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A NEW q-ANALOGUE OF THE BINOMIAL IDENTITY $\sum_k(-1)^k\binom{2n}{n+3k} = 2\cdot 3^{n-1}$

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Abstract. In this paper, we establish a new q-analogue of the binomial identity:

$$\sum_k (-1)^k \binom{2n}{n+3k} = \begin{cases} 1, & \text{if } n=0, \\ 2\cdot 3^{n-1}, & \text{if } n\geq 1. \end{cases}$$

Our proof relies on a weight-preserving and sign-reversing involution due to Guo and Zhang.

1. Introduction

We consider q-analogues of the following binomial identity:

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} (-1)^k \binom{2n}{n+3k} = \begin{cases} 1, & \text{if } n=0, \\ 2\cdot 3^{n-1}, & \text{if } n\geq 1, \end{cases} \tag{1.1}$$

where $\lfloor x\rfloor$ denotes the integral part of real x. Recall that the q-shifted factorials are given by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geq 1$ and $(a;q)_0=1$, and the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\leq k\leq n, \\ 0, & \text{otherwise.} \end{cases}$$

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Here we list two different q-analogues of (1.1) as follows:

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor}(-1)^k q^{\frac{9k^2+3k}{2}}\begin{bmatrix}2n\\ n+3k\end{bmatrix}=\begin{cases}1, & \text{if } n=0,\\ (1+q^n)\dfrac{(q^3;q^3)_{n-1}}{(q;q)_{n-1}}, & \text{if } n\geq 1,\end{cases} \tag{1.2}$$

and

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor}(-1)^k q^{\frac{9k^2+9k}{2}}\begin{bmatrix}2n\\ n+3k\end{bmatrix}=\begin{cases}1, & \text{if } n=0,\\ 1+q, & \text{if } n=1,\\ q^{n-2}(1+q^n)(1+q+q^2)\dfrac{(q^3;q^3)_{n-2}}{(q;q)_{n-2}}, & \text{if } n\geq 2.\end{cases} \tag{1.3}$$

Note that (1.2) is equivalent to Bailey pair $J(2)$ in [5], which was also recorded in [4, Proposition 2(5)] and was used by Berkovich and Warnaar [2] to prove a 'perfect' Rogers-Ramanujan identity.

Letting $n\to\infty$ on both sides of (1.2) reduces to the famous Euler's pentagonal number theorem [1, Corollary 1.7]:

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k+1)}{2}}=(q;q)_\infty. \tag{1.4}$$

However, letting $n\to\infty$ on both sides of (1.3) leads us to the following trivial identity:

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(k+1)}{2}}=0.$$

In 2014, Guo and Zhang [3] gave a nice combinatorial proof of (1.2) and (1.3). Motivated by Guo and Zhang's work, we shall establish a new q-analogue of (1.1) as follows.

Theorem 1.1. *For any nonnegative integer n, we have*

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor}(-1)^k q^{\frac{9k^2+15k}{2}}\begin{bmatrix}2n\\ n+3k\end{bmatrix}=\begin{cases}1, & \textit{if } n=0,\\ 1+q, & \textit{if } n=1,\\ (1+q^2)(1+q+q^2), & \textit{if } n=2,\\ \dfrac{(q^3;q^3)_{n-3}}{(q;q)_{n-3}}f_n(q), & \textit{if } n\geq 3,\end{cases} \tag{1.5}$$

where $f_n(q)$ is given by

$$f_n(q)=q^{-3}(1+q^n)(-q^{4n}+q^{3n}+q^{2n+3}+q^{2n+2}+2q^{2n+1}+q^{2n}+2q^{2n-1}+q^{2n-2}+q^{2n-3}+q^n-1).$$

It is clear that letting $q\to 1$ on both sides of (1.5) reduces to (1.1). Letting $n\to\infty$ on both sides of (1.5) also reduces to an equivalent form of the Euler's pentagonal number theorem (1.4). A combinatorial proof of (1.5) will be given in the next section. It is worth mentioning that the proof of (1.5) is inspired by a weight-preserving and sign-reversing involution in Guo and Zhang's paper [3].

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2. Combinatorial proof of (1.5)

For $n = 0, 1, 2$, it is easy to check (1.5) by hand. In what follows, suppose that $n \geq 3$.

Let $S = \{a_1, \dots, a_{2n}\}$ be a set of $2n$ elements, and let

$$\mathscr{P} = \{A \subseteq S : \#A \equiv n \pmod 3\},$$

$$\mathscr{Q} = \{A \in \mathscr{P} : \#(A \cap \{a_1, \dots, a_{2i+1}\}) \notin \{i-1, i+2\} \quad \text{for } i = 1, \dots, n-3\}. \tag{2.1}$$

For any $A \in \mathscr{P}$, we associate A with a sign $\operatorname{sgn}(A) = (-1)^{(\#A-n)/3}$ and a weight $||A|| = \sum_{a \in A} a$.

For any $A \in \mathscr{Q}$, we shall prove that

$$\#(A \cap \{a_1, \dots, a_{2i+1}\}) \in \{i, i+1\}, \quad \text{for all } i = 1, \dots, n-3. \tag{2.2}$$

In fact, the statement (2.2) holds for $i = 1$. Suppose it holds for $i - 1$, i.e.,

$$\#(A \cap \{a_1, \dots, a_{2i-1}\}) \in \{i-1, i\}.$$

It follows from the above and (2.1) that

$$\#(A \cap \{a_1, \dots, a_{2i+1}\}) \in \{i, i+1\},$$

which confirms our statement (2.2). In particular,

$$\#(A \cap \{a_1, \dots, a_{2n-5}\}) \in \{n-3, n-2\}.$$

Since $\#A \equiv n \pmod 3$, we have

$$\#A \in \{n-3, n, n+3\}. \tag{2.3}$$

For any $A \in \mathscr{P} \setminus \mathscr{Q}$, let $i \leq n-3$ be the smallest number such that

$$\#(A \cap \{a_1, \dots, a_{2i+1}\}) \in \{i-1, i+2\}. \tag{2.4}$$

Next we recall an involution due to Guo and Zhang [3]. The set A' is defined as follows:

- $a_1 \in A'$ if and only if $a_1 \notin A$;
- $a_{2j}, a_{2j+1} \in A'$ if $a_{2j}, a_{2j+1} \notin A \quad (j = 1, \dots, i)$;
- $a_{2j}, a_{2j+1} \notin A'$ if $a_{2j}, a_{2j+1} \in A \quad (j = 1, \dots, i)$;
- $a_{2j} \in A'$ and $a_{2j+1} \notin A'$ if $a_{2j} \in A$ and $a_{2j+1} \notin A \quad (j = 1, \dots, i)$;
- $a_{2j} \notin A'$ and $a_{2j+1} \in A'$ if $a_{2j} \notin A$ and $a_{2j+1} \in A \quad (j = 1, \dots, i)$;
- $a_k \in A'$ if and only if $a_k \in A \quad (2i+2 \leq k \leq 2n)$.

Note that

$$\#(A \cap \{a_1, \dots, a_{2i+1}\}) + \#(A' \cap \{a_1, \dots, a_{2i+1}\}) = 2i+1. \tag{2.5}$$

From (2.4) and (2.5), we deduce that $\#A' = \#A \pm 3$.

Let $a_1 = 0, \{a_{2j}, a_{2j+1}\} = \{-j, j\}$ for $j = 1, \cdots, n-3$, and $\{a_{2n-4}, a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}\} = \{n-2, n-1, n, n+1, n+2\}$. Note that S is obtained by $[2n]$ by a shift $-(n-2)$:

$$S = \{3-n, \ldots, n-3, n-2, n-1, n, n+1, n+2\},$$

where $[n] = \{1, \ldots, n\}$. Then $A \to A'$ is a weight-preserving and sign-reversing involution, and so

$$\sum_{A \in \mathscr{P} \setminus \mathscr{Q}} \operatorname{sgn}(A) q^{||A||} = 0. \tag{2.6}$$

Let

$$\mathscr{Q}^* := \{A \subseteq \{a_1, \ldots, a_{2n-5}\} : \#(A \cap \{a_1, \ldots, a_{2i+1}\}) \notin \{i-1, i+2\} \quad \text{for } i = 1, \ldots, n-3\}.$$

Similarly, the statement (2.2) also holds for any $A \in \mathscr{Q}^*$. If $\#(A \cap \{a_1, \ldots, a_{2i-1}\}) = i-1$, then we have three choices $\{a_{2i}\}, \{a_{2i+1}\}$ and $\{a_{2i}, a_{2i+1}\}$ for $A \cap \{a_{2i}, a_{2i+1}\}$. If $\#(A \cap \{a_1, \ldots, a_{2i-1}\}) = i$, then we also have three choices $\{a_{2i}\}, \{a_{2i+1}\}$ and $\varnothing$ for $A \cap \{a_{2i}, a_{2i+1}\}$. Since $a_{2i} + a_{2i+1} = 0$, we have

$$\sum_{A \in \mathscr{Q}^*} q^{||A||} = \sum_{\substack{A \in \mathscr{Q}^* \\ \#A = n-3}} q^{||A||} + \sum_{\substack{A \in \mathscr{Q}^* \\ \#A = n-2}} q^{||A||} = 2 \prod_{i=1}^{n-3} (q^i + q^{-i} + q^0).$$

Moreover, we can construct a weight-preserving bijection from $\{A \in \mathscr{Q}^* : \#A = n-3\}$ to $\{A \in \mathscr{Q}^* : \#A = n-2\}$ similar to the above involution $A \to A'$ on $\mathscr{P} \setminus \mathscr{Q}$. It follows that

$$\sum_{\substack{A \in \mathscr{Q}^* \\ \#A = n-3}} q^{||A||} = \sum_{\substack{A \in \mathscr{Q}^* \\ \#A = n-2}} q^{||A||} = \prod_{i=1}^{n-3} (q^i + q^{-i} + q^0). \tag{2.7}$$

From (2.3), we deduce that

$$\begin{aligned}
\mathscr{Q} = {} & \{A \in \mathscr{Q}^* : \#A = n-3\} \\
& \biguplus \{A \cup B : A \in \mathscr{Q}^*, \#A = n-3 \quad \& \quad B \subseteq \{a_{2n-4}, a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}\}, \#B = 3\} \\
& \biguplus \{A \cup B : A \in \mathscr{Q}^*, \#A = n-2 \quad \& \quad B \subseteq \{a_{2n-4}, a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}\}, \#B = 2\} \\
& \biguplus \{A \cup \{a_{2n-4}, a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}\} : A \in \mathscr{Q}^*, \#A = n-2\}.
\end{aligned}$$

It follows from the above and (2.7) that

$$\begin{aligned}
\sum_{A \in \mathscr{Q}} \operatorname{sgn}(A) q^{||A||} = {} & (1 + q^n) \prod_{i=1}^{n-3} (q^i + q^{-i} + q^0) \\
& \times (-q^{4n} + q^{3n} + q^{2n+3} + q^{2n+2} + 2q^{2n+1} + q^{2n} + 2q^{2n-1} + q^{2n-2} + q^{2n-3} + q^n - 1).
\end{aligned} \tag{2.8}$$

By the q-binomial theorem [1, Theorem 3.3]:

$$(-qz;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k q^{\binom{k+1}{2}},$$

we have

$$\sum_{\substack{A\subseteq[n] \\ \#A=k}} q^{||A||} = \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}}. \tag{2.9}$$

By using (2.9), we obtain

$$\begin{aligned}
\sum_{A\in\mathscr{P}} \operatorname{sgn}(A) q^{||A||} &= \sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} \sum_{\substack{A\subseteq S \\ \#A=n+3k}} \operatorname{sgn}(A) q^{||A||} \\
&= \sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} (-1)^k \begin{bmatrix} 2n \\ n+3k \end{bmatrix} q^{\binom{n+3k+1}{2}-(n+3k)(n-2)} \\
&= q^{\frac{n(5-n)}{2}} \sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} (-1)^k q^{\frac{9k^2+15k}{2}} \begin{bmatrix} 2n \\ n+3k \end{bmatrix}. && (2.10)
\end{aligned}$$

On the other hand,

$$\sum_{A\in\mathscr{P}} \operatorname{sgn}(A) q^{||A||} = \sum_{A\in\mathscr{Q}} \operatorname{sgn}(A) q^{||A||} + \sum_{A\in\mathscr{P}\setminus\mathscr{Q}} \operatorname{sgn}(A) q^{||A||}. \tag{2.11}$$

Finally, combining (2.6), (2.8), (2.10) and (2.11), we arrive at

$$\begin{aligned}
q^{\frac{n(5-n)}{2}} \sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} (-1)^k q^{\frac{9k^2+15k}{2}} \begin{bmatrix} 2n \\ n+3k \end{bmatrix} &= (1+q^n) \prod_{i=1}^{n-3} (q^i + q^{-i} + q^0) \\
&\times (-q^{4n} + q^{3n} + q^{2n+3} + q^{2n+2} + 2q^{2n+1} + q^{2n} + 2q^{2n-1} + q^{2n-2} + q^{2n-3} + q^n - 1),
\end{aligned}$$

which is equivalent to (1.5).

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