Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 13 No. 2 (2024), pp. 153-164.

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THE HIGMAN-SIMS SPORADIC SIMPLE GROUP AS THE AUTOMORPHISM GROUP OF RESOLVABLE 3-DESIGNS

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Abstract. Presenting sporadic simple groups as an automorphism groups of designs and graphs is an exciting field in finite group theory. In this paper, with two different methods, we present some new resolvable simple 3-designs with Higman-Sims sporadic simple group HS as the full automorphism group. Also, we classify all block-transitive self-orthogonal designs on 176 points with even block size that admit sporadic simple group HS as an automorphism group. Furthermore, with these methods we construct some new resolvable 3-designs on 36, 40, 120 and 176 points.

1. Introduction

A t-design is a combinatorial structure consisting of a collection of blocks over a set of points. The existence of t-designs with given parameters is one of the main problems in design theory that has been studied by many authors. An exciting feature of this study is the interplay of combinatorial designs and finite groups. Using multiply transitive group action is an essential technique to construct t-designs for $t \geq 3$. It is well known to construct t-design from a homogeneous permutation group.

One of the methods for constructing t-designs is the Kramer and Mesner method that introduces the computational approach to construct admissible combinatorial designs using prescribed automorphism groups [10]. Many t-designs are constructed by this method [3, 4, 11, 12]. Another method for constructing t-designs is assembling orbits obtained from the action of a specific permutation group on the

Keywords: t-design, resolvable design, sporadic simple group.

MSC(2010): Primary: 20D08; Secondary: 20B25.

Communicated by Behruz Tayfeh Rezaie.

Article Type: Research Paper.

Received: 19 January 2022, Accepted: 11 April 2023.

Cite this article: A. R. Rahimipour, The Higman-Sims sporadic simple group as the automorphism group of resolvable 3-designs, Trans. Comb., **13** no. 2 (2024) 153–164. http://dx.doi.org/10.22108/toc.2023.132404.1956 .

set of all k-subsets of points. For another computational method which has been applied to construct some t-designs, we refer the reader to [13].

In this paper, we present the Higman-Sims sporadic simple group HS as the automorphism group of some resolvable 3-designs. To the best of our knowledge, these 3-designs are new and we show the existence of them. Also, we classify all block-transitive self-orthogonal designs on 176 points with even block size, that admit sporadic simple group HS as the automorphism group.

We use two different methods for constructing 3-designs from the sporadic simple group HS. The first method is computationally and in the second, we use the extension method of t-designs which Alltop presented in [2].

The paper is organized as follows: Introductory notions and preliminaries are described in Section 2. Consideration of sporadic simple group HS in its primitive action on 176 points and foundation of some non-isomorphic simple block-transitive resolvable 3-designs are done in Section 3. In this section, we consider the codes constructed from these 3-designs. Also, the classification of block-transitive self-orthogonal designs with even block size on 176 points that admit sporadic simple group HS as an automorphism group, is done in this section. In Section 4, we show the existence of some resolvable 3-designs on 36, 40, 120 and 176 points. In the last section, we give the generators of groups and the base blocks of designs which are presented in this paper.

2. **Notation and preliminaries**

Throughout this paper, our notation for designs and codes will be standard, and follows [6]. All computations have been done with the aid of GAP [15] and Magma [5].

Let t, k, v and λ be integers such that $0 \leq t \leq k \leq v$ and $\lambda > 0$. Let V be a set of size v. A t-(v, k, λ) *design* $\mathcal{D}$ is a pair (V, B) consisting of the set V of *points* and a collection B of k-subsets of V, such that every t-subset of points is contained in exactly λ subsets of B. The sets in B are known as *blocks*.

If all blocks of $\mathcal{D}$ are distinct, then $\mathcal{D}$ is called *simple*. Throughout this paper, we are concerned only with simple designs.

A t-design is called *self-orthogonal* if the block intersection numbers, which are the cardinalities of the intersections of any two distinct blocks, have the same parity as the block size k [14].

The *complement* of a design (V, B) is $(V, \overline{B})$, where $\overline{B} = \{V \setminus b : b \in B\}$. Since the complement of a $t-(v, k, \lambda)$ design is a t-design, then throughout the paper we consider designs with $2k \leq v$.

An *isomorphism* from one design to another is a bijective mapping of points to points which preserves the blocks. An isomorphism from a design $\mathcal{D}$ onto itself is called an *automorphism* of $\mathcal{D}$. The set of all automorphisms forms a group called the *full automorphism group* of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$. A subgroup G of $Aut(\mathcal{D})$ is *block-transitive* if G acts transitively on the blocks of $\mathcal{D}$. In this case, $\mathcal{D}$ is called block-transitive. *Point-transitivity* is defined similarly.

A *resolution* in a design is a set of blocks that partitions the point set. A design is called *resolvable* when the blocks can be partitioned into resolutions.

http://dx.doi.org/10.22108/toc.2023.132404.1956

It is convenient to represent a t-(v,k,λ) design by a binary matrix $M = [m_{i,j}]$ of size $|B| \times v$ that is called the *incidence matrix*. The rows of M are indexed by the blocks of design and the columns by the points, where $m_{i,j} = 1$ if and only if j-th point is in i-th block. The *code* of a design $\mathcal{D}$=(V,B) over $GF(q)$ is a linear code over $GF(q)$ spanned by the rows of the incidence matrix M, denoted by $\mathcal{C}_q$.

We use $[n,k,d]_q$ to denote the linear code over $GF(q)$ of length n, dimension k and minimum distance d. The set of non-zero coordinate positions of a codeword c is called *support* of c. For any code $\mathcal{C}$, the dual code $\mathcal{C}^\perp$ is the orthogonal complement under the standard inner product. A code $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C} \subset \mathcal{C}^\perp$ and *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$. A code is *even* if all weights of codewords are even, and *doubly-even* if all weights are divisible by 4. An *automorphism* of a code is a permutation of the coordinate positions that maps codewords to codewords. The set of all automorphisms of the code $\mathcal{C}$ is denoted by $Aut(\mathcal{C})$. The *support design* of a code $\mathcal{C}$ of length n for a given non-zero codeword c, is the design with points the n coordinate indices and blocks the supports of all codewords in the orbit of c under the action of $Aut(\mathcal{C})$.

The following lemma describes the relation between self-orthogonal designs and self-orthogonal codes.

Lemma 2.1. [14, Lemma 2.1] *Let $\mathcal{D}$ be a self-orthogonal $2-(v,k,\lambda)$ design with even k. If v is even, then the code spanned by the rows of incident matrix of $\mathcal{D}$ is self-orthogonal.*

Alltop in [2] presented some extension methods for t-designs. The following lemma from [2] is crucial in constructing resolvable 3-designs in Section 4. Note that the $(t+1)$-design is constructed in the following lemma is resolvable.

Lemma 2.2. [2, Theorem B] *Let $\mathcal{D} = (V,B)$ be a $t-(2k+1,k,\lambda)$ design and x a new point not in V. If t is even, then $\mathcal{D}' = (V \cup \{x\}, \{b \cup \{x\} : b \in B\} \cup \overline{B})$ is a $(t+1)-(2k+2,k+1,\lambda)$ design.*

Chigira et al. [7] defined the binary code $C(G,|\Omega|)$ for a permutation group G as follows: Let G be a permutation group on a set Ω. The binary code $C(G,|\Omega|)$ is defined by $C(G,|\Omega|) = \langle Fix(g) | g \in I(G)\rangle^\perp$, where $I(G)$ is the set of involutions of G and $Fix(g)$ denoted the set of fixed points by g. Here $C(G,|\Omega|)$ is contained in the power set $\mathcal{P}(\Omega)$ of Ω, where $\mathcal{P}(\Omega)$ is regarded as an n-dimensional vector space over $GF(2)$ by defining the sum as the symmetric difference.

We use the following lemma from [7] for classifying self-orthogonal designs in Section 3.

Lemma 2.3. [7, Lemma 2.4] *Let $\mathcal{D}$ be a self-orthogonal $t-(n,k,\lambda)$ design with even k. Suppose that $\mathcal{D}$ is invariant under a permutation group G on the point set Ω of size n. Then the code generated by the rows of incidence matrix of $\mathcal{D}$ is contained in $C(G,|\Omega|)$.*

We refer the reader to [8] for further information and results about designs. Also, for the group theoretical notation of the finite simple groups, we refer the reader to [9].

3. Block-transitive resolvable 3-designs from the group HS

The sporadic simple group HS has order 44352000. The group HS has twelve conjugacy classes of maximal subgroups. Consider the permutation representation of HS on the 176 right cosets of the

http://dx.doi.org/10.22108/toc.2023.132404.1956

maximal subgroup $U_3(5)$:2. The group HS is 2-transitive and primitive in this representation [9]. In this section, we consider the permutation representation of HS on 176 points with generators from [1].

Assume that $\mathcal{D} = (V, B)$ is a $t-(v, k, \lambda)$ design with automorphism group G such that G acts transitively on the blocks. Now, assume that H is the stabilizer of a block. Then, it is well-known that this block is a union of some orbits of the group H. By using this fact, in this section we construct some block-transitive simple 3-designs with full automorphisms group HS as follows: The group HS up to conjugation has 589 subgroups [15]. Let H be a representative of a conjugacy class of subgroups of HS. Consider a union of orbits of H, say X. We construct the orbit of X under the action of HS. Since HS acts 2-transitively, the orbit of X under the action of HS forms a 2-design. We use this method for each union of orbits of H and we will be interested when the constructed designs are t-design for $t \geq 3$. According to the limitation of computational power, we applied this method for each conjugacy class of subgroups of order greater than 32 of HS. Thereby, some non-isomorphic classes of resolvable simple 3-designs with parameters $3-(176, 88, 18920)$ and $3-(176, 88, 85140)$ and one class of $3-(176, 60, 42480)$ design were constructed. The full automorphism groups of these designs are the sporadic simple group HS.

In the following proposition, we study the properties of the member from each isomorphic class of 3-designs. For this purpose, we describe each design with a base block that is the union of some orbits of the stabilizer of a block of the design. For instance, the group HS has four conjugacy classes of subgroups of order 288. But, only the union of some orbits of one of them under the action of HS construct a 3-design.

Proposition 3.1. *Up to isomorphism and complementation, there are two classes of* $3-(176, 88, 18920)$ *designs, thirteen classes of* $3-(176, 88, 85140)$ *designs and one classes of* $3-(176, 60, 42480)$ *design with full automorphism group* HS. *The group* HS *acts* 2*-transitively on the points and transitively on the blocks. These are the only block-transitive* 3*-designs on* 176 *points with block stabilizer of order greater than 32 that admit the group* HS *as an automorphism group. Apart from two, all of these designs are resolvable and two of them (the designs* $\mathcal{D}_2$ *and* $\mathcal{D}_3$ *) are also self-orthogonal.*

Proof. The group HS has one conjugacy class of subgroups of order 288 isomorphic to $(A_4 \times A_4)$:2. The length of this conjugacy class is 38500. Let H be the representative of this conjugacy class. The group H has orbit lengths $4^2, 36^2, 48^2$.

1) Consider the base block X_1 from Table 5. The orbit of X_1 under the action of HS forms a $3-(176, 88, 18920)$ design, denoted by $\mathcal{D}_1$. This design has 154000 blocks. The stabilizer of a block is a subgroup isomorphic to H. The full automorphism group of $\mathcal{D}_1$ is HS. According to the block intersection numbers of this design, each block has an empty intersection with one other block. Since the block size is 88, then the complement of each block is also a block and the union of each block and its complement is a partition of the point set of design. Therefore, $\mathcal{D}_1$ is resolvable.

2) Consider the base block X_2 from Table 5. The orbit of X_2 under the action of HS forms a $3-(176, 88, 18920)$ design, denoted by $\mathcal{D}_2$. This design has 154000 blocks. The stabilizer of a block is a subgroup isomorphic to H. The full automorphism group of $\mathcal{D}_2$ is HS. This design has block

intersection numbers $0^1, 28^{128}, 32^{333}, 36^{1360}, 40^{42634}, 44^{65088}, 48^{42634}, 52^{1360}, 56^{333}, 60^{128}$. According to the block intersection numbers, $\mathcal{D}_2$ is resolvable. Since block size and all intersection numbers are even, then $\mathcal{D}_2$ is also self-orthogonal.

The group HS has one conjugacy class of subgroups of order 64 isomorphic to 4^2:2:2. The length of this conjugacy class is 173250. Let K be the representative of this conjugacy class. The group K has orbit lengths $8^6, 32^4$.

3) Consider the base block X_3 from Table 5. The orbit of X_3 under the action of HS forms a $3-(176, 88, 85140)$ design, denoted by $\mathcal{D}_3$. This design has 693000 blocks. The stabilizer of a block is a subgroup isomorphic to K. The full automorphism group of $\mathcal{D}_3$ is HS. This design has block intersection numbers $0^1, 28^{384}, 32^{1385}, 36^{5288}, 40^{199698}, 44^{279488}, 48^{199698}, 52^{5288}, 56^{1385}, 60^{384}$. According to the block intersection numbers, $\mathcal{D}_3$ is resolvable and self-orthogonal.

4) Consider the base block X_i for $i = 4, \ldots, 10$ from Table 5. The orbit of X_i under the action of HS forms a $3-(176, 88, 85140)$ design, denoted by $\mathcal{D}_i$. These designs have 693000 blocks. The stabilizer of a block is a subgroup isomorphic to K. The full automorphism group of these designs is HS. According to the block intersection numbers, these designs are resolvable.

The group HS has one conjugacy class of subgroups of order 64 isomorphic to $(4^2{\times}2)$:2. The length of this conjugacy class is 86625. Let F be the representative of this conjugacy class. The group F has orbit lengths $8^6, 32^4$.

5) Consider the base block X_i for $i = 11, \ldots, 15$ from Table 6. The orbit of X_i under the action of HS forms a $3-(176, 88, 85140)$ design, denoted by $\mathcal{D}_i$. These designs have 693000 blocks. The stabilizer of a block is a subgroup isomorphic to F. The full automorphism group of these designs is HS. According to the block intersection numbers, the designs $\mathcal{D}_i$ for $i = 11, \ldots, 14$ are resolvable.

The group HS has one conjugacy class of subgroups of order 40 isomorphic to $2^2{\times}\mathrm{D}_{10}$. The length of this conjugacy class is 184800. Let L be the representative of this conjugacy class. The group L has orbit lengths $2^3, 10^3, 20^7$.

6) Consider the base block X_{16} from Table 6. The orbit of X_{16} under the action of HS forms a $3-(176, 60, 42480)$ design, denoted by $\mathcal{D}_{16}$. This design has 1108800 blocks. The stabilizer of a block is a subgroup isomorphic to L. The full automorphism group of this design is HS. □

In the following propositions, we consider the binary linear codes corresponding to the designs $\mathcal{D}_i$ for $i = 1, 2$ presented in the proof of Proposition 3.1.

Proposition 3.2. *Let $\mathcal{D}_i$ for $i = 1, 2$ be the design in the proof of Proposition 3.1. Then, the binary linear codes $\mathcal{C}_i$ associated with $\mathcal{D}_i$ for $i = 1, 2$ have the following properties,*

1) *$\mathcal{C}_1$ is an even $[176, 154, 6]_2$ code,*
2) *$\mathcal{C}_1^{\perp}$ is a self-orthogonal and even $[176, 22, 50]_2$ code and the weight distribution of $\mathcal{C}_1^{\perp}$ is given in Table 1,*
3) *$\mathcal{C}_2$ is a self-orthogonal and doubly even $[176, 21, 56]_2$ code,*
4) *$\mathcal{C}_2 \subset \mathcal{C}_1^{\perp} \subset \mathcal{C}_1$,*

5) $Aut(\mathcal{C}_2) \cong Aut(\mathcal{C}_1^{\perp}) \cong Aut(\mathcal{C}_1) \cong \mathrm{HS}$.

Proof. The results are obtained by computations with the aid of Magma [5]. □

TABLE 1. The weight distribution of $\mathcal{C}_1^{\perp}$

weight of word	number of words
0, 176	1
50, 126	176
56, 120	1100
64, 112	4125
66, 110	5600
70, 106	17600
72, 104	15400
78, 98	193600
80, 96	604450
82, 94	462000
86, 90	369600
88	847000

Proposition 3.3. *Let G be the automorphism group of $\mathcal{C}_1^{\perp}$. Then,*

1) *all words of length l for $l \in \{50, 56, 64, 66, 70, 72, 82, 86, 90, 94, 104, 106, 110, 112, 120, 126\}$ form one orbit under the action of G,*
2) *the action of G on words of length l for $l \in \{78, 88, 98\}$ produces two orbits,*
3) *the action of G on words of length l for $l \in \{80, 96\}$ produces three orbits,*

up to complementation, properties of the support designs of words, in each orbit under the action of G, are given in Table 2. All of these designs are self-orthogonal.

Proof. The results are obtained by computations with the aid of Magma [5]. □

Proposition 3.4. *Up to complementation, the presented self-orthogonal designs in Table 2 are the only block-transitive self-orthogonal designs on 176 points with even block size, that admit automorphism group* HS*.*

Proof. From [7, Table 1], we have $\mathcal{C}_1^{\perp} = C(\mathrm{HS}, 176)$. Therefore, the result follows from Lemma 2.3 and Proposition 3.3. □

Corollary 3.5. *The designs $\mathcal{D}_i$ for $i = 1, \ldots, 14$ are block-transitive resolvable designs invariant under the group* HS*. Therefore, the union of each set of these designs is also resolvable simple* 3*-design. Thereby, there exist resolvable simple* $3-(176, 88, 37840)$ *design and resolvable simple* $3-(176, 88, 18920i + 85140j)$ *designs for* $i = 0, 1, 2$ *and* $j = 1, \ldots, 12$ *with full automorphism group* HS*.*

TABLE 2. The properties of support designs of $\mathcal{C}_1^\perp$

weight	orbit length	stabilizer of block	design	action on blocks
50	176	$U_3(5)$:2	2-(176,50,14)	primitive
56	1100	$L_3(4)$:2	2-(176,56,110)	primitive
64	4125	4^3:$L_2(7)$	2-(176,64,540)	primitive
66	5600	M_{11}	2-(176,66,780)	primitive
70	17600	A_7	2-(176,70,2760)	transitive
72	15400	$2\times A_6\cdot 2^2$	2-(176,72,2556)	primitive
78	61600	A_6:2	2-(176,78,12012)	transitive
78	132000	$L_2(7)$:2	2-(176,78,25740)	transitive
80	3850	2^4:S_6	2-(176,80,790)	primitive
80	231000	$2\times D_8$:2:3:2	2-(176,80,47400)	transitive
80	369600	S_5	2-(176,80,75840)	transitive
82	462000	$SL_2(3)$:2:2	2-(176,82,99630)	transitive
86	369600	S_5	2-(176,86,87720)	transitive
88	154000	$(A_4\times A_4)$:2	3-(176,88,18920)($\mathcal{D}_2$)	transitive
88	693000	$4^2 : 2 : 2$	3-(176,88,85140)($\mathcal{D}_3$)	transitive

4. **The existence of some resolvable 3-designs**

By using Lemma 2.2, in the following proposition we give a method of extending t-design to $(t+1)$-design.

Proposition 4.1. *Let G be a transitive permutation group on a set Ω of size $2k$. Suppose that H is the stabilizer subgroup of point $\alpha \in \Omega$. Let t be even. If the following conditions are satisfied:*

1) *the subgroup H is self-normalizing,*
2) *there exists a $t-(2k-1, k-1, \lambda)$ design $\mathcal{D} = (\Omega \backslash \{\alpha\}, B)$ with automorphism group H,*
3) *the stabilizer of a block of $\mathcal{D}$ does not have any orbit of length 1.*

Then, there is resolvable simple $(t+1)-(2k, k, \lambda i)$ designs $\mathcal{D}_i$ for $i = 1, \ldots, 2k$. Also, the group G is a group of automorphism of the design $\mathcal{D}_{2k}$.

Proof. Since, the subgroup H is self-normalizing with index $2k$ and G is transitive on Ω, then the stabilizer of each point of Ω is a conjugate of H. Let S_β be the stabilizer of point $\beta \in \Omega$. By (2), for each $\beta \in \Omega$ there is $\mathcal{R}_\beta = (\Omega \backslash \{\beta\}, B_\beta)$, which is a $t-(2k-1, k-1, \lambda)$ design with automorphism group S_β. By Lemma 2.2 each $\mathcal{R}_\beta$, by adding the point β to the set $\Omega \backslash \{\beta\}$, can be extended to a resolvable simple $(t+1)-(2k, k, \lambda)$ design, denoted by $\mathcal{E}_\beta = (\Omega, B'_\beta)$. By (3), the stabilizer of a block of design $\mathcal{R}_\beta$ does not have any orbit of length 1. So, by considering the method of construction in Lemma 2.2, the stabilizer of a block of design $\mathcal{E}_\beta$ has only the fixed point β on Ω. Then, the stabilizer of a block of design $\mathcal{E}_\beta$ is only the subgroup of S_β. Therefore, the stabilizer of a block of $\mathcal{E}_\beta$ and of $\mathcal{E}_\alpha$ are different for each pair $\alpha, \beta \in \Omega$, so the blocks of $\mathcal{E}_\beta$ and of $\mathcal{E}_\alpha$ are distinct. Now consider the design $\mathcal{D}_{|I|} = (\Omega, \cup_{\beta \in I} B'_\beta)$, the union of some $\mathcal{E}_\beta$ for $I \subseteq \Omega$. The design $\mathcal{D}_{|I|}$ is the union of some resolvable simple $(t+1)-(2k, k, \lambda)$ designs with distinct blocks then is resolvable simple $(t+1)-(2k, k, \lambda|I|)$ design. Since the group G is transitive on the set of conjugates of H, therefore G is a group of automorphism of design $\mathcal{D}_{|\Omega|}$. □

We apply Proposition 4.1 to the sporadic simple group HS. Consider the 2-transitive action of the group HS on 176 right cosets of maximal subgroup $U_3(5){:}2$. Let H be the stabilizer of a point, in this action. Then, $H \cong U_3(5){:}2$ and has orbit lengths $1^1, 175^1$. In the following proposition, we present some 2−designs such that can be extended to some resolvable 3-designs.

Proposition 4.2. *Table 3 lists parameters of some 2-designs with automorphism group* $U_3(5){:}2$.

Proof. Consider the permutation representation of $U_3(5){:}2$ on 175 points with generators a and b from Table 4 in Section 5. Taking the orbits of the base block Y_i for $i = 1, \ldots, 8$ from Table 7 under the action of $U_3(5){:}2$, forms the design $\mathcal{R}_i$ in Table 3. □

TABLE 3. Some 2-designs from $U_3(5){:}2$

design	parameters	st. of block	orbit lengths of st.
$\mathcal{R}_1$	$2-(175, 87, 2580)$	$2^2 \times S_3$	$2^2, 3^1, 6^4, 12^6, 24^3$
$\mathcal{R}_2$	$2-(175, 87, 5160)$	A_4	$3^1, 4^4, 6^6, 12^{10}$
$\mathcal{R}_3$	$2-(175, 87, 5160)$	12	$3^5, 4^1, 12^{13}$
$\mathcal{R}_4$	$2-(175, 87, 5160)$	D_{12}	$2^2, 3^3, 6^7, 12^{10}$
$\mathcal{R}_5$	$2-(175, 87, 5160)$	6×2	$2^2, 3^1, 6^4, 12^{12}$
$\mathcal{R}_6$	$2-(175, 87, 5160)$	D_{12}	$2^2, 3^3, 6^{11}, 12^8$
$\mathcal{R}_7$	$2-(175, 87, 5160)$	D_{12}	$3^3, 4^1, 6^9, 12^9$
$\mathcal{R}_8$	$2-(175, 87, 5160)$	D_{12}	$2^2, 3^3, 6^9, 12^9$

Proposition 4.3.

1) *There are resolvable simple* $3-(176, 88, \lambda i)$ *designs for* $\lambda = 2580, 5160$ *and* $i = 1, ..., 175$.
2) *There are resolvable simple* $3-(176, 88, 176\lambda)$ *designs for* $\lambda = 2580, 5160$ *with* HS *as an automorphism group.*

Proof. The results are achieved by Proposition 4.1 and Proposition 4.2. □

In the following, we examine Proposition 4.1 for some simple groups in their 2-transitive action.

Example 4.4. *Consider the* 2*-transitive action of the simple group* $S_6(2)$ *on 36 points* [9]. *A point stabilizer in this action is a maximal subgroup isomorphic to* S_8.

1) There is a simple $2-(35, 17, 768)$ *design with automorphism group* S_8. *The stabilizer of a block of this design is isomorphic to* D_{12} *with orbit lengths* $2^1, 3^1, 6^5$. *Therefore by Proposition 4.1, there exist resolvable simple* $3-(36, 18, 768i)$ *designs for* $i = 1, \ldots, 36$. *The group* $S_6(2)$ *is a group of automorphism of* $3-(36, 18, 27648)$ *design.*

2) There is a simple $2-(35, 17, 1536)$ *design with automorphism group* S_8. *The stabilizer of a block of this design is isomorphic to* S_3 *with orbit lengths* $2^1, 3^3, 6^4$. *Therefore by Proposition 4.1*], there exist resolvable simple $3-(36, 18, 1536i)$ designs for $i = 1, \ldots, 36$. The group $S_6(2)$ is a group of automorphism of $3-(36, 18, 55296)$ design.

http://dx.doi.org/10.22108/toc.2023.132404.1956

Example 4.5. *The simple group* $\mathrm{S}_8(2)$ *acts* 2*-transitively on the 120 right cosets of maximal subgroup* $\mathrm{O}_8^-(2){:}2$ [9].

1) There is a self-orthogonal and anti-flag transitive $2-(119,59,11136)$ *design with automorphism group* $\mathrm{O}_8^-(2){:}2$*. The stabilizer of a block of this design is isomorphic to* $\mathrm{S}_5{\times}\mathrm{S}_3{\times}\mathrm{S}_3{:}2$ *with orbit lengths* $5^1, 9^1, 45^1, 60^1$*. Therefore by Proposition 4.1, there exist resolvable simple* $3-(120,60,11136i)$ *designs for* $i = 1,\ldots,120$*. The group* $\mathrm{S}_8(2)$ *is a group of automorphism of* $3-(120,60,1336320)$ *design. The design* $3-(120,60,11136)$ *is also self-orthogonal.*

2) There is a simple $2-(119,59,133632)$ *design with automorphism group* $\mathrm{O}_8^-(2){:}2$*. The stabilizer of a block of this design is isomorphic to* $\mathrm{S}_5{\times}\mathrm{S}_3$ *such that has orbit lengths* $3^1, 5^1, 6^1, 15^1, 30^3$*. Then by Proposition 4.1, there exist resolvable simple* $3-(120,60,133632i)$ *designs for* $i = 1,\ldots,120$*. The group* $\mathrm{S}_8(2)$ *is a group of automorphism of* $3-(120,60,16035840)$ *design.*

Example 4.6. *Consider the* 2*-transitive action of the simple group* $\mathrm{L}_4(3)$ *on the 40 points* [9]*. A point stabilizer in this action is a maximal subgroup isomorphic to* $3^3{:}\mathrm{L}_3(3)$*.*

1) There is a simple $2-(39,19,1458)$ *design with automorphism group* $3^3{:}\mathrm{L}_3(3)$*. The stabilizer of a block of this design is isomorphic to* S_4 *such that has orbit lengths* $3^1, 4^1, 6^2, 8^1, 12^1$*. Therefore by Proposition* **??**, there exist resolvable simple $3-(40,20,1458i)$ designs for $i = 1,\ldots,40$. The group $\mathrm{L}_4(3)$ is a group of automorphism of $3-(40,20,29160)$ design.

2) There is a simple $2-(39,19,2916)$ design with automorphism group $3^3{:}\mathrm{L}_3(3)$. The stabilizer of a block of this design is isomorphic to A_4 such that has orbit lengths $3^1, 4^3, 6^2, 12^1$. Therefore by Proposition 4.1, there exist resolvable simple $3-(40,20,2916i)$ designs for $i = 1,\ldots,40$. The group $\mathrm{L}_4(3)$ is a group of automorphism of $3-(40,20,58320)$ design.

5. Generators and base blocks

This section contains generators of the group $\mathrm{U}_3(5){:}2$ that was used in Section 4. Also, the base blocks of constructed designs in Section 3 and 4 are given in three tables.

TABLE 4. The generators of group $\mathrm{U}_3(5){:}2$

a	(1,110,17,92,78,37,111,24,157,118)(2,163,136,103,84,98,109,71,23,150) (3,42,127,143,35,9,15,59,106,75)(4,142,173,79,46,141,108,169,85,121) (5,61,166,10,138,160,155,16,6,113)(7,72,151,13,137,168,62,43,83,144) (8,140,95,100,132,58,124,139,63,90)(11,67,41,174,145,48,167,39,123,22) (12,116,158,126,122,91,82,68,57,34)(14,26,147,20,148,131,52,154,21,76) (18,32,36,88,107,120,87,134,28,104)(19,97,130,70,119,146,65,50,29,27) (25,133,60,94,69,80,175,125,54,117)(30,128,73,47,81)(31,51,135,77,129) (33,74,44,115,64,161,99,152,114,105)(40,45,162,172,149,153,165,159,102,89) (49,66,86,164,55,170,96,56,156,171)(38,101,53,93,112)
b	(1,133,13,64,163,144,33,121)(3,94,53,54,169,93,123,166)(4,61,135,162,89,116,78,128) (5,131,158,122,145,40,76,11)(6,114,10,85,30,86,126,21)(7,90,127,91,109,107,84,34) (8,120,18,97,28,74,22,68)(9,124,174,141,73,118,24,81)(12,60,39,62,44,111,14,20) (15,95,101,82,167,159,49,77)(17,67,173,50)(19,151,102,26,25,143,38,175) (23,48,112,41,171,147,35,104)(27,88,125,160,113,65,115,70)(29,129,58,100,99,80,148,172) (31,170,98,51,66,130,96,42)(32,75,150,47,71,142,161,103)(36,37,119,117,110,87,79,59) (43,138,165,56,105,72,164,134)(45,157,46,156,52,106,92,137)(55,146,108,63) (69,132,136,152,83,153,168,154)(139,140,155,149)(2,16)

Table 5. The base blocks of 3-designs of Section 3

X_1	1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 15, 17, 18, 21, 23, 25, 26, 29, 31, 32, 34, 39, 41, 43, 46, 55, 56 57, 58, 59, 60, 61, 62, 65, 66, 67, 74, 75, 80, 81, 82, 88, 92, 93, 95, 97, 99, 101, 104, 105, 108 109, 112, 113, 114, 115, 116, 117, 118, 120, 121, 123, 125, 128, 131, 133, 136, 137, 140, 141, 142, 144, 145, 146, 147, 155, 156, 157, 158, 159, 162, 163, 164, 165, 166, 168, 173, 174
X_2	1, 2, 3, 5, 6, 7, 8, 9, 11, 15, 17, 18, 21, 23, 25, 26, 29, 31, 32, 34, 39, 41, 42, 43, 46, 55, 56, 57, 58, 59, 60, 61, 62, 65, 66, 67, 74, 75, 80, 81, 83, 88, 92, 93, 95, 97, 99, 101, 104, 105, 107, 108, 109, 112, 113, 114, 115, 116, 117, 118, 121, 123, 125, 128, 131, 133, 136, 137, 140, 141, 142, 144, 145, 146, 147, 154, 155, 156, 157, 159, 162, 163, 164, 165, 166, 168, 173, 174
X_3	1, 2, 3, 9, 10, 12, 16, 17, 18, 19, 20, 21, 22, 26, 27, 29, 32, 34, 35, 36, 38, 39, 40, 42, 49, 50, 51, 52, 53, 56, 58, 61, 63, 65, 66, 67, 68, 72, 75, 78, 79, 82, 83, 84, 85, 86, 99, 104, 106, 108, 109, 110, 111, 112, 117, 118, 119, 120, 125, 126, 127, 130, 132, 135, 136, 138, 139, 140, 143, 147, 149, 150, 151, 153, 154, 155, 157, 159, 160, 161, 165, 166, 167, 168, 169, 171, 172, 174
X_4	1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 20, 21, 22, 24, 26, 27, 29, 33, 34, 35, 36, 38, 39, 40, 42, 44, 48, 49, 50, 51, 53, 56, 58, 60, 62, 63, 67, 68, 70, 73, 74, 78, 79, 81, 82, 84, 87, 99, 101, 104, 106, 108, 109, 110, 111, 112, 116, 117, 118, 120, 121, 122, 126, 127, 129, 135, 139, 143, 147, 149, 150, 151, 153, 155, 156, 157, 160, 161, 163, 165, 168, 169, 170, 174
X_5	1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 20, 21, 22, 24, 26, 27, 29, 34, 35, 36, 38, 39, 40, 42, 44, 49, 50, 51, 53, 56, 58, 61, 62, 63, 65, 67, 68, 70, 72, 74, 78, 79, 81, 82, 84, 85, 99, 104, 106, 108, 109, 110, 111, 112, 116, 117, 118, 119, 120, 121, 122, 126, 127, 135, 139 143, 147, 149, 150, 151, 153, 155, 156, 157, 159, 160, 161, 165, 167, 168, 169, 170, 171, 174
X_6	1, 2, 3, 4, 7, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 22, 26, 27, 29, 32, 33, 34, 35, 36, 38, 39, 40, 42, 48, 49, 50, 51, 52, 53, 56, 58, 60, 63, 67, 68, 70, 73, 75, 78, 79, 81, 82, 83, 84, 87, 99, 101, 104, 106, 108, 109, 110, 111, 112, 116, 117, 118, 120, 122, 125, 126, 127, 129, 132, 135, 136, 139, 143, 147, 149, 150, 151, 153, 155, 157, 160, 161, 163, 165, 168, 169, 174
X_7	1, 2, 3, 4, 7, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 22, 26, 27, 29, 32, 34, 35, 36, 38, 39, 40, 42, 49, 50, 51, 52, 53, 56, 58, 61, 63, 65, 67, 68, 70, 72, 75, 78, 79, 81, 82, 83, 84, 85, 99, 104, 106, 108, 109, 110, 111, 112, 116, 117, 118, 119, 120, 122, 125, 126, 127, 132, 135, 136, 139, 143, 147, 149, 150, 151, 153, 155, 157, 159, 160, 161, 165, 167, 168, 169, 171, 174
X_8	1, 2, 3, 8, 9, 10, 12, 16, 17, 18, 20, 21, 22, 24, 26, 27, 29, 33, 34, 35, 36, 38, 39, 40, 42, 44,48, 49, 50, 51, 53, 56, 58, 60, 62, 63, 66, 67, 68, 73, 74, 78, 79, 82, 84, 86, 87, 99, 101, 104, 106, 108, 109, 110, 111, 112, 117, 118, 120, 121, 126, 127, 129, 130, 135, 138, 139, 140, 143, 147, 149, 150, 151, 153, 154, 155, 156, 157, 160, 161, 163, 165, 166, 168, 169, 170, 172, 174
X_9	1, 2, 3, 8, 9, 10, 12, 16, 17, 18, 20, 21, 22, 24, 26, 27, 29, 34, 35, 36, 38, 39, 40, 42, 44, 49, 50, 51, 53, 56, 58, 61, 62, 63, 65, 66, 67, 68, 72, 74, 78, 79, 82, 84, 85, 86, 99, 104, 106, 108, 109, 110, 111, 112, 117, 118, 119, 120, 121, 126, 127, 130, 135, 138, 139, 140, 143, 147, 149, 150, 151, 153, 154, 155, 156, 157, 159, 160, 161, 165, 166, 167, 168, 169, 170, 171, 172, 174
X_{10}	1, 2, 3, 9, 10, 12, 16, 17, 18, 19, 20, 21, 22, 26, 27, 29, 32, 33, 34, 35, 36, 38, 39, 40, 42, 48, 49, 50, 51, 52, 53, 56, 58, 60, 63, 66, 67, 68, 73, 75, 78, 79, 82, 83, 84, 86, 87, 99, 101, 104, 106, 108, 109, 110, 111, 112, 117, 118, 120, 125, 126, 127, 129, 130, 132, 135, 136, 138, 139, 140, 143, 147, 149, 150, 151, 153, 154, 155, 157, 160, 161, 163, 165, 166, 168, 169, 172, 174

http://dx.doi.org/10.22108/toc.2023.132404.1956

Table 6. The base blocks of 3-designs of Section 3

X_{11}	1, 2, 4, 5, 7, 8, 11, 13, 15, 16, 17, 21, 22, 24, 25, 29, 30, 31, 32, 34, 37, 38, 42, 45, 47, 48, 49, 53, 57, 58, 61, 63, 67, 70, 71, 72, 73, 74, 75, 76, 78, 80, 81, 82, 83, 89, 92, 94, 98, 99, 101, 102, 103, 105, 106, 107, 109, 110, 112, 114, 116, 117, 118, 120, 122, 124, 128, 133, 136, 137, 139, 141, 145, 146, 149, 151, 152, 155, 157, 159, 160, 161, 162, 163, 164, 170, 171, 176
X_{12}	1, 2, 4, 5, 7, 8, 11, 13, 15, 16, 17, 21, 22, 24, 25, 29, 30, 31, 33, 34, 37, 38, 42, 44, 45, 47, 48, 49, 53, 57, 58, 60, 62, 63, 67, 70, 71, 73, 74, 76, 78, 80, 81, 82, 87, 89, 92, 94, 98, 99, 101, 102, 103, 105, 106, 107, 109, 110, 112, 114, 116, 117, 118, 120, 121, 122, 124, 128, 129, 133, 137, 139, 141, 145, 146, 149, 151, 152, 155, 156, 157, 160, 161, 162, 163, 164, 170, 176
X_{13}	1, 2, 4, 5, 7, 11, 13, 15, 16, 17, 19, 21, 22, 25, 29, 30, 31, 32, 34, 37, 38, 42, 45, 47, 49, 52, 53, 57, 58, 61, 63, 65, 67, 70, 71, 72, 75, 76, 78, 80, 81, 82, 83, 85, 89, 92, 94, 98, 99, 102, 103, 105, 106, 107, 109, 110, 112, 114, 116, 117, 118, 119, 120, 122, 124, 125, 128, 132, 133, 136, 137, 139, 141, 145, 146, 149, 151, 152, 155, 157, 159, 160, 161, 162, 164, 167, 171, 176
X_{14}	1, 2, 5, 8, 15, 16, 17, 21, 22, 24, 25, 29, 30, 31, 32, 34, 37, 38, 42, 45, 47, 48, 49, 53, 57, 58, 61, 63, 66, 67, 71, 72, 73, 74, 75, 76, 78, 80, 82, 83, 86, 89, 92, 94, 98, 99, 101, 102, 103, 105, 106, 107, 109, 110, 112, 114, 117, 118, 120, 124, 128, 130, 133, 136, 137, 138, 139, 140, 141, 145, 146, 149, 151, 152, 154, 155, 157, 159, 160, 161, 162, 163, 164, 166, 170, 171, 172, 176
X_{15}	1, 2, 4, 5, 7, 11, 13, 15, 16, 17, 21, 22, 25, 29, 30, 31, 32, 33, 34, 37, 38, 42, 44, 45, 47, 49, 53, 57, 58, 60, 61, 62, 63, 67, 70, 71, 72, 75, 76, 78, 80, 81, 82, 83, 87, 89, 92, 94, 98, 99, 102, 103, 105, 106, 107, 109, 110, 112, 114, 116, 117, 118, 120, 121, 122, 124, 128, 129, 133, 136, 137, 139, 141, 145, 146, 149, 151, 152, 155, 156, 157, 159, 160, 161, 162, 164, 171, 176
X_{16}	1, 3, 6, 10, 11, 13, 14, 17, 27, 29, 34, 39, 41, 44, 45, 50, 51, 55, 56, 62, 67, 69, 76, 77, 79, 84, 85, 86, 87, 89, 96, 99, 101, 102, 104, 105, 107, 111, 112, 114, 117, 118, 121, 123, 125, 131, 132, 134, 139, 142, 146, 149, 160, 161, 162, 163, 164, 165, 171, 176

Table 7. The base blocks of 2-designs of Section 4

Y_1	2, 4, 5, 6, 10, 11, 12, 14, 15, 16, 17, 18, 22, 23, 24, 26, 27, 29, 31, 34, 35, 36, 39, 40, 41, 42, 46, 47, 52, 56, 57, 59, 62, 63, 67, 69, 73, 74, 75, 76, 77, 79, 80, 81, 82, 83, 87, 92, 95, 101, 104, 107, 110, 111, 112, 113, 114, 115, 116, 118, 120, 122, 123, 125, 127, 128, 130, 139, 140, 141, 142, 143, 146, 148, 151, 155, 156, 157, 159, 161, 164, 166, 168, 169, 170, 171, 175
Y_2	1, 4, 5, 6, 9, 14, 16, 17, 18, 20, 23, 24, 28, 30, 31, 32, 34, 35, 37, 38, 39, 42, 44, 45, 47, 48, 52, 54, 57, 59, 60, 61, 62, 63, 64, 65, 66, 70, 73, 74, 75, 76, 79, 81, 82, 86, 88, 89, 90, 94, 101, 103, 104, 105, 106, 107, 109, 110, 111, 112, 113, 117, 119, 120, 122, 124, 126, 127, 128, 129, 130, 132, 136, 139, 140, 145, 149, 152, 153, 156, 158, 159, 161, 164, 169, 170, 173
Y_3	3, 4, 6, 7, 8, 11, 13, 16, 17, 18, 20, 21, 22, 23, 24, 26, 28, 29, 31, 32, 35, 40, 43, 44, 45, 46, 47, 48, 49, 51, 59, 61, 64, 67, 68, 73, 74, 75, 76, 80, 81, 82, 83, 85, 89, 90, 91, 92, 95, 96, 97, 99, 100, 104, 105, 106, 107, 109, 110, 114, 115, 116, 118, 122, 130, 133, 134, 138, 139, 141, 142, 144, 146, 147, 152, 153, 154, 159, 160, 163, 166, 168, 170, 171, 172, 174, 175
Y_4	1, 2, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 21, 22, 23, 25, 29, 33, 35, 36, 37, 40, 41, 44, 46, 48, 51, 52, 53, 58, 59, 62, 67, 68, 74, 78, 80, 83, 84, 88, 90, 91, 92, 94, 95, 96, 99, 104, 106, 107, 108, 109, 114, 115, 116, 117, 118, 119, 121, 122, 126, 128, 129, 130, 132, 134, 135, 138, 140, 143, 144, 146, 149, 150, 160, 162, 164, 165, 166, 167, 168, 169, 170, 172, 173, 174
Y_5	1, 3, 4, 8, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 26, 35, 37, 39, 40, 43, 45, 46, 48, 49, 51, 52, 53, 54, 57, 59, 62, 63, 65, 68, 70, 71, 75, 76, 77, 78, 80, 81, 88, 89, 90, 92, 98, 99, 101, 106, 107, 108, 114, 116, 117, 118, 119, 122, 126, 128, 129, 132, 134, 135, 137, 138, 139, 140, 141, 142, 144, 146, 147, 148, 151, 152, 153, 154, 161, 162, 163, 166, 169, 170, 172, 173, 174
Y_6	1, 2, 3, 6, 9, 10, 12, 13, 16, 17, 19, 20, 22, 23, 25, 27, 28, 30, 31, 32, 33, 37, 38, 39, 40, 42, 46, 49, 50, 52, 53, 56, 59, 60, 61, 62, 64, 71, 72, 74, 77, 79, 82, 83, 84, 85, 87, 88, 91, 95, 97, 98, 99, 101, 103, 104, 105, 106, 107, 109, 110, 111, 113, 115, 117, 118, 120, 121, 123, 124, 126, 129, 132, 136, 139, 146, 152, 153, 155, 156, 160, 161, 164, 165, 168, 170, 175
Y_7	2, 3, 4, 5, 7, 9, 10, 13, 15, 17, 20, 22, 23, 24, 25, 29, 33, 35, 36, 38, 41, 43, 45, 47, 48, 58, 59, 64, 66, 67, 72, 74, 76, 79, 80, 82, 83, 84, 89, 90, 92, 93, 94, 95, 97, 99, 100, 103, 104, 105, 106, 109, 110, 114, 115, 116, 117, 121, 127, 130, 131, 132, 133, 134, 138, 141, 143, 145, 146, 149, 150, 152, 155, 156, 158, 159, 162, 164, 165, 166, 167, 169, 171, 172, 173, 174, 175
Y_8	7, 8, 10, 11, 14, 15, 16, 19, 22, 25, 26, 27, 31, 33, 34, 36, 37, 38, 39, 40, 41, 44, 46, 47, 48, 49, 50, 60, 61, 63, 65, 66, 69, 70, 71, 72, 73, 74, 75, 77, 79, 84, 85, 87, 91, 94, 95, 97, 98, 99, 103, 106, 107, 108, 110, 113, 114, 116, 118, 122, 124, 127, 128, 129, 130, 131, 134, 135, 139, 142, 144, 148, 151, 152, 154, 155, 156, 157, 159, 161, 162, 165, 167, 168, 169, 174, 175

http://dx.doi.org/10.22108/toc.2023.132404.1956

Acknowledgments

The author gratefully thanks the School of Computer Science, Institute for Research in Fundamental Science (IPM), where the computations were done there.

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