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COMPARING UPPER BROADCAST DOMINATION AND BOUNDARY INDEPENDENCE BROADCAST NUMBERS OF GRAPHS

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Abstract. A broadcast on a nontrivial connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \dots, d\}$, where $d = \mathrm{diam}(G)$, such that $f(v) \le e(v)$ (the eccentricity of v) for all $v \in V$. The weight of f is $\sigma(f) = \sum_{v \in V} f(v)$. A vertex u hears f from v if $f(v) > 0$ and $d(u, v) \le f(v)$. A broadcast f is dominating if every vertex of G hears f. The upper broadcast domination number of G is $\Gamma_b(G) = \max\{\sigma(f) : f \text{ is a minimal dominating broadcast of } G\}$.

A broadcast f is boundary independent if, for any vertex w that hears f from vertices $v_1, \dots, v_k$, $k \ge 2$, the distance $d(w, v_i) = f(v_i)$ for each i. The maximum weight of a boundary independent broadcast is the boundary independence broadcast number $\alpha_{\mathrm{bn}}(G)$.

We compare α_{bn} to Γ_b, showing that neither is an upper bound for the other. We show that the differences $\Gamma_b - \alpha_{\mathrm{bn}}$ and $\alpha_{\mathrm{bn}} - \Gamma_b$ are unbounded, the ratio $\alpha_{\mathrm{bn}}/\Gamma_b$ is bounded for all graphs, and $\Gamma_b/\alpha_{\mathrm{bn}}$ is bounded for bipartite graphs but unbounded in general.

1. Introduction

The study of broadcast domination and broadcast independence was initiated by Erwin in his doctoral dissertation [11]. To generalize the concept of an independent set X in a graph G to an independent broadcast, Erwin focussed on the property that no vertex in X belongs to the neighbourhood of another vertex in X. Focussing instead on the property that no edge of G is incident with (or

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covered by) more than one vertex in X, Neilson [16] and Mynhardt and Neilson [14] defined boundary independent broadcasts as an alternative to Erwin's independent broadcasts. We explain below why this definition results in a parameter, called the boundary independence number $\alpha_{\rm bn}(G)$, that is, in some sense, "better behaved" than Erwin's broadcast independence number, which we denote here by $\alpha_h(G)$.

We compare the upper broadcast domination number Γ_b, also defined by Erwin [11], to $\alpha_{\rm bn}$, showing that neither is an upper bound for the other. We denote this incomparability by $\alpha_{\rm bn} \diamond \Gamma_b$. We show that the differences $\Gamma_b - \alpha_{\rm bn}$ and $\alpha_{\rm bn} - \Gamma_b$ are unbounded, the ratio $\alpha_{\rm bn}/\Gamma_b$ is bounded for all graphs, and $\Gamma_b/\alpha_{\rm bn}$ is bounded for bipartite graphs but unbounded in general.

2. Definitions and background

For undefined concepts we refer the reader to [9]. A *broadcast* on a connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \text{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$ if $|V| \geq 2$, and $f(v) = 1$ if $V = \{v\}$. When G is disconnected, we define a broadcast on G as the union of broadcasts on its components. Define $V_f^+ = \{v \in V : f(v) > 0\}$ and partition V_f^+ into the two sets $V_f^1 = \{v \in V : f(v) = 1\}$ and $V_f^{++} = V_f^+ - V_f^1$. A vertex in V_f^+ is called a *broadcasting vertex*. A vertex u *hears* f from $v \in V_f^+$, and v *f-dominates* u, if the distance $d(u, v) \leq f(v)$. Denote the set of all vertices that do not hear f by U_f. A broadcast f is *dominating* if $U_f = \varnothing$. For any subset U of V, we define $f(U) = \sum_{u \in U} f(u)$. The *weight* of f is $\sigma(f) = f(V)$, and the *broadcast domination number* of G is

$$\gamma_b(G) = \min \{\sigma(f) : f \text{ is a dominating broadcast of } G\}.$$

If f and g are broadcasts on G such that $g(v) \leq f(v)$ for each $v \in V$, we write $g \leq f$. If in addition $g(v) < f(v)$ for at least one $v \in V$, we write $g < f$. A dominating broadcast f on G is a *minimal dominating broadcast* if no broadcast $g < f$ is dominating. The *upper broadcast domination number* of G is

$$\Gamma_b(G) = \max \{\sigma(f) : f \text{ is a minimal dominating broadcast of } G\},$$

and a dominating broadcast f of G such that $\sigma(f) = \Gamma_b(G)$ is called a *$\Gamma_b(G)$-broadcast* (abbreviated to *Γ_b-broadcast* if the graph G is obvious). Introduced by Erwin [11], the upper broadcast domination number was also studied by, for example, Ahmadi, Fricke, Schroeder, Hedetniemi and Laskar [1], Bouchemakh and Fergani [6], Bouchouika, Bouchemakh and Sopena [8], Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [10], and Mynhardt and Roux [15].

We denote the independence number of G by $\alpha(G)$; an independent set of G of cardinality $\alpha(G)$ is called an *$\alpha(G)$-set*, often abbreviated to *α-set*. To generalize the concept of independent sets, Erwin [11] defined a broadcast f to be *independent*, or, for our purposes, *hearing independent*, abbreviated to *h-independent*, if no vertex $u \in V_f^+$ hears f from any other vertex $v \in V_f^+$; that is, broadcasting vertices only hear themselves. The maximum weight of an h-independent broadcast is the *h-independent*

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broadcast number, denoted by $\alpha_h(G)$; such a broadcast is called an $\alpha_h(G)$-*broadcast* (or α_h-*broadcast* for short). This version of broadcast independence was also considered by, among others, Ahmane, Bouchemakh and Sopena [2, 3], Bessy and Rautenbach [4, 5], Bouchemakh and Zemir [7], Bouchouika et al. [8] and Dunbar et al. [10]. For a survey of broadcasts in graphs, see the chapter by Henning, MacGillivray and Yang [12].

Before continuing, we define a class of trees often used as examples. For $k \geq 3$ and $n_i \geq 1$ for $i \in \{1, \dots, k\}$, the *(generalized) spider* $\mathrm{Sp}(n_1, \dots, n_k)$ is the tree which has exactly one vertex b, called the *head*, having $\deg(b) = k$, and for which the k components of $\mathrm{Sp}(n_1, \dots, n_k) - b$ are paths of lengths $n_1 - 1, \dots, n_k - 1$, respectively. The *legs* $L_1, \dots, L_k$ of the spider are the paths from b to the leaves. If $n_i = r$ for each i, we write $\mathrm{Sp}(r^k)$ for $\mathrm{Sp}(n_1, \dots, n_k)$. An *endpath* in a tree is a path ending in a leaf and having all internal vertices (if any) of degree 2; the legs of a spider are examples of endpaths.

2.1. **Neighbourhoods and boundaries.** For a broadcast f on a graph G and $v \in V_f^+$, we define the

$$\left.\begin{array}{l} f\text{-}\textit{neighbourhood} \\ f\text{-}\textit{boundary} \\ f\text{-}\textit{private} \\ \quad\textit{neighbourhood} \\ \\ f\text{-}\textit{private boundary} \\ \\ \\ \end{array}\right\} \text{ of } v \text{ by } \left\{\begin{array}{rcl} N_f(v) & = & \{u \in V : d(u,v) \leq f(v)\} \\ B_f(v) & = & \{u \in V : d(u,v) = f(v)\} \\ & & \\ \mathrm{PN}_f(v) & = & \{u \in N_f(v) : u \notin N_f(w) \\ & & \text{for all } w \in V_f^+ - \{v\}\} \\ \mathrm{PB}_f(v) & = & \{u \in N_f(v) : u \text{ is not} \\ & & \text{dominated by } (f - \{(v, f(v))\}) \\ & & \cup\{(v, f(v) - 1)\}. \end{array}\right. \tag{2.1}$$

If $v \in V_f^1$ and v does not hear f from any vertex $u \in V_f^+ - \{v\}$, then $v \in \mathrm{PB}_f(v)$, and if $v \in V_f^{++}$, then $\mathrm{PB}_f(v) = B_f(v) \cap \mathrm{PN}_f(v)$. Since $f(v) \leq e(v)$ for each vertex v of a non-trivial graph G, $B_f(v) \neq \varnothing$. Therefore, if there exists a vertex v such that $\mathrm{PB}_f(v) = \varnothing$, then each vertex in $B_f(v)$ also hears f from another broadcasting vertex and we deduce that $|V_f^+| \geq 2$. If f is a broadcast such that every vertex x that hears more than one broadcasting vertex also satisfies $d(x,u) \geq f(u)$ for all $u \in V_f^+$, we say that the *broadcast overlaps only in boundaries*. Equivalently, f overlaps only in boundaries if $N_f(u) \cap N_f(v) \subseteq B_f(u) \cap B_f(v)$ for all distinct $u, v \in V_f^+$. If $uv \in E(G)$ and $u, v \in N_f(x)$ for some $x \in V_f^+$ such that at least one of u and v does not belong to $B_f(x)$, we say that the edge uv is *covered* in f, or f-*covered*, by x. If uv is not covered by any $x \in V_f^+$, we say that uv is *uncovered by* f or f-*uncovered*. We denote the set of f-uncovered edges by U_f^E.

Erwin [11] determined a necessary and sufficient condition for a dominating broadcast to be minimal dominating. We restate it here in terms of private boundaries.

Proposition 2.1. [11] *A dominating broadcast f is a minimal dominating broadcast if and only if $\mathrm{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$.*

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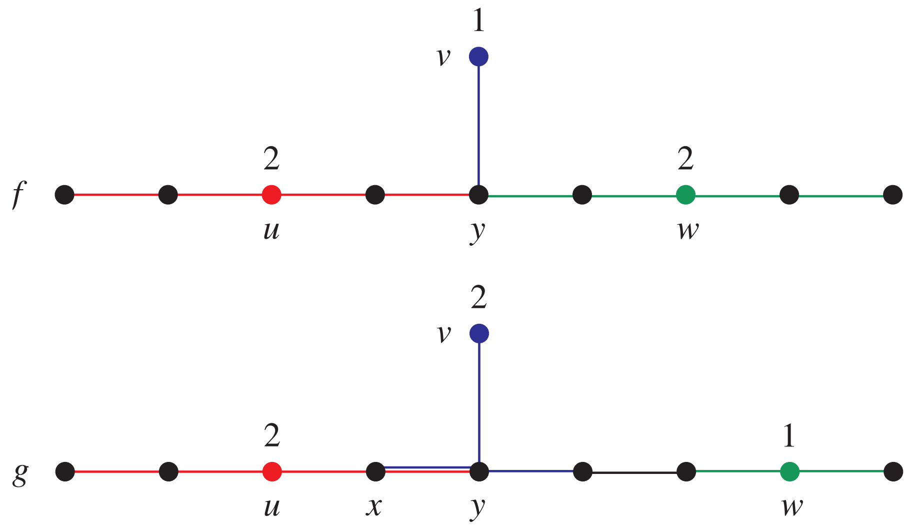

FIGURE 1. The broadcast f (top) is bn-independent, but g is not.

Ahmadi et al. [1] define a broadcast f to be *irredundant* if $\mathrm{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$. Therefore, a dominating broadcast is minimal dominating if and only if it is irredundant.

2.2. **Boundary independent broadcasts.** The characteristic function of an independent set of a graph G is a broadcast on G that overlaps only in boundaries. This feature was used in [14, 16] to define three types of *boundary independent broadcasts*; since no edge is covered by more than one broadcasting vertex, these types of broadcasts are, in some sense, more independent than h-independent broadcasts.

Definition 2.1. [14, 16] A broadcast is *bn-independent* if it overlaps only in boundaries. The maximum weight of a bn-independent broadcast on G is $\alpha_{\mathrm{bn}}(G)$; such a broadcast is called an $\alpha_{\mathrm{bn}}(G)$-*broadcast*.

For example, the broadcast f in Figure 1 is bn-independent, because y is the only vertex that hears f from more than one vertex in V_f^+, and $y \in B_f(u) \cap B_f(v) \cap B_f(w)$. On the other hand, the broadcast g is not boundary independent because x hears g from u and v, but $x \notin B_g(u)$ (and $y \notin B_g(v)$); the edge xy is covered by u and by v.

Note that in a bn-independent broadcast, no broadcasting vertex u hears another broadcasting vertex v (otherwise an edge incident with u or v would be covered by both u and v), hence any bn-independent broadcast is also hearing independent.

Definition 2.2. [14, 16] A broadcast is *bnr-independent* if it is bn-independent and irredundant. The maximum weight of a bnr-independent broadcast is $\alpha_{\mathrm{bnr}}(G)$; such a broadcast is called an $\alpha_{\mathrm{bnr}}(G)$-*broadcast*

Definition 2.3. [14, 16] A broadcast is *bnd-independent* if it is minimal dominating and bn-indepen–dent. The maximum weight of a bnd-independent broadcast is $\alpha_{\text{bnd}}(G)$; such a broadcast is called an $\alpha_{\text{bnd}}(G)$*-broadcast*.

When the graph G is clear, an $\alpha_{\text{bn}}(G)$-broadcast is also called an α_{bn}-*broadcast*; the same comment holds for $\alpha_{\text{bnr}}(G)$ and $\alpha_{\text{bnd}}(G)$. Since the characteristic function of an independent set is a bnd-, bnr-, bn- and h-independent broadcast, it follows from Definitions 2.1 – 2.3 that

$$\alpha(G) \le \alpha_{\text{bnd}}(G) \le \alpha_{\text{bnr}}(G) \le \alpha_{\text{bn}}(G) \le \alpha_h(G) \tag{2.2}$$

for any graph G. As shown in [14, 16], if G is a 2-connected bipartite graph, then

$$\alpha(G) = \alpha_{\text{bnd}}(G) = \alpha_{\text{bnr}}(G) = \alpha_{\text{bn}}(G). \tag{2.3}$$

Denote the $m \times n$ grid by $G_{m,n}$. Bouchemakh and Zemir [7] showed that $\alpha_h(G_{5,5}) = 15 > \lceil \frac{25}{2} \rceil = \alpha(G_{5,5})$, hence the equality string (2.3) does not extend to α_h. If v is a peripheral vertex of a connected graph $G \ne K_1$, then broadcasting with a strength of $\text{diam}(G)$ from v and 0 from any other vertex results in a minimal dominating bn-independent broadcast, hence we also have

$$\alpha_{\text{bn}}(G) \ge \text{diam}(G) \text{ and } \Gamma_b(G) \ge \text{diam}(G)$$

for all nontrivial connected graphs G.

Suppose f is a bn-, bnr- or bnd-independent broadcast on G and an edge uv of G is covered by vertices $x, y \in V_f^+$. By the definition of covered, $\{u, v\} \nsubseteq B_f(x)$ and $\{u, v\} \subseteq N_f(x) \cap N_f(y)$. This violates the bn-independence of f. Hence we have the following property, a primary motivation for the definition of bn-independent broadcasts.

Observation 2.2. *If f is a bn-, bnr- or bnd-independent broadcast on a graph G, then each edge of G is covered by at most one vertex in V_f^+.*

Bouchemakh and Fergani [6] showed that if G is a graph of order n and minimum degree $\delta(G)$, then $\Gamma_b(G) \le n - \delta(G)$; the bound is sharp, for example for paths, stars and complete graphs. Mynhardt and Neilson [14] showed that $\alpha_{\text{bn}}(G) \le n-1$ for all graphs G, and that equality holds for a connected graph G if and only if G is a path or a generalized spider. Hence $\Gamma_b(G)$ and $\alpha_{\text{bn}}(G)$ never exceed the order of G. On the other hand, it follows from a result in [10] that $\alpha_h(\text{Sp}(r^k)) = k(2r-1)$. Since $\text{Sp}(r^k)$ has order $kr+1$, we see that there exist graphs whose h-independent broadcast number is almost double their order. One may thus regard α_{bn} as being "better behaved" than α_h.

A bn-independent broadcast f on G is *maximal bn-independent* if no broadcast g on G such that $g > f$ is bn-independent. The minimum weight of a maximal bn-independent broadcast on G is the *lower broadcast independence number*, denoted by $i_{\text{bn}}(G)$. This parameter was introduced by Neilson [16] and further investigated by Marchessault and Mynhardt [13].

The rest of the paper is organized as follows. We present previous results in Section 2.3. In Section 3 we show that $\alpha_{\rm bnr} \diamond \Gamma_b$ and $\alpha_{\rm bn} \diamond \Gamma_b$, that is, neither pair of parameters are comparable. In Section 4 we show that the differences $\Gamma_b - \alpha_{\rm bn}$ and $\Gamma_b - \alpha_{\rm bnr}$ can be arbitrary for general graphs, while $\Gamma_b - \alpha_{\rm bnr}$ can also be arbitrary for trees. In the other direction we show that $\alpha_{\rm bnr} - \Gamma_b$, hence also $\alpha_{\rm bn} - \Gamma_b$, can be arbitrary for trees as well as for cyclic graphs. We consider the ratios of these parameters in Section 5, showing that $\alpha_{\rm bnr}(G)/\Gamma_b(G) \leq \alpha_{\rm bn}(G)/\Gamma_b(G) < 2$ for all graphs, whereas $\Gamma_b(G)/\alpha_{\rm bnr}(G)$ and $\Gamma_b(G)/\alpha_{\rm bn}(G)$ are unbounded for general graphs, but $\Gamma_b(G)/\alpha_{\rm bn}(G) \leq \Gamma_b(G)/\alpha_{\rm bnr}(G) < 2$ for connected bipartite graphs. We conclude with questions for future consideration in Section 6.

2.3. **Known results.** In this subsection we present known results that will be used later on. It is often useful to know when a bn-independent broadcast is maximal bn-independent.

Proposition 2.3. (*i*) [14] *A bn-independent broadcast f on a graph G is maximal bn-independent if and only if (a) it is dominating, and (b) either $V_f^+ = \{v\}$ or $B_f(v) - {\rm PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$.*

(*ii*) [13] *Let f be a bn-independent broadcast on a connected graph G such that $|V_f^+| \geq 2$. Then f is maximal bn-independent if and only if each component of $G - U_f^E$ contains at least two broadcasting vertices.*

When f is a minimal dominating or a bnr-broadcast, ${\rm PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$, but when f is an h- or bn-independent broadcast, it is possible that ${\rm PB}_f(v) = \varnothing$. The next observations follow from (2.1) in Section 2.1 and Definitions 2.1 – 2.3 in Section 2.2; we state them here for referencing.

Observation 2.4. [14, 16] (*i*) *If f is a bn-, bnr- or h-independent broadcast and $v \in V_f^1$, then $v \in {\rm PB}_f(v)$.*

(*ii*) *If f is an h- or a bn-independent broadcast such that ${\rm PB}_f(v) = \varnothing$ for some $v \in V_f^+$, then $v \in V_f^{++}$ and $|V_f^+| \geq 2$.*

Let f be an $\alpha_{\rm bn}$-broadcast on a graph G and let T be a spanning tree of G. Removing the edges in $E(G) - E(T)$ does not affect bn-independence, hence f is also a bn-independent broadcast on T. This leads to the following observation.

Observation 2.5. *If T is a spanning tree of a graph G, then $\alpha_{\rm bn}(T) \geq \alpha_{\rm bn}(G)$.*

The above inequality can be strict, for example for K_n, $n \geq 3$, because $\alpha_{\rm bn}(K_n) = 1$ and $\alpha_{\rm bn}(T) \geq {\rm diam}(T) \geq 2$ for all trees of order at least 3.

When determining boundary independence numbers of trees, the following results from [**?**, 16], which we summarize in a single theorem, can be quite useful.

Theorem 2.6. *Consider a tree T.*

(*i*) [16, Lemma 2.3.12] *If f is an $\alpha_{\rm bn}(T)$-broadcast, then no leaf hears f from a non-leaf.*

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(*ii*) [16, Theorem 2.3.13] *There exists an $\alpha_{\rm bnr}(T)$-broadcast f such that no leaf hears f from a non-leaf.*

(*iii*) [16, Theorem 2.3.14] *There exists an $\alpha_{\rm bn}(T)$-broadcast f such that for all $v \in V^+$, $f(v) = 1$ or* $\deg(v) = 1$.

(*iv*) [16, Theorem 2.3.14] *There exists an $\alpha_{\rm bnr}(T)$-broadcast f such that for all $v \in V^+$, $f(v) = 1$ or* $\deg(v) = 1$.

(*v*) [16, Lemma 2.3.15] *If f is an $\alpha_{\rm bn}(T)$-broadcast such that $|V_f^1|$ is maximum, then* $\mathrm{PB}_f(v) = \varnothing$ *for all $v \in V_f^{++}$.*

We next summarize results on the above-mentioned parameters for spiders. Most of the proofs can be found elsewhere, as indicated, and we only prove (*ii*).

Proposition 2.7. (*i*) [14] *For any $r \geq 1$ and $k \geq 3$, $\alpha_{\rm bn}(\mathrm{Sp}(r^k)) = kr$.*

(*ii*) *For any $r \geq 2$ and $k \geq 3$, $\alpha_{\rm bnd}(\mathrm{Sp}(r^k)) = \alpha_{\rm bnr}(\mathrm{Sp}(r^k)) = k(r-1)+1$.*

(*iii*) [16, Proposition 2.3.16] *For any $k \geq 3$, $\Gamma_b(\mathrm{Sp}(2^k)) = k+1$.*

(*iv*) [10] *For any $r \geq 1$ and $k \geq 3$, $\alpha_h(\mathrm{Sp}(r^k)) = k(2r-1)$.*

Proof of (*ii*)**.** Let $S = \mathrm{Sp}(r^k)$. Denote the head of S by b and its legs by $L_1, \dots, L_k$. Say $L_i = b, v_{i,1}, \dots, v_{i,r-1}, \ell_i$, $i = 1, \dots, k$. Define the broadcast f_0 on S by $f_0(\ell_1) = r$, $f_0(\ell_i) = r-1$ for $2 \leq i \leq k$, and $f_0(x) = 0$ for all other vertices of S. Then $b \in \mathrm{PB}_{f_0}(\ell_1)$ while $b_i \in \mathrm{PB}_{f_0}(\ell_i)$ for $2 \leq i \leq k$, hence f_0 is irredundant. It is clear that f_0 is a dominating broadcast; since f_0 is also irredundant, it is minimal dominating. Since no vertex is dominated by more than one broadcasting vertex, f_0 is bn-independent. Therefore, f_0 is a bnr-broadcast and a bnd-broadcast, from which it follows that $\alpha_{\rm bnr}(S) \geq \alpha_{\rm bnd}(S) \geq \sigma(f_0) = k(r-1)+1$.

To prove the upper bound, let F be the set of all $\alpha_{\rm bnr}$-broadcasts f' such that no leaf hears f' from a non-leaf vertex. By Theorem 2.6(*ii*), F is nonempty. Among all broadcasts in F, let f be one having the fewest non-leaf broadcasting vertices.

Suppose f has at least one non-leaf broadcasting vertex. Among all of these, let u be one nearest to a leaf. Since the proof works the same for b and for any vertex $v_{i,j}$ for some $i \in \{1, \dots, k\}$ and some $j \in \{1, \dots, r-1\}$, we assume without loss of generality that $u = v_{1,\ldots,wherewetake}$j=0*if*u=b.*Since*$v_{1,j}$ not dominate ℓ_1, $f(\ell_1) > 0$. Say $f(\ell_1) = t$. Since $\mathrm{PB}_f(\ell_1) \neq \varnothing$ and $v_{1,j} \in V_f^+$, the vertex $v_{1,r-t}$ on L_1 belongs to $\mathrm{PB}_f(\ell_1)$. By the choice of $v_{1,j}$ as being a broadcasting vertex nearest to a leaf, either $v_{1,r-t-1}$ is undominated, or $v_{1,r-t-1} \in \mathrm{PB}_f(v_{1,j})$. Since $v_{1,j}$ also does not dominate ℓ_2, there exists a vertex $x \in B_f(v_{1,j})$ that does not lie on the $\ell_1 - v_{1,j}$ subpath of L_1; indeed, either (a) x lies on L_1 between $v_{1,j}$ and b, or (b) $x = b$ if $f(v_{1,j}) = d(v_{1,j}, \ldots,$ or (c) if $v_{1,j}$ overdominates b, then each leg L_i, $2 \leq i \leq k$, contains such a vertex x (all at the same distance from b).

Let $d = d(v_{1,r-t}, x)$ and note that $d \geq 2f(v_{1,j}) + 1 > f(v_{1,j}) + 1$. Define the broadcast g on S by

$$g(\ell_1) = f(\ell_1) + d - 1,\ g(v_{1,j}) = 0, \text{ and } g(v) = f(v) \text{ otherwise.}$$

Then $\sigma(g) = \sigma(f) - f(v_{1,j}) + d - 1 > \sigma(f)$. Moreover, since ℓ_1 does not broadcast to x, $N_g(\ell_1)$ is a proper subset of $N_f(\ell_1) \cup N_f(v_{1,j})$, whereas $N_g(v) = N_f(v)$ for all other vertices $v \in V_g^+$. This implies that $\mathrm{PB}_g(v) \neq \varnothing$ for each vertex in V_g^+; consequently, g is a bnr-broadcast. But $\sigma(g) > \sigma(f) = \alpha_{\mathrm{bnr}}(S)$, a contradiction. We conclude that V_f^+ consists of leaves.

Suppose some leaf ℓ overdominates b; assume without loss of generality that $\ell = \ell_1$. Say $f(\ell_1) = d(\ell_1, b) + t = r + t$ for some t such that $1 \leq t \leq r$. Since f is irredundant and only leaves are broadcasting vertices, it follows that $f(\ell_i) \leq r - t - 1$ for each $i \in \{2, \ldots, k\}$. Note that in this case, $\sigma(f) \leq r + t + (k-1)(r-t-1) = k(r-1) - t(k-2) + 1$. But then $\alpha_{\mathrm{bnr}}(S) = \sigma(f) < \sigma(f_0) = k(r-1) + 1$, where f_0 is the bnr-broadcast defined in the first part of the proof. This is impossible.

We deduce that no leaf overdominates b. Thus $f(\ell_i) \leq r$ for each i, and, since f is irredundant and $\mathrm{PB}_f(\ell_1) \neq \varnothing$ for each i, $f(\ell_i) = r$ for at most one i. Therefore, $\sigma(f) \leq k(r-1) + 1$. This proves that $\alpha_{\mathrm{bnr}}(S) = k(r-1) + 1$. Since $\alpha_{\mathrm{bnd}}(S) \leq \alpha_{\mathrm{bnr}}(S)$, the proof is complete. ■

By Proposition 2.7, the differences $\alpha_h - \alpha_{\mathrm{bn}}$, $\alpha_h - \alpha_{\mathrm{bnr}}$ and $\alpha_{\mathrm{bn}} - \alpha_{\mathrm{bnr}}$ can be arbitrary, because

$$\begin{aligned} \alpha_h(\mathrm{Sp}(r^k)) - \alpha_{\mathrm{bn}}(\mathrm{Sp}(r^k)) &= k(2r-1) - kr = k\,(r-1)\,, \\ \alpha_h(\mathrm{Sp}(r^k)) - \alpha_{\mathrm{bnr}}(\mathrm{Sp}(r^k)) &= k(2r-1) - (kr - k + 1) = kr - 1 \\ \text{and } \alpha_{\mathrm{bn}}(\mathrm{Sp}(r^k)) - \alpha_{\mathrm{bnr}}(\mathrm{Sp}(r^k)) &= kr - (kr - k + 1) = k - 1. \end{aligned}$$

Mynhardt and Neilson [14] proved that

$$\alpha_{\mathrm{bn}}(G)/\alpha_{\mathrm{bnr}}(G) < 2, \alpha_h(G)/\alpha_{\mathrm{bn}}(G) < 2 \text{ and } \alpha_h(G)/\alpha_{\mathrm{bnr}}(G) < 3$$

for any graph G, and used spiders to illustrate that all bounds are asymptotically best possible. We consider corresponding results for α_{bn} and α_{bnr} versus Γ_b in Sections 4 and 5.

3. Non-comparability of parameters

While it is clear from the definitions that $\alpha_{\mathrm{bnd}}(G) \leq \Gamma_b(G)$ for all graphs G, we demonstrate below that $\alpha_{\mathrm{bnr}} \diamond \Gamma_b$ and $\alpha_{\mathrm{bn}} \diamond \Gamma_b$.

As shown in [6], $\Gamma_b(G_{m,n}) = m(n-1)$ for $2 \leq m \leq n$. A minimal dominating broadcast with weight 6 of $G_{3,3}$ is depicted in Figure 2, where the square vertices belong to the private boundaries of the broadcasting vertices of the same colour (on the same vertical line when seen in monochrome). On the other hand, since grids are 2-connected bipartite graphs, (2.3) gives $\alpha(G_{3,3}) = \alpha_{\mathrm{bn}}(G_{3,3}) = \alpha_{\mathrm{bnr}}(G_{3,3}) = \lceil \frac{9}{2} \rceil = 5 < 6 = \Gamma_b(G_{3,3})$. For the path P_n, $n \geq 2$, $\alpha_{\mathrm{bnd}}(P_n) = \alpha_{\mathrm{bn}}(P_n) = \alpha_{\mathrm{bnr}}(P_n) = \Gamma_b(P_n) = n - 1$, where the results for $\alpha_{\mathrm{bnd}}(P_n)$, $\alpha_{\mathrm{bn}}(P_n)$ and $\alpha_{\mathrm{bnr}}(P_n)$ are stated in [16, Section 3.2], and the result for $\Gamma_b(P_n)$ follows from the bound of Bouchemakh and Fergani [6].

For an example of a graph for which $\alpha_{\mathrm{bn}} \geq \alpha_{\mathrm{bnr}} > \Gamma_b$, consider the graph in Figure 3. As shown in [16], $\alpha_{\mathrm{bnr}}(G) \geq 9$ (it can be shown that equality holds) and $\Gamma_b(G) = 7$. We omit the proofs here as

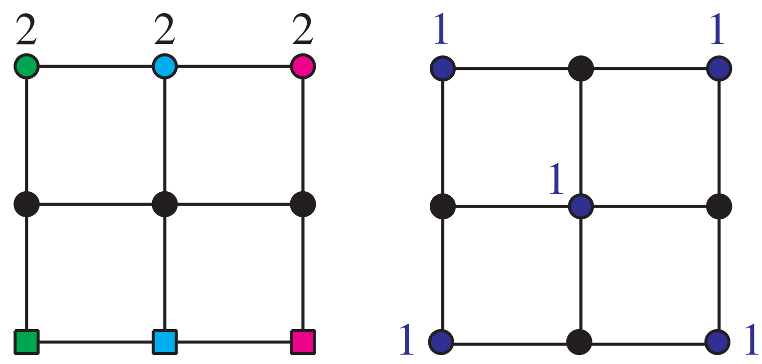

FIGURE 2. A Γ_b-broadcast of $G_{3,3}$ (left) of weight 6 and an α-set (right) of cardinality 5.

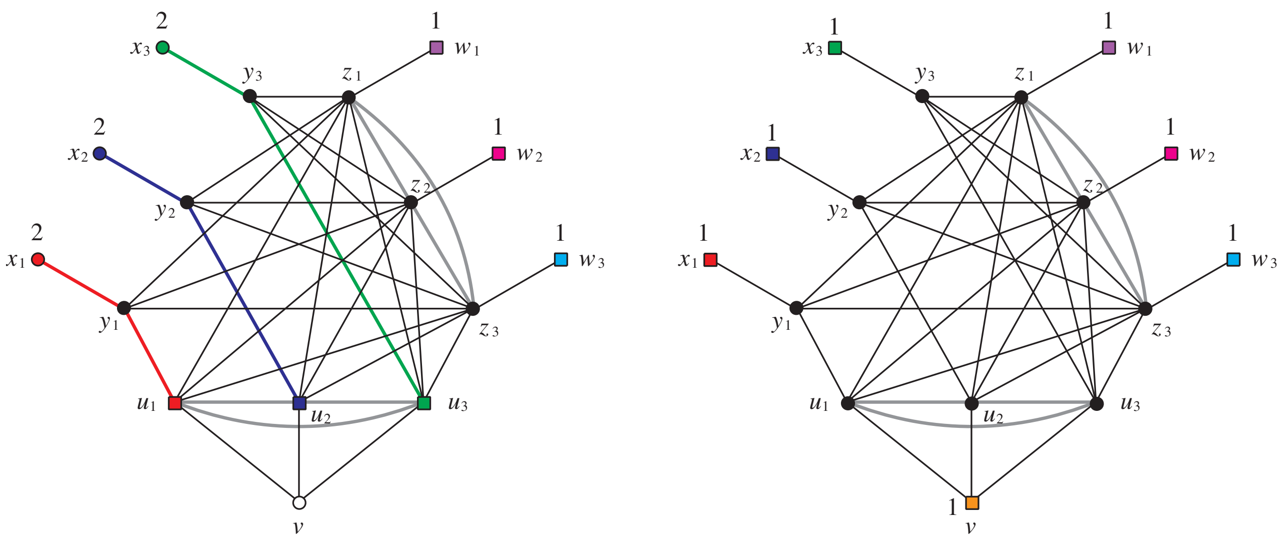

FIGURE 3. A graph G with $\alpha_{\rm bnr}(G) = 9$ and $\Gamma_b(G) = 7$. A non-dominating $\alpha_{\rm bnr}$-broadcast is shown on the left, and a Γ_b-broadcast on the right.

they are special cases of the proofs of Propositions 4.7 and 4.8 in Section 4.2. It follows that $\alpha_{\rm bnr} \diamond \Gamma_b$ and $\alpha_{\rm bn} \diamond \Gamma_b$.

4. Differences

Having demonstrated that $\alpha_{\rm bnr} \diamond \Gamma_b$ and $\alpha_{\rm bn} \diamond \Gamma_b$, we proceed to show that the differences $\Gamma_b - \alpha_{\rm bn}$ and $\Gamma_b - \alpha_{\rm bnr}$ can be arbitrary for general graphs G, while $\Gamma_b - \alpha_{\rm bnr}$ can also be arbitrary for trees. In the other direction we show that $\alpha_{\rm bnr} - \Gamma_b$, hence also $\alpha_{\rm bn} - \Gamma_b$, can be arbitrary for trees as well as for cyclic graphs.

4.1. $\Gamma_b(G) - \alpha_{\rm bn(bnr)}(G)$**.** In this subsection we show that $\Gamma_b(G) - \alpha_{\rm bn}(G)$ is unbounded. Since $\alpha_{\rm bn}(G) \geq \alpha_{\rm bnr}(G)$ it will follow that $\Gamma_b(G) - \alpha_{\rm bnr}(G)$ is unbounded for graphs in general. We also show that $\Gamma_b(T) - \alpha_{\rm bnr}(T)$ is unbounded for trees.

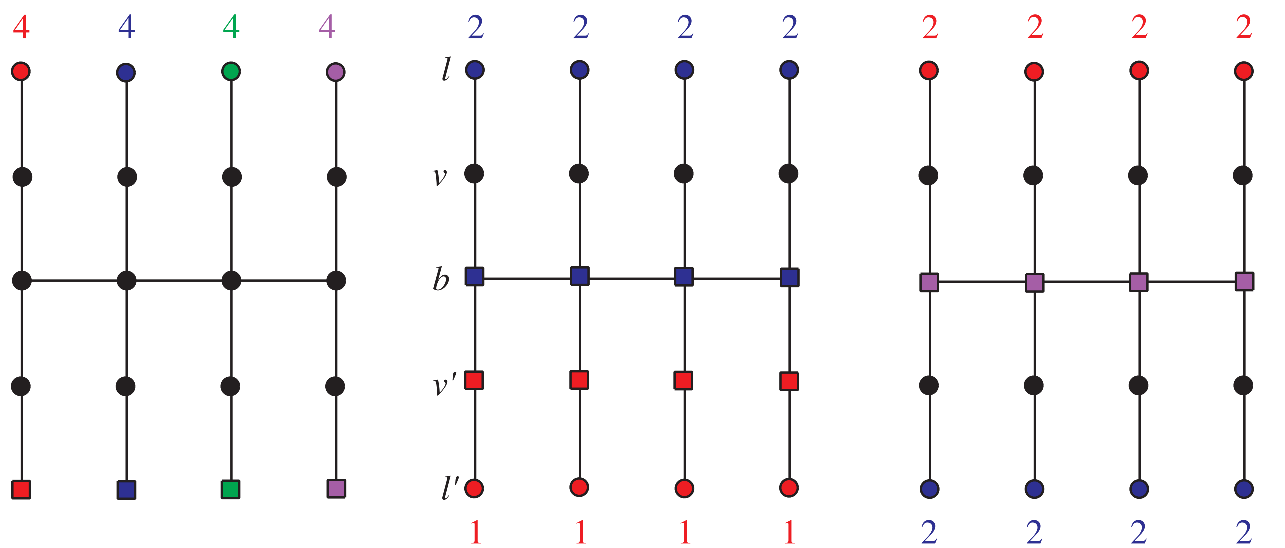

Figure 4. The tree T_4 with a minimal dominating broadcast on the left, an α_{bnr}-broadcast in the middle, and an α_{bn}-broadcast on the right.

For the grid $G_{m,n} = P_m \,\square\, P_n$, where $2 \leq m \leq n$, Bouchemakh and Fergani [6] showed that $\Gamma_b(G_{m,n}) = m(n-1)$. Hence $\Gamma_b(G_{3,k}) = 3(k-1)$ when $k \geq 3$. We use this fact in the proof of the next result.

Proposition 4.1. *For any integer $k \geq 3$, there exists a graph G_k such that $\Gamma_b(G_k) - \alpha_{\text{bn}}(G_k) \geq \lfloor \frac{3k-6}{2} \rfloor$.*

Proof. Let G_k be the grid $G_{3,k}$. Since grids are 2-connected bipartite graphs, $\alpha_{\text{bn}}(G_k) = \lceil \frac{3k}{2} \rceil$ by (2.3). As stated above, $\Gamma_b(G_{3,k}) = 3(k-1)$, and the result follows. ■

By Observation 2.5, $\alpha_{\text{bn}}(T) \geq \alpha_{\text{bn}}(G)$ when T is a spanning tree of G. Hence it is possible that $\Gamma_b(T) - \alpha_{\text{bn}}(T)$ is bounded. We leave this as an open question; see Question 2 in Section 6.

To show that $\Gamma_b(T) - \alpha_{\text{bnr}}(T)$ is unbounded for trees in general, we use the tree T_k described below and shown in Figure 4 for $k = 4$. To form T_k, take k copies $P_5^i = (l_i, v_i, b_i, v_i', l_i')$, $i = 1, \ldots, k$, of P_5 and add the edges $b_i b_{i+1}$, $i = 1, \ldots, k-1$. Define the broadcast f on T_k by $f(l_i) = 4$, $i = 1, \ldots, k$, and $f(x) = 0$ otherwise. Note that f is dominating and $\text{PB}_f(l_i) = \{l_i'\}$. Therefore, f is minimal dominating, so $\Gamma_b(T_k) \geq \sigma(f) = 4k$.

Proposition 4.2. *For $k \geq 2$, $\alpha_{\text{bnr}}(T_k) = 3k$, where T_k is the tree described above and shown in Figure 4 (for $k = 4$).*

Proof. For $1 \leq i \leq k$, let $A_i = V(P_5^i) = \{l_i, v_i, b_i, v_i', l_i'\}$. Define the broadcast g on T_k by $g(l_i) = 2$, $i = 1, \ldots, k$, $g(l_i') = 1$, $i = 1, \ldots, k$, and $g(x) = 0$ otherwise. Then g is bn-independent and dominating. Furthermore, $\text{PB}_g(l_i) = \{b_i\}$ and $\text{PB}_g(l_i') = \{l_i', v_i'\}$. Therefore g is bn-independent and irredundant, hence $\alpha_{\text{bnr}}(T_k) \geq \sigma(g) = 3k$.

We now show that $\alpha_{\text{bnr}}(T_k) \le 3k$. Suppose to the contrary that $\alpha_{\text{bnr}}(T_k) > 3k$. By Theorem 2.6($ii$), there exists an $\alpha_{\text{bnr}}(T_k)$-broadcast such that no leaf hears a non-leaf. Of all such broadcasts, let f be one such that $L = \{v : v \in V(T_k) \text{ and } f(v) \ge 4\}$ is a minimum. Let $M = \max\{f(v) : v \in V_f^+\}$. Assume by symmetry that $f(l_i) = \max\{f(l_i), f(l_i')\}$ for each i. Then $f(l_i') = \min\{f(l_i), f(l_i')\}$. Since (a) no leaf hears a non-leaf, (b) f is bn-independent, and (c) f is irredundant,

$$\text{(4.1)}\qquad \begin{array}{ll} \text{(a)} & f(v_i) = f(v_i') = 0 \text{ and } f(b_i) \le 1 \text{ for } 1 \le i \le k, \\ \text{(b)} & f(b_i) + f(b_{i+1}) \le 1 \text{ for } 1 \le i \le k-1, \\ \text{(c)} & f(l_i') \le 1 \text{ for } 1 \le i \le k. \end{array}$$

Moreover, the bn-independence of f also implies that

$$\text{(4.2)}\qquad \text{if } f(b_i) = 1 \text{ for some } i, \text{ then } f(l_i) = f(l_i') = 1.$$

Suppose $L = \varnothing$. If $M < 3$ then, by (4.1) and (4.2), $f(A_i) \le 3$ for all $1 \le i \le k$. But then $\sigma(f) \le 3k$, a contradiction. Hence assume that $M = 3$. We obtain a contradiction by showing that if $f(A_i) > 3$ for some i, then $f(A_1) + f(A_2) \le 6$ if $i = 1$, $f(A_{k-1}) + f(A_k) \le 6$ if $i = k$, and $f(A_{i-1}) + f(A_i) + f(A_{i+1}) \le 8$ otherwise.

By (4.1) and (4.2), if $1 \le f(l_i) \le 2$ for some i, then $f(A_i) \le 3$. Assume therefore that $f(l_i) = 3$. If $i = 1$, then $B_f(l_1) = \{v_1', b_2\}$ and $f(b_1) = f(b_2) = 0$. Now $f(l_1') \le 1$, otherwise $\text{PB}_f(l_1') = \varnothing$. Similarly, $f(l_2) \le 1$ and $f(l_2') \le 1$. Hence $f(A_1)+f(A_2) \le 6$. By symmetry, if $f(l_k) = 3$, then $f(A_k)+(A_{k-1}) \le 6$. If $f(l_i) = 3$ and $i \ne 1, k$, then $B_f(l_i) = \{b_{i-1}, v_i', b_{i+1}\}$, $f(l_{i-1}) = f(l_{i-1}') = f(l_i') = f(l_{i+1}) = f(l_{i+1}') = 1$ and $f(b_{i-1}) = f(b_i) = f(b_{i+1}) = 0$. Hence $f(A_{i-1}) + f(A_i) + f(A_{i+1}) = 8$. It follows that $\sigma(f) \le 3k$, contrary to our assumption.

Suppose $L \ne \varnothing$ and $M = 4$.

- If $k = 2$, then, without loss of generality, $f(l_1) = 4$. Then $B_f(l_1) = \{v_2', v_2, l_1'\}$ and, since f is bnr-independent, $f(l_2) = f(l_2') = 1$ and $f(b_1) = f(b_2) = f(l_1') = 0$. Hence $f(A_1) + f(A_2) = 6 = 3k$.
- If $k = 3$ and $f(l_2) = 4$, then $f(l_1) = f(l_1') = f(l_3) = f(l_3') = 1$ and $f(b_1) = f(b_2) = f(b_3) = f(l_2') = 0$, hence $\sigma(f) = 8 < 3k$.
- If $f(l_i) = 4$ and either $k = 3$ and $i \ne 2$, or $k > 3$, then, without loss of generality, $\{l_i', v_{i+1}, v_{i+1}', b_{i+2}\} \subseteq B_f(l_i)$ and $f(l_{i+1}) = f(l_{i+1}') = 1$ and $f(b_i) = f(b_{i+1}) = f(b_{i+2}) = 0$. Create a new broadcast g with $g(l_i) = 3$, $g(l_i') = 1$ and $g(x) = f(x)$ otherwise. Notice that $b_{i+1} \in \text{PB}_g(l_i)$. Since $g(l_i') = 1$, $\text{PB}_g(l_i') = \{l_i'\}$. Also, $N_g(l_i) \cup N_g(l_i') \subseteq N_f(l_i)$. Hence, g is a bnr-independent broadcast. Since $\sigma(f) = \sigma(g)$, either g can be extended and violates the maximality of f,..., since it has fewer vertices broadcasting with strength 4, it violates the choice of f.

Assume that $M \ge 5$. Let l_i be a vertex such that $f(l_i) = M$. Since $f(l_i) \le e(l_i)$ and by the structure of T_k, there are two leaves l_t, l_t', such that $d(l_t, l_i) = d(l_t', l_i) = M$. Assume by symmetry that $t > i$.

Since $d(l_i, l_{i+1}) = 5$, $t = (i+1)+(M-5) = i+M-4$. Create a new broadcast g with $g(l_j) = 2$, $g(l'_j) = 1$ and $g(b_j) = 0$ for all $i \le j \le t$, and $g(x) = f(x)$ otherwise. Notice that $\bigcup_{j=i}^{t}(N_g(l_j) \cup N_g(l'_j)) \subset N_f(l_i)$. For $i \le j \le t$, $b_j \in \mathrm{PB}_g(l_j)$ and $l'_j \in \mathrm{PB}_g(l'_j)$. Hence g is a bnr-independent broadcast with

$$\sigma(g) = \sigma(f) - M + 3(t - i + 1) = \sigma(f) - M + 3(M - 3) = \sigma(f) + 2M - 9.$$

Since $M \ge 5$, $\sigma(g) > \sigma(f)$ and g violates the maximality of f. We conclude that $\alpha_{\mathrm{bnr}}(T_k) \le 3k$. ■

Since $\Gamma_b(T_k) \ge 4k$, the following result is an immediate consequence of Proposition 4.2.

Theorem 4.3. *For any integer $k \ge 1$ there exists a tree T such that $\Gamma_b(T) - \alpha_{\mathrm{bnr}}(T) \ge k$.*

4.2. $\alpha_{\mathrm{bnr(bn)}}(G) - \Gamma_b(G)$**.** We next consider the differences $\alpha_{\mathrm{bn}} - \Gamma_b$ and $\alpha_{\mathrm{bnr}} - \Gamma_b$ for both trees and cyclic graphs. By Proposition 2.7, when $k \ge 3$,

$$\alpha_{\mathrm{bn}}(\mathrm{Sp}(2^k)) = 2k \text{ and } \alpha_{\mathrm{bnd}}(\mathrm{Sp}(2^k)) = \alpha_{\mathrm{bnr}}(\mathrm{Sp}(2^k)) = \Gamma_b(\mathrm{Sp}(2^k)) = k + 1.$$

Therefore $\alpha_{\mathrm{bn}}(\mathrm{Sp}(r^k)) - \Gamma_b(\mathrm{Sp}(r^k)) = k-1$, and it follows that the difference $\alpha_{\mathrm{bn}} - \Gamma_b$ can be arbitrary for trees. We strengthen this result by constructing a tree H_k such that $\alpha_{\mathrm{bnr}}(H_k) - \Gamma_b(H_k) \ge k$.

If a tree T has an α_{bnr}-broadcast which is dominating, then $\alpha_{\mathrm{bnr}}(T) \le \Gamma_b(T)$. However, not all trees have such a broadcast and there exist trees such that $\alpha_{\mathrm{bnr}}(T) > \Gamma_b(T)$. Figure 5 gives an example of a bnr-independent broadcast on a tree T which is not dominating. By using symmetry and examining a few cases, it can be shown that $\alpha_{\mathrm{bnr}}(T) = 14$ and $\Gamma_b(T) = 13$. We use this tree as a basis to construct bigger trees H_k mentioned above.

To construct H_k, take $3k$ copies $T_1, \ldots, T_{3k}$ of the tree in Figure 5 and label the central vertex and its neighbours in the i^{th} copy as u_i, v_i, w_i. Let H_k be the tree formed by joining v_i to v_{i+1} for each $i = 1, \ldots, 3k-1$. For $i = 1, \ldots, 3k$, let f_i be the broadcast on T_i illustrated in the top copy of T in Figure 5, and define $f = \bigcup_{i=1}^{3k} f_i$. Then f is a bnr-broadcast, hence $\alpha_{\mathrm{bnr}}(H_k) \ge 42k$. We show that $\Gamma_b(H_k) = 41k$.

Proposition 4.4. *For $k \ge 1$, $\Gamma_b(H_k) = 41k$, where H_k is the tree described above.*

Proof. Let f_i, g_i and h_i be the broadcasts on T_i illustrated in the top, middle and bottom copy, respectively, of T in Figure 5. Define the broadcast λ on H_k by

$$\lambda(x) = \begin{cases} g_i(x) & \text{if } x \in V(T_i) \text{ and } i \equiv 2 \pmod 3 \\ f_i(x) & \text{otherwise.} \end{cases}$$

Since $\sigma(f_i) = 14$ and $\sigma(g_i) = 13$, $\sigma(\lambda) = 28k + 13k = 41k$. It is easy to see that λ is a dominating broadcast. Suppose $i \equiv 2 \pmod 3$. Then $\mathrm{PB}_\lambda(v_i) = \{u_i, v_i, w_i, v_{i-1}, v_{i+1}\}$, and if ℓ is a leaf, then $\mathrm{PB}_\lambda(\ell)$ consists of the vertex at distance 2 from ℓ. Suppose $i \equiv 0$ or $1 \pmod 3$. If $\lambda(\ell) = 3$, then $\mathrm{PB}_g(\ell) = \{u_i\}$ or $\{w_i\}$, as the case may be, and if $\lambda(\ell) = 2$, then, as before, $\mathrm{PB}_\lambda(\ell)$ consists of

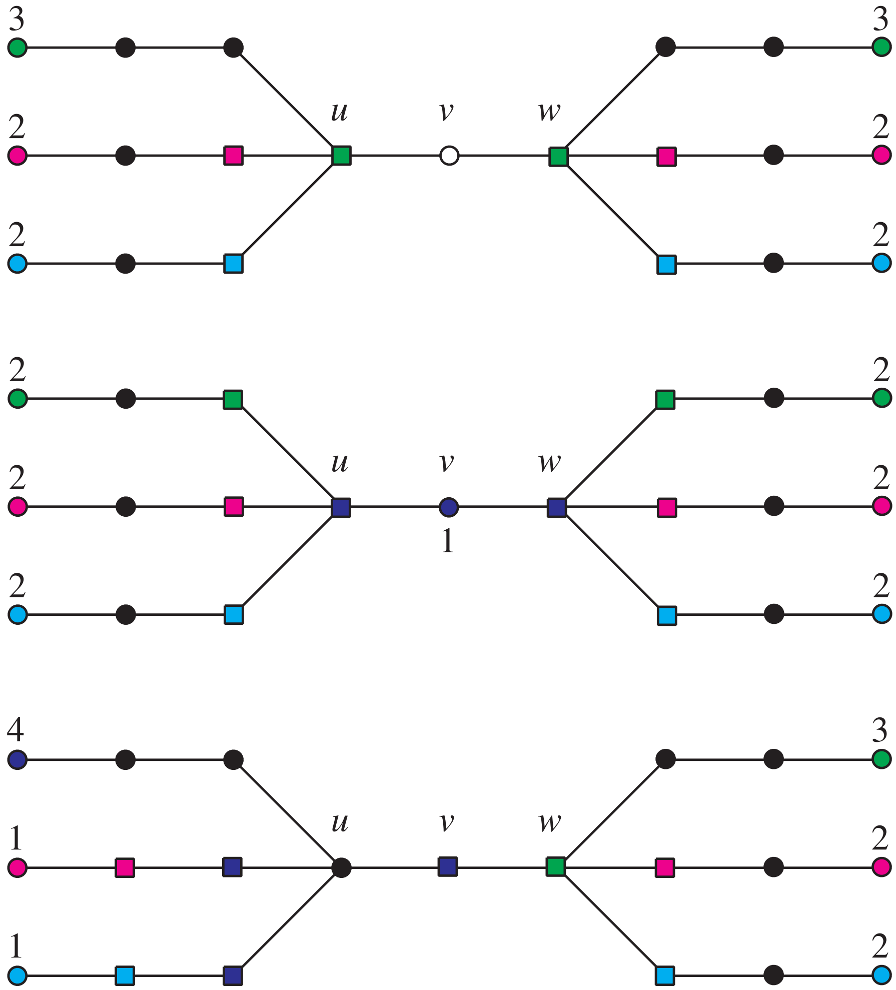

FIGURE 5. A tree T with a non-dominating α_{bnr}-broadcast (top) and two Γ_b-broadcasts (middle and bottom).

the vertex at distance 2 from ℓ. Hence λ is a minimal dominating broadcast and we deduce that $\Gamma_b(H_k) \geq 41k$. We show that $\Gamma_b(H_k) = 41k$.

Among all $\Gamma_b(H_k)$-broadcasts on H_k, let ρ be one such that V_ρ^+ contains the maximum number of leaves. For $i = 1, \dots, 3k$, define the function $\rho_i : V(T_i) \to \{1, \dots, \text{diam}(H_k)\}$ by $\rho_i(x) = \rho(x)$ for each $x \in V(T_i)$. (Possibly, ρ_i is not, strictly speaking, a broadcast on T_i because $\rho_i(x)$ may exceed the eccentricity of x in T_i.) If each v_i is dominated only by a vertex of T_i, then ρ_i is a broadcast on T_i and, as in the case of T, $\sigma(\rho_i) \leq 13$ for each i, so that $\sigma(\rho) \leq 39k < 41k$. Hence assume a vertex v_i is dominated by a vertex $x \in V(T_j)$, $j \neq i$.

Claim 4.5. *If $x \in V(T_j)$ dominates v_i, where $j \neq i$, then x does not overdominate v_i.*

Proof of Claim 4.5 Say x overdominates v_i by exactly t, where $t \geq 0$, that is, $\rho(x) = t + d(x, v_i)$. The possible values of $\sigma(\rho_i)$ and $\sigma(\rho_j)$ are tabled below.

t	0	1	2	3	≥ 4
$\sigma(\rho_i)$	≤ 14	≤ 12	≤ 6	≤ 6	0
$\sigma(\rho_j)$	$\leq 12 + \rho(x)$	$\leq 6 + \rho(x)$	$\leq 6 + \rho(x)$	$\rho(x)$	$\rho(x)$

Suppose $t > 0$. Let j be the smallest index such that T_j contains a vertex x which overdominates a vertex v_l, $l \neq j$. By symmetry we may assume that $j \leq \lceil \frac{3k}{2} \rceil$. We show that x overdominates some vertex v_i by exactly 1.

- Suppose x overdominates v_{3k} by at least 4. Since $j \leq \lceil \frac{3k}{2} \rceil$, x overdominates each v_i, $i = 1, \dots, 3k$, by at least 4. Then, regardless of the value of j, x dominates H_k and $\sigma(\rho) = e(x) \leq \text{diam}(H_k) = 3k + 7 < 41k$.
- Suppose x overdominates v_{3k} by 3.
  - If k is odd and $j = \lceil \frac{3k}{2} \rceil$, then $\rho(x) \leq \lfloor \frac{3k}{2} \rfloor + 7$ and x dominates all of H_k except for the twelve leaves of T_1 and T_{3k}, so $\sigma(\rho) \leq \lfloor \frac{3k}{2} \rfloor + 19 < 41k$ for all k.
  - In all other cases, $\rho(x) \leq \text{diam}(H_k) - 1$ and x dominates all of H_k except for six leaves, so $\sigma(\rho) \leq 3k + 12 < 41k$ for all k.
- Similarly, if x overdominates v_{3k} by 2, we obtain that $\sigma(\rho) \leq \text{diam}(H_k) - 2 + 24 = 3k + 29 < 41k$ for all k.

In each case, $\sigma(\rho) < \sigma(\lambda)$ and we have a contradiction of the assumption that ρ is a $\Gamma_b(H_k)$-broadcast. We deduce that x does not overdominate v_{3k} by more than 1, that is, $\rho(x) \leq d(x, v_{3k}) + 1$. Say $\rho(x) = d(x, v_{3k}) + 1 - \beta$, for some β such that $0 \leq \beta \leq 3k - j - 1$. That is, $\rho(x) = 3k - j - \beta + 1$. Observe that if $\beta = 0$, then x overdominates v_{3k} by exactly 1, if $\beta = 1$, then x overdominates v_{3k-1} by exactly 1, etc., and if $\beta = 3k - j - 1$, then x overdominates v_{j+1} by exactly 1. By symmetry and the choice of j, it is also possible – but not necessary – that x overdominates some vertex $v_{i'}$ by exactly 1, where $i' < j$. Since x overdominates at least one v_i by exactly 1, there exists a smallest index $i \neq j$ such that x overdominates v_i by exactly 1. We consider two cases, depending on the value of i relative to j.

Case 1: $i < j$. Since $j \leq \lceil \frac{3k}{2} \rceil$, by symmetry x also overdominates v_{2j-i} by exactly 1. Then $\sigma(\rho_i), \sigma(\rho_{2j-i}) \leq 12$. If the indices $i - 1$ and $2j - i + 1$ exist, x dominates v_{i-1} and v_{2j-i+1}. In this case, $\sigma(\rho_{i-1}), \sigma(\rho_{2j-i+1}) \leq 14$. It is possible that one or both of v_{i-1} and v_{2j-i+1} hear ρ from vertices different from x. Define the broadcast η on H_k by

$$\eta(y) = \begin{cases} h_\lambda(y) & \text{if } y \in V(T_\lambda) \text{ and } \lambda \in \{i-1, 2j-i+1\} \\ f_\lambda(y) & \text{if } y \in V(T_\lambda) \text{ and } \lambda \in \{i, 2j-i\} \\ \rho(y) - 1 & \text{if } y = x \\ \rho(y) & \text{otherwise.} \end{cases}$$

Then

$$\sigma(\eta) = \sigma(\rho) - [\sigma(\rho_i) + \sigma(\rho_{2j-i}) + \sigma(\rho_{i-1}) + \sigma(\rho_{2j-i+1})] + [\sigma(f_i) + \sigma(f_{2j-i}) + \sigma(h_{i-1}) + \sigma(h_{2j-i+1})] - 1.$$

Since $\sigma(f_i) - \sigma(\rho_i) \geq 2$ and $\sigma(\rho_{i-1}) - \sigma(h_{i-1}) \leq 1$ (and similarly for the other indices), we see that $\sigma(\eta) > \sigma(\rho)$. Note that $v_i \in \mathrm{PB}_\eta(x)$. The private boundaries of the vertices in $V(T_{i-1}) \cap V_\eta^+$ and $V(T_{2j-i+1}) \cap V_\eta^+$ are as shown in Figure 5 but possibly excluding v_{i-1} or v_{2j-i+1}. Hence η is bnr-independent. All vertices v_l are dominated, so all undominated vertices lie on endpaths. Since x ρ-dominates these vertices and $v_i \in \mathrm{PB}_\eta(x)$, these vertices can be dominated by extending η on an appropriate leaf to get a minimal dominating broadcast η^* such that $\sigma(\rho) < \sigma(\eta^*)$. But again, this contradicts ρ being a $\Gamma_b(H_k)$-broadcast.

Case 2: $i > j$. By the choice of i, x overdominates each v_l, $1 \leq l < i$, by at least 2. Now $\sigma(\rho_i) \leq 12$ and $\sigma(\rho_{i-1}) = 6$; moreover, if $i < 3k$, then $\sigma(\rho_{i+1}) \leq 14$. Consider an endpath P in T_{i-1} from u_{i-1} to a leaf ℓ, say $P = (u_{i-1}, \ldots, \ell)$. Then $a \in B_\rho(x)$. If $a \notin \mathrm{PB}_\rho(x)$, then $\rho(c) = 1$ and thus $\rho(\ell) = 0$. But then $\rho' = (\rho - \{(c,1),(\ell,0)\}) \cup \{(c,0),(\ell,1)\}$ is a $\Gamma_b(H_k)$-broadcast such that $V_{\rho'}^+$ contains more leaves than V_ρ^+ does, contrary to the choice of ρ. We deduce that $a \in \mathrm{PB}_\rho(x)$ and $\rho(\ell) = 1$. Define the broadcast η' on H_k by

$$\eta'(y) = \begin{cases} h_{i+1}(y) & \text{if } y \in V(T_{i+1}) \\ f_i(y) & \text{if } y \in V(T_i) \\ \rho(y) - 1 & \text{if } y = x \\ \rho(y) & \text{otherwise.} \end{cases}$$

Then

$$\sigma(\eta') = \sigma(\rho) - [\sigma(\rho_i) + \sigma(\rho_{i+1})] + [\sigma(f_i) + \sigma(h_{i+1})] - 1.$$

As above, $\sigma(f_i) - \sigma(\rho_i) \geq 2$ and $\sigma(\rho_{i+1}) - \sigma(h_{i+1}) \leq 1$, hence $\sigma(\rho) \leq \sigma(\eta')$. Moreover, since $a \in \mathrm{PB}_\rho(x)$, η' is not dominating. Following the reasoning above we can extend η' on appropriate leaves (including ℓ) to get a minimal dominating broadcast η'^* such that $\sigma(\rho) < \sigma(\eta'^*)$, again a contradiction. ♦

Therefore, if $x \in V(T_j)$ dominates v_i for $i \neq j$, then $i = j \pm 1$ and $\{v_{j-1}, v_{j+1}\} \subseteq B_\rho(x)$. (Assume $j-1$ and $j+1$ both exist; the proof is the same if only one of them exists.) Then $\sigma(\rho_j) \leq 13$ and $\sigma(\rho_{j-1}), \sigma(\rho_{j+1}) \leq 14$. Consequently,

$$\sigma(\rho_l) \leq \begin{cases} 13 & \text{if some vertex of } T_l \text{ dominates a vertex of } T_i,\ i \neq l \\ 14 & \text{otherwise.} \end{cases}$$

Since each v_i is dominated and the subtree of T induced by $\{v_1, \ldots, v_{3k}\}$ is the path P_{3k}, there are at least $\gamma(P_{3k}) = k$ indices j such that some vertex of T_j dominates v_i, $i \neq j$. This implies that $\sigma(\rho) \leq 13k + 14 \cdot 2k = 41k$ and we conclude that $\Gamma_b(H_k) = 41k$. ■

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Since $\alpha_{\rm bnr}(H_k) \geq 42k$ and $\Gamma_b(H_k) = 41k$, the next theorem follows.

Theorem 4.6. *For any integer $k \geq 1$ there exists a tree H_k such that $\alpha_{\rm bn}(H_k) - \Gamma_b(H_k) \geq \alpha_{\rm bnr}(H_k) - \Gamma_b(H_k) \geq k$.*

By Observation 2.5, $\alpha_{\rm bn}(G) \leq \alpha_{\rm bn}(T)$ if T is a spanning tree of G. Therefore it is possible that $\alpha_{\rm bn}(G) - \Gamma_b(G)$ is bounded for cyclic graphs. Again, we use the unboundedness of $\alpha_{\rm bnr}(G) - \Gamma_b(G)$ to show that this is not the case. To show that $\alpha_{\rm bnr}(G) - \Gamma_b(G)$ is unbounded, we generalize the construction of the graph G in Figure 3.

Denote the corona of G and K_1 by $G \circ K_1$. For $k \geq 1$, construct the graph G_k as follows. Let $U = \{u_1, \ldots, u_{k+1}\}$, $W = \{w_1, \ldots, w_{k+1}\}$, $X = \{x_1, \ldots, x_{k+1}\}$, $Y = \{y_1, \ldots, y_{k+1}\}$, $Z = \{z_1, \ldots, z_{k+1}\}$ and $\{v\}$ be disjoint sets of vertices. Add edges so that

$$G_k[X] \cong G_k[Y] \cong G_k[W] \cong \overline{K_{k+1}},$$

$$G_k[U \cup \{v\}] \cong K_{k+2},\ G_k[Z] \cong K_{k+1},\ G_k[U \cup Z] \cong K_{2(k+1)},$$

$$G_k[\{y_i\} \cup Z] \cong K_{k+2} \text{ for each } i \in \{1, \ldots, k+1\},$$

$$G_k[Y \cup U] \cong G_k[W \cup Z] \cong K_{k+1} \circ K_1,$$

$$G_k[X \cup Y] \cong \overline{K_{k+1}} \,\square\, K_2 \cong (k+1)K_2.$$

Assume that the perfect matchings of $G_k[U \cup Y]$, $G_k[X \cup Y]$ and $G_k[W \cup Z]$ are $\{u_i y_i : i = 1, \ldots, k+1\}$, $\{x_i y_i : i = 1, \ldots, k+1\}$ and $\{w_i z_i : i = 1, \ldots, k+1\}$, respectively. The graph G_2 is illustrated in Figure 3.

Proposition 4.7. *Let G_k be the graph described above and shown in Figure 3 for $k = 2$. For any integer $k \geq 1$, $\alpha_{\rm bnr}(G_k) \geq 3(k+1)$.*

Proof. Define the broadcast f by

$$f(x) = \begin{cases} 2 & \text{if } x \in X \\ 1 & \text{if } x \in W \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sigma(f) = 3(k+1)$ and

$$\begin{aligned} N_f(x_i) &= \{x_i, y_i, u_i\} \cup Z \\ B_f(x_i) &= \{u_i\} \cup Z \\ \mathrm{PB}_f(x_i) &= \{u_i\} \\ N_f(w_i) &= \{w_i, z_i\} \\ \mathrm{PB}_f(w_i) &= \{w_i\} \end{aligned}$$

for each $i \in \{1, \ldots, k+1$. Thus we see that f is a bnr-independent broadcast, hence $\alpha_{\rm bnr}(G_k) \geq 3(k+1)$. ■

Proposition 4.8. *Let G_k be the graph described above and shown in Figure 3 for $k = 2$. For any integer $k \geq 1$, $\Gamma_b(G_k) = 2k + 3$.*

Proof. The set $X \cup W \cup \{v\}$ is an independent dominating set of G_k of cardinality $2k + 3$, and its characteristic function is a minimal dominating broadcast. Hence $\Gamma_b(G_k) \geq 2k + 3$.

Consider any minimal dominating broadcast f on G_k. By symmetry, there are six possible ways to dominate the vertex v.

Case 1: v dominates itself. If $f(v) = 3 = e(v)$, then $N_f(v) = V(G_k)$, hence $\sigma(f) = 3$. Suppose $f(v) = 2$. Then $N_f(v) = V(G_k) - X - W$. Since f is irredundant there exists an index l such that $\{y_l, z_l\} \cap \mathrm{PB}_f(v) \neq \varnothing$. Hence either $f(Z) = f(y_l) = f(x_l) = f(u_l) = 0$ (so that $y_l \in \mathrm{PB}_f(v)$) or $f(U \cup Y) = f(z_l) = f(w_l) = 0$ (so that $z_l \in \mathrm{PB}_f(v)$) or both. But then x_l or w_l is not f-dominated, a contradiction. If $f(v) = 1$, then $f(U) = 0$. In addition, $f(z_j) + f(w_j) \leq 1$ and $f(x_j) + f(y_j) \leq 1$ for all $1 \leq j \leq k + 1$. Hence $\sigma(f) \leq 2(k + 1) + 1 = 2k + 3$.

Case 2: v is dominated by u_l for some l. If $f(u_l) = 3 = e(u_l)$, then u_l dominates G_k and $\sigma(f) = 3$. Suppose $f(u_l) = 2$. Then $N_f(u_l) = (V(G_k) - X) \cup \{x_l\}$. Since f is irredundant, $f(v) = 0$ and $f(z_i) = f(w_i) = 0$ for all $1 \leq i \leq k + 1$. Moreover, $f(u_i) = 0$ or $f(u_i) = 2$ for $i \neq l$. Let $U = \{u_i : f(u_i) = 2\}$. For all $1 \leq j \leq k + 1$, if $u_j \in U$ then $f(x_j) = 0$, and if $u_j \notin U$ then $f(x_j) + f(y_j) \leq 1$. Hence $\sigma(f) \leq 2|U| + k + 1 - |U| = |U| + k + 1 \leq 2k + 2$. Suppose $f(u_l) = 1$. Then $N_f(u_l) = U \cup Z \cup \{v, y_l\}$. If $f(u_{l'}) = 1$ for $l' \neq l$, then $\mathrm{PB}_f(u_l) = \{y_l\}$ and x_l is not f-dominated, a contradiction. Hence $f(U) = 1$. Moreover, $f(z_j) + f(w_j) \leq 1$ and $f(x_j) + f(y_j) \leq 1$ for all $1 \leq j \leq k+1$. Hence $\sigma(f) \leq 2(k + 1) + 1 = 2k + 3$.

Cases 3–4: v is dominated by $s \in \{x_l, y_l\}$ for some l, $1 \leq l \leq k + 1$. In either case $N_f(s) = (V(G_k) - X) \cup \{x_l\}$. By irredundance, $f(x_i) \leq 1$ for $i \neq l$, $f(s) \leq e(s)$ and $f(x) = 0$ otherwise. Hence $\sigma(f) \leq f(s) + k \leq 4 + k \leq 2k + 3$.

Cases 5–6: v is dominated by $s \in \{z_l, w_l\}$ for some l, $1 \leq l \leq k + 1$. Since $d(s, v) = e(s)$, $N_f(s) = V(G_k)$. Hence $\sigma(f) = f(s) = e(s) \leq 3$.

This exhausts all possibilities, hence $\Gamma_b(G_k) \leq 2k + 3$. ■

Propositions 4.7 and 4.8 imply the following theorem.

Theorem 4.9. *For any integer $k \geq 1$ there exists a cyclic graph G_k such that $\alpha_{\mathrm{bn}}(G_k) - \Gamma_b(G_k) \geq \alpha_{\mathrm{bnr}}(G_k) - \Gamma_b(G_k) \geq k$.*

5. Ratios

5.1. $\alpha_{\mathrm{bnr(bn)}}(G)/\Gamma_b(G)$**.** We show that the ratios $\alpha_{\mathrm{bn}}(G)/\Gamma_b(G)$ and $\alpha_{\mathrm{bnr}}(G)/\Gamma_b(G)$ are bounded. We need the following lemma.

Lemma 5.1. *Let f be an α_{bn}-broadcast on a graph G such that $\mathrm{PB}_f(v) = \varnothing$ for some $v \in V_f^+$. The broadcast g on G defined by $g(v) = f(v) - 1$ and $g(x) = f(x)$ otherwise, is a dominating bn-independent broadcast, $v \in V_g^+$ and $B_g(v) = \mathrm{PB}_g(v) \neq \varnothing$.*

Proof. By Observation 2.4, $v \in V_f^{++}$, hence $v \in V_g^+$. If some vertex u of G is g-undominated, then $u \in \mathrm{PB}_f(v)$, a contradiction because $\mathrm{PB}_f(v) = \varnothing$. Hence g is dominating. Since $f(v) \le e(v)$, $B_f(v) \neq \varnothing$. Therefore $B_g(v) \neq \varnothing$. Consider any $u \in B_g(v)$ and suppose u also hears g from $w \in V_g^+ - \{v\}$. Let x be any neighbour of u on a $u-w$ geodesic. Then x hears f from v, which means that ux is f-covered by both v and w, which is impossible because f is bn-independent. Therefore $u \in \mathrm{PB}_g(v)$. It follows that $B_g(v) = \mathrm{PB}_g(v) \neq \varnothing$. ■

Theorem 5.2. *For any graph G,*

$$\alpha_{\mathrm{bnr}}(G)/\Gamma_b(G) \le \alpha_{\mathrm{bn}}(G)/\Gamma_b(G) < 2.$$

The bound for $\alpha_{\mathrm{bn}}(G)/\Gamma_b(G)$ is asymptotically best possible.

Proof. If there exists an $\alpha_{\mathrm{bn}}(G)$-broadcast that is minimal dominating, then $\alpha_{\mathrm{bn}}(G) \le \Gamma_b$. Hence we assume that no such broadcast exists. Every α_{bn}-broadcast is dominating, and an irredundant dominating broadcast is minimal dominating. Thus we may also assume that no α_{bn}-broadcast on G is irredundant. We describe a strategy for turning a non-irredundant dominating α_{bn}-broadcast into an irredundant dominating broadcast with weight large enough to achieve the desired result.

Consider an α_{bn}-broadcast f on G. Since f is not irredundant, there exists a vertex $v_1 \in V_f^+$ such that $\mathrm{PB}_f(v_1) = \varnothing$. By Observation 2.4, $v_1 \in V_f^{++}$ and $|V_f^+| \ge 2$. Define the broadcast f_1 by

$$f_1(v_1) = f(v_1) - 1 \text{ and } f_1(x) = f(x) \text{ otherwise.}$$

By Lemma 5.1, f_1 is a dominating bn-independent broadcast. Notice that $V_{f_1}^+ = V_f^+$. Since $f(v_1) \le e(v_1)$, $B_f(v_1) \neq \varnothing$. Therefore the set $B_{f_1}(v_1) = \{v : d(v, v_1) = f(v_1) - 1\}$ is also nonempty. By the bn-independence of f and the definition of f_1, $\mathrm{PB}_{f_1}(v_1) = B_{f_1}(v_1) \neq \varnothing$. Moreover, for each $u \in V_f^+ - \{v_1\}$, $\mathrm{PB}_f(u) \subseteq \mathrm{PB}_{f_1}(u)$, so if $\mathrm{PB}_f(u) \neq \varnothing$, then $\mathrm{PB}_{f_1}(u) \neq \varnothing$. We conclude that $V_{f_1}^+$ contains more vertices with non-empty private boundaries than V_f^+ does.

If f_1 is not irredundant, we repeat this process, choosing a vertex $v_2 \in V_{f_1}^{++} - \{v_1\}$ having $\mathrm{PB}_{f_1}(v_2) = \varnothing$, until we have a smallest index $k \ge 1$ such that f_k is a dominating irredundant broadcast. Then $\sigma(f_k) \le \Gamma_b(G)$. We show that $\sigma(f_k) > \frac{1}{2}\sigma(f)$.

Clearly, $\sigma(f) \ge |V_f^1| + 2|V_f^{++}|$, hence $|V_f^{++}| \le \frac{1}{2}\sigma(f)$.

- If $V_f^1 \neq \varnothing$, then $|V_f^{++}| < \frac{1}{2}\sigma(f)$. Since $k \le |V_f^{++}|$,

$$\sigma(f_k) = \sigma(f) - k \ge \sigma(f) - |V_f^{++}| > \frac{1}{2}\sigma(f). \tag{5.1}$$

- Assume therefore that $V_f^1 = \varnothing$. In what follows, if $k = 1$, we take f_{k-1} to be f and ignore the reference to v_{k-1}. By the construction of the broadcasts f_i, $i = 1, \ldots, k-1$, $\mathrm{PB}_{f_i}(v_i) = B_{f_i}(v_i) \neq \varnothing$. Indeed, we also have that

$$\mathrm{PB}_{f_i}(v_j) = B_{f_i}(v_j) = \mathrm{PB}_{f_j}(v_j) \neq \varnothing \text{ for each } j \text{ such that } 1 \leq j \leq i. \tag{5.2}$$

Now, $\mathrm{PB}_{f_{k-1}}(v_k) = \varnothing$ but $B_{f_{k-1}}(v_k) \neq \varnothing$, hence there exists a vertex $u \in V_{f_{k-1}}^+$ such that $B_{f_{k-1}}(u) \cap B_{f_{k-1}}(v_k) \neq \varnothing$. Since $V_{f_{k-1}}^+ = V_f^+$, $u \in V_f^+$, and since $V_f^1 = \varnothing$, $u \in V_f^{++}$. Moreover, since $B_{f_{k-1}}(u) \cap B_{f_{k-1}}(v_k) \neq \varnothing$, $\mathrm{PB}_{f_{k-1}}(u) \neq B_{f-1}(u)$. By (5.2), $u \notin \{v_1, \ldots, v_k\}$, that is, $u \in V_f^+ - \{v_1, \ldots, v_k\}$, and since $V_f^+ = V_f^{++}$, we deduce that $V_f^{++} - \{v_1, \ldots, v_k\} \neq \varnothing$. Hence $k < |V_f^{++}|$. Similar to (5.1),

$$\sigma(f_k) = \sigma(f) - k > \sigma(f) - |V_f^{++}| \geq \frac{1}{2}\sigma(f). \tag{5.3}$$

Therefore, by (5.1) and (5.3),

$$\Gamma_b(G) \geq \sigma(f_k) > \sigma(f) - \frac{1}{2}\sigma(f) = \frac{1}{2}\sigma(f) = \frac{1}{2}\alpha_{\mathrm{bn}}(G). \tag{5.4}$$

Since we also have that $\alpha_{\mathrm{bnr}}(G) \leq \alpha_{\mathrm{bn}}(G)$, it follows from (5.4) that $\alpha_{\mathrm{bnr}}(G)/\Gamma_b(G) \leq \alpha_{\mathrm{bn}}(G)/\Gamma_b(G) < 2$.

To show that the bound for $\alpha_{\mathrm{bn}}(G)/\Gamma_b(G)$ is asymptotically best possible, we consider the spider $S = \mathrm{Sp}(2^k)$ for $k \geq 3$. By Proposition 2.7(*i*) and (*iii*), $\alpha_{\mathrm{bn}}(S) = 2k$ and $\Gamma_b(S) = k+1$. Hence

$$\lim_{k\to\infty} \alpha_{\mathrm{bn}}(S)/\Gamma_b(S) = \lim_{k\to\infty} 2k/(k+1) = 2. \blacksquare$$

5.2. $\Gamma_b(G)/\alpha_{\mathrm{bn(bnr)}}(G)$**.** Bouchemakh and Fergani [6] showed that if G is a graph of order n and minimum degree $\delta(G)$, then $\Gamma_b(G) \leq n - \delta(G)$. Since $\alpha_{\mathrm{bnr}}(G) \geq \alpha(G)$, it follows that

$$\frac{\Gamma_b(G)}{\alpha_{\mathrm{bn}}(G)} \leq \frac{\Gamma_b(G)}{\alpha_{\mathrm{bnr}}(G)} \leq \frac{n - \delta(G)}{\alpha(G)}.$$

This leads to the following result.

Proposition 5.3. *For any connected bipartite graph G, $\Gamma_b(G)/\alpha_{\mathrm{bn}}(G) \leq \Gamma_b(G)/\alpha_{\mathrm{bnr}}(G) < 2$.*

Proof. Say G has order n. Since G is bipartite, $\alpha(G) \geq \frac{n}{2}$. If $n = 1$, the result is obvious. If $n \geq 2$, then

$$\frac{\Gamma_b(G)}{\alpha_{\mathrm{bn}}(G)} \leq \frac{\Gamma_b(G)}{\alpha_{\mathrm{bnr}}(G)} \leq \frac{n - \delta(G)}{\alpha(G)} \leq \frac{2(n-1)}{n} < 2. \blacksquare$$

Theorem 5.4. *For general graphs, the ratios $\Gamma_b(G)/\alpha_{\mathrm{bnr}}(G)$ and $\Gamma_b(G)/\alpha_{\mathrm{bn}}(G)$ are unbounded.*

Proof. Let $G_n \cong K_n \,\square\, P_3$, where $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$ are the vertex sets of the copies of K_n, and $Q_i = (x_i, y_i, z_i)$ the copies of P_3. We begin by showing that $\Gamma_b(G_n) = 2n$. If $n = 1$, then $G_1 = P_3$, hence $\Gamma_b(G_1) = 2$. Assume that $n \geq 2$. Define the broadcast f on G_n by $f(x) = 2$ for $x \in X$ and $f(v) = 0$ for $v \in Y \cup Z$. Then each x_i broadcasts to all of Q_i and $\mathrm{PB}_f(x_i) = \{z_i\}$. Hence f is a minimal dominating broadcast. Consequently, $\Gamma_b(G_n) \geq \sigma(f) = 2n$.

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Suppose there exists a minimal dominating broadcast g on G_n such that $\sigma(g) > 2n$. Then $n \geq 2$. By the pigeonhole principle, there exists an index i such that $\sigma(Q_i) > 2$, and by symmetry we may assume $i = 1$. Since $\text{diam}(G) = 3$ and $e(x_1) = e(z_1) = 3$ while $e(y_1) = 2$, there are only three cases to consider.

Case 1: Without loss of generality, $g(x_1) = 2$ and $g(z_1) = 1$. Since $N_g(x_1) = X \cup Y \cup \{z_1\}$ and $N_g(z_1) = Z \cup \{y_1\}$, every vertex of G_n hears g from x_1 or z_1, hence $\sigma(g) = 3$.

Case 2: Without loss of generality, $g(x_1) = 3$. Since $e(x_1) = 3$, x_1 dominates G_n and $\sigma(g) = 3$.

Case 3: Otherwise, $g(x_1) = g(y_1) = g(z_1) = 1$. Then $\{x_1, y_1, z_1\}$ is a dominating set of G_n, hence $\sigma(g) = 3$.

In each case we have a contradiction. We conclude that $\Gamma_b(G_n) = 2n$.

On the other hand, we show that $\alpha_{\text{bn}}(G_n) = 3$ for each $n \geq 2$. Let h be the characteristic function of a maximum independent set $\{x_i, y_j, z_k\}$. Then h is bn-independent, irredundant and dominating. Note that $\{x_j, y_i\} \subseteq B_h(x_i) \cap B_h(y_j)$ and $\{x_k, z_i\} \subseteq B_h(x_i) \cap B_h(z_k)$. By Proposition 2.3($i$), h is a maximal bn-independent broadcast, thus $\alpha_{\text{bn}}(G_n) \geq \alpha_{\text{bnr}}(G) \geq \sigma(h) = \alpha(G_n) = 3$. Consider any bn-independent broadcast h' on G. To maintain bn-independence, h' has at most one broadcasting vertex in each of X, Y and Z. Moreover, if a vertex v broadcasts with strength 2, then either $v \in Y$ and dominates the entire graph,..., without loss of generality, $v \in X$ and dominates X and Y. In the latter case, there is at most one other broadcasting vertex, say z, which belongs to Z, and $h'(z) = 1$. Hence $\sigma(h') \leq 3$ and thus $\alpha_{\text{bn}}(G_n) = \alpha_{\text{bnr}}(G) = 3$. It follows that the ratios $\Gamma_b(G)/\alpha_{\text{bnr}}(G)$ and $\Gamma_b(G)/\alpha_{\text{bn}}(G)$ are unbounded. ■

6. Open questions

As mentioned above, $\Gamma_b(G) \leq n - \delta(G)$ [6] for all graphs G of order n and minimum degree $\delta(G)$. Also, $\alpha(G) \leq n - \delta(G)$ and, when G is connected, $\text{diam}(G) \leq n - \delta(G)$.

Question 1. *Is it true that $\alpha_{\text{bn}}(G) \leq n - \delta(G)$ for all graphs G?*

In Proposition 4.1 we used $3 \times n$ grids to show that $\Gamma_b - \alpha_{\text{bn}}$ can be arbitrary for 2-connected graphs. Since $\alpha_{\text{bn}}(T) \geq \alpha_{\text{bn}}(G)$ when T is a spanning tree of G, this result does not automatically extend to trees.

Question 2. *Is $\Gamma_b - \alpha_{\text{bn}}$ bounded for trees?*

We showed in Theorem 5.2 that $\alpha_{\text{bnr}}(G)/\Gamma_b(G) < 2$, but not that the bound is asymptotically best possible. For the tree in Figure 5, $\alpha_{\text{bnr}}(T)/\Gamma_b(T) = 14/13$, and for the general class of trees H_k constructed using T, $\alpha_{\text{bnr}}(H_k)/\Gamma_b(H_k) = 42/41$. For the graph G_2 in Figure 3, $\alpha_{\text{bnr}}(G_2)/\Gamma_b(G_2) = 9/7$, and for the general class of graphs G_k constructed in Section 4.2, $\alpha_{\text{bnr}}(G_k)/\Gamma_b(G_k) = (3k+3)/(2k+3) < 3/2$. Also, $\delta(G_k) = 1$.

Question 3. *What is an asymptotically tight upper bound for* $\alpha_{\mathrm{bnr}}(T)/\Gamma_b(T)$ *for trees? Can the ratio* $\alpha_{\mathrm{bnr}}(G)/\Gamma_b(G) < 3/2$ *for cyclic graphs be improved?*

We showed in Theorem 4.9 that $\alpha_{\mathrm{bn}} - \Gamma_b$ and $\alpha_{\mathrm{bnr}} - \Gamma_b$ can be arbitrary for cyclic graphs, but the graphs used in the proof have end-vertices.

Question 4. *Is it true that* $\alpha_{\mathrm{bnr}}(G) \leq \Gamma_b(G)$ *when* G *is* 2*-connected? Is it true that* $\alpha_{\mathrm{bnr}}(G_k) \leq \Gamma_b(G_k)$ *when* $\delta(G) \geq 2$*?*

Dunbar et al. [10, Section 3.3] also considered hearing independent dominating broadcasts and denoted the maximum cost of a minimal independent dominating broadcast of G, which they called the *upper broadcast independent domination number*, by $\Gamma_{\mathrm{ib}}(G)$. This parameter was denoted $\alpha_{\mathrm{hd}}(G)$ in [16]. By Definition 2.3, $\alpha_{\mathrm{bnd}}(G) \leq \alpha_{\mathrm{hd}}(G)$ for all graphs G. The inequality can be strict: let (u_i, v_i, w_i), $i = 1, 2$, be two copies of P_3 and add the edges v_1v_2 and w_1w_2 to form the graph G. The broadcast f defined by $f(u_i) = 2$ for each i is hearing independent and dominating. Since $w_i \in \mathrm{PB}_f(u_i)$ for each i, it is minimal dominating. Hence $\alpha_{\mathrm{hd}}(G) \geq 4$. However, $\alpha_{\mathrm{bnd}}(G) = \alpha(G) = \mathrm{diam}(G) = 3$. Bouchouika, Bouchemakh and Sopena [8] determined the values of the parameters defined in [10] for paths and cycles, but very little else is known about either of the parameters α_{bnd} and α_{hd}. We encourage interested readers to investigate them.

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