphs*, (Springer Monographs in Mathematics) (2013).

[22] M. S. Jacobson, K. Peters and D. F. Rall, On n-irredundance and n-domination, *Ars. Combin.*, **29** (1990) 151–160.

[23] C.-H. Liu and G. J. Chang,Roman domination on strongly chordal graphs, *J. Comb. Optim.*, **26** (2013) 608–619.

[24] N. Meddah and M. Chellali, Roman domination and 2-independence in trees, *Discrete Math. Algorithms Appl.*, **9** (2017) 6 pp.

[25] C. S. ReVelle and K. E. Rosing, Defendens imperium romanum: A classical problem in military strategy, *Amer. Math. Monthly*, **107** (2000) 585–594.

[26] I. Stewart, Defend the Roman empire!, *Sci. Amer.*, **281** (1999) 136–139.

[27] I. G. Yero, On Clark and Suen bound-type results for k-domination and Roman domination of Cartesian product graphs, *Int. J. Comput. Math.*, **90** (2013) 522–526.

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http://dx.doi.org/10.22108/toc.2023.134483.2005