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ON VARIABLE SUM EXDEG ENERGY OF GRAPHS

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Abstract. In this paper, we put forward the idea of variable sum exdeg energy of graphs. We study the algebraic properties of variable sum exdeg energy. Some properties related to spectral radius of variable sum exdeg matrix are determined. We determine some Nordhaus-Gaddum-type results for variable sum exdeg spectral radius and energy. Some classes of variable sum exdeg equienergetic graphs are also determined.

1. Introduction

Let $\mathcal{G}$ be a graph with vertex set $V_{\mathcal{G}}$ and edge set $E_{\mathcal{G}}$. The number of neighbours of a vertex w in $\mathcal{G}$ is called the degree of w, denoted by $d_{\mathcal{G}}^{(w)}$. If vertices w and z are connected by an edge, we denote it by wz. The order $m_{\mathcal{G}}$ of a graph $\mathcal{G}$ is given by $m_{\mathcal{G}} = |V_{\mathcal{G}}|$. The size $e_{\mathcal{G}}$ of a graph $\mathcal{G}$ is defined by $e_{\mathcal{G}} = |E_{\mathcal{G}}|$. The largest (smallest) degree of $\mathcal{G}$ is the largest (smallest) vertex degree in $\mathcal{G}$, represented as $\nabla_{\mathcal{G}}$ ($\delta_{\mathcal{G}}$). A graph of order $m_{\mathcal{G}}$, size $e_{\mathcal{G}}$, maximum degree $\nabla_{\mathcal{G}}$ and minimum degree $\delta_{\mathcal{G}}$ is denoted by $\mathcal{G}(m_{\mathcal{G}}, e_{\mathcal{G}}, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ and a graph of order $m_{\mathcal{G}}$, size $e_{\mathcal{G}}$ is denoted by $\mathcal{G}(m_{\mathcal{G}}, e_{\mathcal{G}})$. In this paper, we consider simple and connected graphs.

An m-vertex path $\mathcal{P}_m$, ($m \geq 1$), is a graph with vertex set $\{w_1, \ldots w_m\}$ and edge set $\{w_j w_{j+1} \mid j = 1, 2, \ldots, m-1\}$. A star graph $\mathcal{S}_m$ on m vertices is a tree consisting of a central vertex adjacent to $m-1$ pendant vertices. An m-vertex cycle $\mathcal{C}_m$ ($m \geq 3$) is a graph with $V_{\mathcal{C}_m} = \{w_1, \ldots, w_m\}$ and $E_{\mathcal{C}_m} = \{w_j w_{j+1} \mid j = 1, 2, \ldots, m-1\} \cup \{w_m w_1\}$. A complete graph of order m, denoted by $\mathcal{K}_m$, is a

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simple graph whose each pair of vertices is joined by an edge. A graph $\mathcal{G}$ is partite if its vertex set $V_\mathcal{G}$ can be partitioned into two independent sets, say Y_1 and Y_2. The independent sets Y_1 and Y_2 are called partite sets of the bipartite graph. A complete bipartite graph $\mathcal{K}_{m,n}$ is a bipartite graph with partite sets Y_1 and Y_2 satisfying $|Y_1| = m$, $|Y_2| = n$ and each vertex in Y_1 is adjacent to each vertex in Y_2. The complement of a simple graph $\mathcal{G}$ is a graph represented by $\overline{\mathcal{G}}$ with the property that $V_\mathcal{G} = V_{\overline{\mathcal{G}}}$ and $wz \in E_\mathcal{G}$ if and only if $wz \notin E_{\overline{\mathcal{G}}}$. If $d_\mathcal{G}^{(w)} = b$ for each vertex $w \in V_\mathcal{G}$ then $\mathcal{G}$ is called a b-regular graph.

A topological index TI($\mathcal{G}$) of a graph $\mathcal{G}$ is a molecular descriptor which is a conversion of a molecular structure into some real number. Topological indices are the quantities of molecular graph, which are constant under graph isomorphism [10]. In theoretical chemistry, many molecular descriptors are considered and have found applications, see [11, 27]. For a recent study on directed graph energy, we refer [17].

The variable sum exdeg index was introduced by Vukičević [28] for anticipating the octanol-water partition coefficient of some chemical compounds. The variable sum exdeg index of a graph $\mathcal{G}$ is defined as:

$$\mathrm{SEI}_r(\mathcal{G}) = \sum_{wz\in E_\mathcal{G}} (r^{\,d_\mathcal{G}^{(w)}} + r^{\,d_\mathcal{G}^{(z)}}), \tag{1.1}$$

where $r \neq 1$ is a positive real number.

The adjacency matrix $A(\mathcal{G}) = [a_{ij}]_{m\times m}$ of a graph $\mathcal{G}$ of order m is given as

$$a_{ij} = \begin{cases} 1 & \text{if } w_iw_j \in E_\mathcal{G}, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency characteristic polynomial of $\mathcal{G}$ is of the form:

$$\begin{aligned} \Psi(\mathcal{G}, z) &= \det(A(\mathcal{G}) - zI_m) \\ &= z^m + \sum_{j=1}^{m} b_j\, z^{m-j}, \end{aligned}$$

where I_m is $m \times m$ identity matrix.The zeros of $\Psi(\mathcal{G}, z)$ are called A-eigenvalues of the graph $\mathcal{G}$.

Let $z_1, \ldots, z_m$ be the A-eigenvalues of a graph $\mathcal{G}$. Gutman [9] defined the energy of $\mathcal{G}$ as

$$E(\mathcal{G}) = \sum_{j=1}^{m} |z_j|.$$

In the present paper, we introduce variable sum exdeg energy of graphs. Firstly we define a variable sum exdeg matrix $A_r(\mathcal{G}) = [b_{ij}]_{m\times m}$ of an m-vertex graph $\mathcal{G}$ as

$$b_{ij} = \begin{cases} r^{\,d_\mathcal{G}^{(w_i)}} + r^{\,d_\mathcal{G}^{(w_j)}} & \text{if } w_iw_j \in E_\mathcal{G}, \\ 0 & \text{otherwise.} \end{cases}$$

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The A_r -characteristic polynomial of $\mathcal{G}$ is defined by:

$$\begin{aligned}\phi(\mathcal{G},\theta) &= \det(A_r(\mathcal{G})-\theta I_m)\\ &= \theta^m+\sum_{j=1}^{m} c_j\,\theta^{m-j}.\end{aligned}$$

The zeros of $\phi(\mathcal{G},\theta)$ are called A_r -eigenvalues of $A_r(\mathcal{G})$. The set of A_r -eigenvalues of $\mathcal{G}$ together with their multiplicities is called the A_r-spectrum, $\text{spec}_{A_r}(\mathcal{G})$, of $\mathcal{G}$. The A_r-eigenvalues are real, since its corresponding matrix is symmetric and real. If $\theta_1,\theta_2,\ldots,\theta_k$ are distinct A_r-eigenvalues of $\mathcal{G}$ and $q_1,q_2,\ldots,q_k$ are their respective multiplicities then A_r-spectrum of $\mathcal{G}$ is written by $\text{spec}_{A_r}(\mathcal{G}) = \{(\theta_1)^{q_1},(\theta_2)^{q_2},\ldots,(\theta_k)^{q_k}\}$.

The theory of Randic energy was put forward by Bozkurt et al. [2, 3]. In 2014, Gutman et al. [12] proved few of the properties of Randić matrix and Randić energy. Vukičević [29] gave the idea of the mathematical study of the variable sum exdeg index SEI_r. For $r>1$, the author finds the maximal and minimal graphs with respect to the SEI_r among different classes of graphs. Yarahmadi and Ashrafi [30] introduced the concept of a polynomial form of this graph invariant which is applied in nanoscience. Various forms of graph energies are summarized by Das et al. [7] in 2018. Das et al. [7] calculated some bounds for these graph energies. We refer [5, 6, 8, 18, 21, 22, 24, 26, 32, 20, 25, 33] for some old and new results on graph energies.

In this paper, we put forward the idea of variable sum exdeg energy of graphs. Motivated by the paper [14], we study the algebraic properties of variable sum exdeg energy. Some properties related to spectral radius of variable sum exdeg matrix are determined. We determine some Nordhaus-Gaddum-type results for variable sum exdeg spectral radius and energy. Some classes of variable sum exdeg equienergetic graphs are also determined.

2. Auxiliary results

Let $\theta_1,\ldots,\theta_m$ be the A_r -eigenvalues of an m-vertex graph $\mathcal{G}$. Then we define the variable sum exdeg energy of graph $\mathcal{G}$ as

$$E_r(\mathcal{G})=\sum_{j=1}^{m}|\theta_j|. \tag{2.1}$$

For convenience, we define two notations. Let

$$\varOmega=\sum_{1\le i<j\le m}\left(r^{\,d_{\mathcal{G}}^{(w_i)}}+r^{\,d_{\mathcal{G}}^{(w_j)}}\right)^2,\ \varUpsilon=\det(A_r(\mathcal{G})).$$

The trace of $A_r(\mathcal{G})=[b_{ij}]_{m\times m}$ is defined by $\sum_{j=1}^{m} b_{jj}$ and is represented by $tr(A_r(\mathcal{G}))$. Let $(A_r(\mathcal{G}))_{ij}$ represent the (i,j)-entry of the A_r matrix. Now we prove the following lemma.

Lemma 2.1. *Let $\mathcal{G}$ be an m-vertex graph and let $\theta_1,\ldots,\theta_m$ be its A_r-eigenvalues. Then*

(1) $\sum_{j=1}^{m} \theta_j = 0$,

(2) $tr(A_r^2(\mathcal{G})) = \sum_{j=1}^{m} \theta_j^2 = 2\Omega$.

Proof. (1). Since $\mathcal{G}$ is simple graph, the sum of diagonal entries of $A_r(\mathcal{G})$ is 0. Thus $tr(A_r(\mathcal{G})) = 0$. But $tr(A_r(\mathcal{G})) = \sum_{j=1}^{m} \theta_j$. Therefore $\sum_{j=1}^{m} \theta_j = 0$.

(2). For $i = j$, we have

$$\begin{aligned}(A_r^2(\mathcal{G}))_{jj} &= \sum_{k=1}^{m} (A_r)_{jk}(A_r)_{kj} = \sum_{k=1}^{m} ((A_r)_{jk})^2 \\ &= \sum_{w_iw_j \in E_{\mathcal{G}}} ((A_r)_{ij})^2 = 2 \sum_{1 \le i < j \le m} (r^{d_{\mathcal{G}}^{(w_i)}} + r^{d_{\mathcal{G}}^{(w_j)}})^2 = 2\Omega.\end{aligned}$$

□

Theorem 2.2. *Suppose $\mathcal{G}(m, q)$ is a simple connected graph and $r > 1$ is a real number. Then*

$$tr(A_r^2(\mathcal{G})) \le tr(A_r^2(\mathcal{K}_m)).$$

Proof. Suppose $\mathcal{G} \ncong \mathcal{K}_m$. Then $d_{\mathcal{G}}^{(w_j)} \le m - 1$ for each $w_j \in V_{\mathcal{G}}$, $j = 1, \ldots, m$. Now we have

$$r^{d_{\mathcal{G}}^{(w_i)}} + r^{d_{\mathcal{G}}^{(w_j)}} \le r^{m-1} + r^{m-1} = 2r^{m-1}.$$

Then

$$\begin{aligned}tr(A_r^2(\mathcal{G})) &= 2 \sum_{1 \le i < j \le m} (r^{d_{\mathcal{G}}^{(w_i)}} + r^{d_{\mathcal{G}}^{(w_j)}})^2 \\ &\le 2q(2r^{m-1})^2 = 8\, q\, r^{2m-2}.\end{aligned}$$

Since $\mathcal{G} \ncong \mathcal{K}_m$, therefore $q < \frac{m(m-1)}{2}$. Now

$$\begin{aligned}tr(A_r^2(\mathcal{G})) \le 8\, q\, r^{2m-2} &< \frac{8m(m-1)}{2} r^{2m-2} \\ &= 4m(m-1)r^{2m-2}.\end{aligned}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}tr(A_r^2(\mathcal{K}_m)) &= 2\left[\frac{m(m-1)}{2} \times (r^{m-1} + r^{m-1})\right] \\ &= 4m(m-1)r^{2m-2}.\end{aligned}$$

Hence $tr(A_r^2(\mathcal{G})) \le tr(A_r^2(\mathcal{K}_m))$. The result is proved. □

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are the graphs having disjoint vertex sets. Then the graph union $\mathcal{H}_1 \cup \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ is a graph with vertex set $V_{\mathcal{H}_1} \cup V_{\mathcal{H}_2}$ and edge set $E_{\mathcal{H}_1} \cup E_{\mathcal{H}_2}$. Also $d_{\mathcal{H}_1 \cup \mathcal{H}_2}^{(w)} = d_{\mathcal{H}_1}^{(w)}$ if $w \in V_{\mathcal{H}_1}$ and $d_{\mathcal{H}_1 \cup \mathcal{H}_2}^{(w)} = d_{\mathcal{H}_2}^{(w)}$ if $w \in V_{\mathcal{H}_2}$. A square diagonal matrix that contains square matrices as diagonal entries and 0 as its non-diagonal entries is known as block diagonal matrix.

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The relationship between variable sum exdeg energy of a graph and its components is given in next theorem.

Theorem 2.3. *Suppose graph $\mathcal{G}$ has components $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_t$. Then $E_r(\mathcal{G}) = \sum_{j=1}^{t} E_r(\mathcal{H}_j)$.*

Proof. Since graph $\mathcal{G}$ has components $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_t$., we can write $\mathcal{G} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_t$. Then $A_r(\mathcal{G})$ is a block diagonal matrix with diagonal entries of $A_r(\mathcal{G})$ are $A_r(\mathcal{H}_1), A_r(\mathcal{H}_2), \ldots, A_r(\mathcal{H}_t)$. Therefore

$$\mathrm{spec}_{A_r}(\mathcal{G}) = \mathrm{spec}_{A_r}(\mathcal{H}_1) \cup \mathrm{spec}_{A_r}(\mathcal{H}_2) \cup \cdots \cup \mathrm{spec}_{A_r}(\mathcal{H}_t).$$

Hence

$$E_r(\mathcal{G}) = \sum_{j=1}^{t} E_r(\mathcal{H}_j).$$

□

Next theorem gives some interesting fact about variable sum exdeg energy of a non-trivial graph. The proof is similar to the proof of [13, Theorem 2.5] and hence omitted.

Theorem 2.4. *The variable sum exdeg energy of $\mathcal{G}$ must be an even positive integer if it is an integer.*

3. Variable sum exdeg energy of some graphs

In this section, we prove variable sum exdeg energy formula for certain graph classes. The A-spectrum of $\mathcal{K}_m$ and $\mathcal{K}_{t,s}$ is given by

$$\begin{aligned} \mathrm{spec}_A(\mathcal{K}_m) &= \{(-1)^{m-1}, (m-1)\}, \\ \mathrm{spec}_A(\mathcal{K}_{t,s}) &= \{(0)^{t+s-2}, \pm\sqrt{ts}\}. \end{aligned}$$

The energy formula for $\mathcal{C}_m$ was proved by Bhat and Pirzada [1], which is given by

$$E(\mathcal{C}_m) = \begin{cases} 4\ \cot\frac{\pi}{m} & \text{if } m \equiv 0 (\mathrm{mod} 4) \\ 4\ \csc\frac{\pi}{m} & \text{if } m \equiv 2 (\mathrm{mod} 4) \\ 2\ \csc\frac{\pi}{2m} & \text{if } m \equiv 1 (\mathrm{mod} 2). \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose $\mathcal{G}(m, e)$ be a b-regular graph. Then $E_r(\mathcal{G}) = 2\, r^b\ E(\mathcal{G})$, where $r \neq 1$ is a positive real number.*

Proof. Let $\theta_1, \theta_2, \ldots, \theta_m$ be A_r-eigenvalues of $\mathcal{G}$ and $z_1, z_2, \ldots, z_m$ be A-eigenvalues of $\mathcal{G}$. Note that non-zero entries in $A_r(\mathcal{G})$ are equal to $2r^b$. This implies that $A_r(\mathcal{G}) = 2\, r^b\, A(\mathcal{G})$. Hence $\theta_j = 2\, r^b\, z_j$

for $j = 1, 2, \ldots, m$. This implies that

$$\begin{aligned} E_r(\mathcal{G}) &= \sum_{j=1}^{m} |\theta_j| = \sum_{j=1}^{m} |2\, r^b\, z_j| \\ &= 2\, r^b \sum_{j=1}^{m} |z_j| = 2\, r^b\, E(\mathcal{G}). \end{aligned}$$

This finishes the proof. □

Example 3.2. *Let $r \neq 1$ is a positive real number. We can easily calculate the following:*

$$\begin{aligned} \mathrm{spec}_A(\mathcal{K}_{6,6}) &= \{(0)^{10}, \pm 6\}, \\ \mathrm{spec}_{A_r}(\mathcal{K}_{6,6}) &= \{(0)^{10}, \pm 12 r^5\}. \end{aligned}$$

Then

$$E_r(\mathcal{K}_{6,6}) = 2r^5(12) = 2r^5 E(\mathcal{K}_{6,6}),$$

which is consistent with Theorem 3.1.

Using Theorem 3.1, we have the following results.

Corollary 3.3. $E_r(\mathcal{C}_m) = 2\, r^2\, E(\mathcal{C}_m)$

Corollary 3.4. $E_r(\mathcal{K}_m) = 4\,(m-1)\, r^{m-1}$.

Remark 3.5. *Let $m \equiv 2 (\mathrm{mod}\ \ 4)$. Then from equation (3.1), one can see that $E_r(\mathcal{C}_m) = 2E_r(\mathcal{C}_{\frac{m}{2}})$.*

We now obtain the variable sum exdeg energy formula for $\mathcal{K}_{t,s}$.

Theorem 3.6. *Let $r \neq 1$ be a positive real number. Then $E_r(\mathcal{K}_{t,s}) = 2(r^t + r^s)\sqrt{ts}$.*

Proof. Let us represent $t \times s$ and an $s \times t$ matrices whose each entry is $r^t + r^s$ by C and D, respectively. Let O be a t-square and O' be an s-square matrix, whose each entry is zero. Then

$$A_r(\mathcal{K}_{t,s}) = \begin{bmatrix} O & C \\ D & O' \end{bmatrix}.$$

That is,

$$A_r(\mathcal{K}_{t,s}) = (r^t + r^s)\ A(\mathcal{K}_{t,s}).$$

Hence

$$\mathrm{spec}_{A_r}(\mathcal{K}_{t,s}) = \left\{ (r^t + r^s)\, \sqrt{ts},\ 0^{(t+s-2)},\ -(r^t + r^s)\, \sqrt{ts} \right\}.$$

Therefore

$$\begin{aligned} E_r(\mathcal{K}_{t,s}) &= \sum_{j=1}^{t+s} |\theta_j| \\ &= \left|(r^t + r^s)\sqrt{ts}\right| + \left| -(r^t + r^s)\sqrt{ts}\right| \\ &= 2(r^t + r^s)\sqrt{ts}. \end{aligned}$$

This proves the result. □

Using Theorem 3.6, we obtain next corollary.

Corollary 3.7. $E_r(\mathcal{S}_m) = 2(r + r^{m-1})\sqrt{m-1}$.

Remark 3.8. *Using Theorem 2.3 and Theorem 3.4, one can easily find that*

$$E_r(\overline{\mathcal{K}}_{t,s}) = \frac{4}{r}\left[(t-1)r^t + (s-1)r^s\right].$$

Let $C = (c_{st})_{p\times p}$ be a p-square matrix with eigenvalues τ_j and D be a q-square matrix with eigenvalues κ_f, $s,t,j = 1,\ldots,p$, $f = 1,\ldots,q$. The tensor product of C and D, represented by $C\otimes D$, is the matrix attained by substituting each entry c_{st} of C by $c_{st}D$. The eigenvalues of $C\otimes D$ are $\tau_j\kappa_f$.

Suppose $\mathcal{G}$ is a graph with vertex set $V_\mathcal{G}$ and $\widetilde{U}$ be a set such that $V_\mathcal{G}\cap\widetilde{U} = \phi$, $|V_\mathcal{G}| = |\widetilde{U}|$ and $h : V_\mathcal{G} \longrightarrow \widetilde{U}$ is a bijection. For $w \in V_\mathcal{G}$, let $h(w) = \widetilde{w}$. The duplication of $\mathcal{G}$, represented by $\mathcal{G}^\star$, is the graph with $V_{\mathcal{G}^\star} = V_\mathcal{G}\cup\widetilde{U}$ and its edges are defined as: $wz \in E_\mathcal{G}$ if and only if $w\widetilde{z} \in E_{\mathcal{G}^\star}$ and $z\widetilde{w} \in E_{\mathcal{G}^\star}$.

Next theorem gives the relationship of variable sum exdeg energy of $\mathcal{G}$ with variable sum exdeg energy of $\mathcal{G}^\star$. We would like to point out that the idea of proof is taken from [13].

Theorem 3.9. *Let $\mathcal{G}$ be an m-vertex graph. Then $E_r(\mathcal{G}^\star) = 2E_r(\mathcal{G})$.*

Proof. Suppose O be an m-square zero matrix. Using definition of $G^\star$ with appropriate labeling, we obtain

$$A_r(\mathcal{G}^\star) = \begin{bmatrix} O & A_r(\mathcal{G}) \\ A_r(\mathcal{G}) & O \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A_r(\mathcal{G}).$$

Therefore $\mathrm{spec}_{A_r}(\mathcal{G}^\star) = \{\pm\theta_j \mid j = 1,\ldots,m.\}$. Thus $E_r(\mathcal{G}^\star) = 2E_r(\mathcal{G})$.

Example 3.10. *Let $r \neq 1$ be a positive real number. Since*

$$\begin{aligned} \mathrm{spec}_{A_r}(\mathcal{C}_6) &= \{(4r^2)^1, (-2r^2)^2, (2r^2)^2, (-4r^2)^1\}, \\ \mathrm{spec}_{A_r}(\mathcal{P}_3) &= \{((r^2+r)\sqrt{2})^1, (-(r^2+r)\sqrt{2})^1, (0)^1\}, \end{aligned}$$

we have

$$\begin{aligned} \text{spec}_{A_r}(\mathcal{C}_6^*) &= \{\pm(4r^2)^1, \pm(-2r^2)^2, \pm(2r^2)^2, \pm(-4r^2)^1\}, \\ \text{spec}_{A_r}(\mathcal{P}_3^*) &= \{\pm((r^2+r)\sqrt{2})^1, \pm(-(r^2+r)\sqrt{2})^1, \pm(0)^1\}. \end{aligned}$$

One can easily show that

$$E_r(\mathcal{C}_6^*) = 2E_r(\mathcal{C}_6),$$
$$E_r(\mathcal{P}_3^*) = 2E_r(\mathcal{P}_3),$$

which is in accordance with Theorem 3.9.

4. Spectral radius and spread of the variable sum exdeg matrix

In this section, we give bounds on spectral radius and spread of graphs with respect to variable sum exdeg matrix. For any complex $m \times m$ matrix M with eigenvalues $\xi_1, \ldots, \xi_m$, the spread $s(M)$ of M is introduced in [19] and is defined as $s(M) = \max_{j,k} |\xi_j - \xi_k|$.

Let $\theta_1 \geq \cdots \geq \theta_m$ be the A_r -eigenvalues of a simple graph $\mathcal{G}$. Then spread of $A_r(\mathcal{G})$ is defined as $s(A_r(\mathcal{G})) = \theta_1 - \theta_m$, since the eigenvalues $\theta_1 \geq \cdots \geq \theta_m$ are all real.

We first give some lemmas that are used to prove our required results.

Lemma 4.1. [31, Zhang] *If C is a symmetric matrix of order m with eigenvalues $\omega_1 \geq \cdots \geq \omega_m$, then for any $z \in \mathbb{R}^m$ with $z \neq 0$.*

$$z^T C z \leq \omega_1 z^T z,$$

where z^T is the transpose of z. Equality holds if and only if z is an eigenvector of C corresponding to the eigenvalue ω_1.

Lemma 4.2. [16, Horn and Johnson] *Let $D_1 = [c_{ij}]_{m\times m}$ and $D_2 = [d_{ij}]_{m\times m}$ be $m \times m$ symmetric and non-negative matrices, respectively. If $D_1 \geq D_2$, that is, $c_{ij} \geq d_{ij}$ for all $i, j = 1, \ldots, m$, then $\omega_1(D_1) \geq \omega_1(D_2)$, where $\omega_1(D_k)$, $k = 1, 2$ is the largest eigenvalue of the respective matrix.*

Theorem 4.3. [15, Hong] *Let $\mathcal{G}(m, e)$ be a connected graph with A-eigenvalues $z_1 \geq \cdots \geq z_m$. Then*

$$z_1 \leq \sqrt{2e - m + 1},$$

where the equality holds if and only if $\mathcal{G}(m, e) \cong \mathcal{S}_m$ or $\mathcal{G}(m, e) \cong \mathcal{K}_m$.

Theorem 4.4. [4, Cao] *Let $\mathcal{G}(m, e, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ be a graph with A-eigenvalues $z_1 \geq \cdots \geq z_m$ and $\delta_{\mathcal{G}} \geq 1$. Then*

$$z_1 \leq \sqrt{2e - \delta_{\mathcal{G}}(m-1) + (\delta_{\mathcal{G}} - 1)\nabla_{\mathcal{G}}}.$$

Now we give bounds on largest A_r-eigenvalue of a graph.

Theorem 4.5. *Let $m \geq 2$. Also let $\mathcal{G}(m, e, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ be a connected simple graph with A_r-eigenvalues $\theta_1 \geq \cdots \geq \theta_m$ and $r > 1$ is a real number. Then*

$$\frac{2\, e\, r^{\delta_{\mathcal{G}}}}{m} \;\leq\; \theta_1 \;\leq\; 2\, r^{m-1}\sqrt{2e - m + 1}.$$

Proof. Let $z \in \mathbb{R}^m$ such that $z = (z_1, z_2, \ldots, z_m)^T$. Then

$$z^T \mathcal{A}_r(\mathcal{G})\, z \;=\; \sum_{w_i w_j \,\in\, E_{\mathcal{G}}} (r^{\,d_{\mathcal{G}}^{(w_i)}} + r^{\,d_{\mathcal{G}}^{(w_j)}})\, z_i z_j \;\geq\; \sum_{w_i w_j \,\in\, E_{\mathcal{G}}} (r^{\delta_{\mathcal{G}}} + r^{\delta_{\mathcal{G}}})\, z_i z_j.$$

Taking $z = (\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}, \ldots, \frac{1}{\sqrt{m}})^T$, we get $\sum_{w_i w_j \,\in\, E_{\mathcal{G}}} (r^{\delta_{\mathcal{G}}} + r^{\delta_{\mathcal{G}}})\, z_i z_j = \frac{2e}{m} r^{\delta_{\mathcal{G}}}$. Therefore by Lemma 4.1, we have $\theta_1 \geq \frac{2\, e\, r^{\delta_{\mathcal{G}}}}{m}$. Now for any vertex $w_j \in V_{\mathcal{G}}$, $j = 1, 2 \ldots, m$, we have $1 \leq \delta_{\mathcal{G}} \leq d_{\mathcal{G}}(w_i) \leq \nabla_{\mathcal{G}} \leq (m-1)$. Therefore

$$r^{\,d_{\mathcal{G}}^{(w_i)}} + r^{\,d_{\mathcal{G}}^{(w_j)}} \;\leq\; r^{\nabla_{\mathcal{G}}} + r^{\nabla_{\mathcal{G}}} \;=\; 2r^{\nabla_{\mathcal{G}}} \;\leq\; 2r^{m-1}.$$

If ω_1 is the spectral radius of a matrix $2r^{m-1}A(\mathcal{G})$, then by Lemma 4.2 and Theorem 4.3, we obtain

$$\theta_1 \;\leq\; 2r^{m-1} z_1 \;\leq\; 2r^{m-1}\sqrt{2e - m + 1},$$

where z_1 is the spectral radius of $A(\mathcal{G})$. $\square$

Example 4.6. *Let $r > 1$ be a real number. One can easily see that* $\text{spec}_{A_r}(\mathcal{K}_{6,5}) = \{(0)^{10} \pm (r^5 + r^6)\sqrt{30}\}$ *and $\theta_1 = (r^5 + r^6)\sqrt{30}$. Then*

$$\frac{60}{11} r^5 \;\leq\; (r^5 + r^6)\sqrt{30} \;\leq\; 2\sqrt{50} r^{10},$$

which is consistent with Theorem 4.5.

Also one can easily calculate that $\text{spec}_{A_r}(\mathcal{C}_6) = \{\pm(2r^2)^2, \pm(4r^2)\}$ *and $\theta_1 = 4r^2$. Then*

$$2r^2 \;\leq\; 4r^2 \;\leq\; 2\sqrt{7} r^5,$$

which in accordance with Theorem 4.5.

Next theorem gives bounds on the smallest A_r-eigenvalue of a graph. The proof is similar to the proof of [14, Theorem 4.7] and thus omitted.

Theorem 4.7. *Let $\mathcal{G}(m, e, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ be a graph with A_r-eigenvalues $\theta_1 \geq \cdots \geq \theta_m$. Then*

$$\sqrt{\frac{2\Omega + (m-1)\,(m-2)\,\Upsilon^{2/m-1}}{2}} \leq \theta_m \leq \sqrt{\frac{2(m-1)\Omega}{m}},$$

where $r \neq 1$ is a positive real number.

In the following theorem, we give bounds on spread of the variable sum exdeg matrix of a graph.

Theorem 4.8. *Let $\mathcal{G}(m, e, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ be a connected graph with A_r-eigenvalues $\theta_1 \geq \cdots \geq \theta_m$. Then*

$$\begin{aligned} s(A_r(\mathcal{G})) &\geq 2\left(\frac{er}{m} - r^{m-1}\sqrt{\frac{2e(m-1)}{m}}\right), \\ s(A_r(\mathcal{G})) &\leq 2r^{m-1}\sqrt{2e-m+1} - \frac{1}{\sqrt{2}}\sqrt{8er^2 + (m-1)(m-2)\Upsilon^{\frac{2}{m-1}}}, \end{aligned}$$

where $r > 1$ is a real number.

Proof. We have $1 \leq \delta_{\mathcal{G}} \leq d_{\mathcal{G}}^{(w_j)} \leq \nabla_{\mathcal{G}} \leq (m-1)$ for any vertex $w_j \in V_{\mathcal{G}}$, $j = 1, \ldots, m$. Therefore

$$2\,\Omega \;=\; 2\sum_{1\leq i<j\leq m}(r^{d_{\mathcal{G}}^{(w_i)}} + r^{d_{\mathcal{G}}^{(w_j)}})^2 \;\geq\; 2\sum_{1\leq i<j\leq m}(r^{\delta_{\mathcal{G}}} + r^{\delta_{\mathcal{G}}})^2 \;\geq\; 8er^2. \tag{4.1}$$

Also

$$2\,\Omega \;=\; 2\sum_{1\leq i<j\leq m}(r^{d_{\mathcal{G}}^{(w_i)}} + r^{d_{\mathcal{G}}^{(w_j)}})^2 \;\leq\; 2\sum_{1\leq i<j\leq m}(r^{\nabla_{\mathcal{G}}} + r^{\nabla_{\mathcal{G}}})^2 \;\leq\; 8er^{2m-2}. \tag{4.2}$$

Hence using Theorems 4.5 and 4.7, and Equations (4.1) and (4.2), we get

$$\begin{aligned} s(A_r(\mathcal{G})) &= \theta_1 - \theta_m \\ &\leq 2r^{m-1}\sqrt{2e-m+1} - \sqrt{\frac{2\Omega + (m-1)\,(m-2)\,\Upsilon^{2/m-1}}{2}} \\ &\leq 2r^{m-1}\sqrt{2e-m+1} - \sqrt{\frac{8er^2 + (m-1)\,(m-2)\,\Upsilon^{2/m-1}}{2}} \\ &= 2r^{m-1}\sqrt{2e-m+1} - \frac{1}{\sqrt{2}}\sqrt{8er^2 + (m-1)\,(m-2)\,\Upsilon^{2/m-1}}. \end{aligned}$$

Since $\delta_{\mathcal{G}} \geq 1$, we have

$$\begin{aligned} s(A_r(\mathcal{G})) &= \theta_1 - \theta_m \geq \frac{2er^{\delta_{\mathcal{G}}}}{m} - \sqrt{\frac{2(m-1)\Omega}{m}} \\ &\geq \frac{2er}{m} - \sqrt{\frac{8er^{2m-2}(m-1)}{m}} = 2\left(\frac{er}{m} - r^{m-1}\sqrt{\frac{2e(m-1)}{m}}\right). \end{aligned}$$

The proof is complete. □

Example 4.9. *Let $r > 1$ be a real number. We see that*

$$s(A_r(\mathcal{K}_{6,5})) = 2\sqrt{30}(r^5 + r^6).$$

Then one can easily calculate that

$$\begin{aligned} s(A_r(\mathcal{K}_{6,5})) &\geq \frac{60r}{11} - 2\sqrt{\frac{600}{11}}r^{10}, \\ s(A_r(\mathcal{K}_{6,5})) &\leq 2\sqrt{50}r^{10} - 2\sqrt{15}r, \end{aligned}$$

which is in accordance with Theorem 4.8.

5. Upper and lower bounds of variable sum exdeg energy

We find upper and lower bounds of the variable sum exdeg energy of graphs.

Theorem 5.1. *Suppose $\mathcal{G}(m, e, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ be a connected graph with A_r-eigenvalues given by $\theta_1 \geq \cdots \geq \theta_m$ and $r \neq 1$ is a positive real number. Then*

$$\frac{2er}{m} \;\leq\; E_r(\mathcal{G}) \;\leq\; r^{m-1}\sqrt{8em}.$$

Proof. Assume, without loss of generality, that $\theta_1, \ldots, \theta_t$ are positive and $\theta_{t+1}, \ldots, \theta_m$ are negative. By Theorem 4.5, we obtain

$$E_r(\mathcal{G}) \;=\; \sum_{j=1}^{m} |\theta_j| \;=\; 2\sum_{j=1}^{t} \theta_j \;\geq\; 2\theta_1 \;\geq\; \frac{2er^{\delta_{\mathcal{G}}}}{m} \;\geq\; \frac{2er}{m}.$$

Using Cauchy-Schwartz inequality, Part (2) of Lemma 2.1 and Equation (4.2), we get

$$E_r(\mathcal{G}) \;=\; \sum_{j=1}^{m} |\theta_j| \;\leq\; \sqrt{m\sum_{j=1}^{m} \theta_j^2} \;=\; \sqrt{2m\,\Omega} \;\leq\; \sqrt{8emr^{2m-2}} \;=\; r^{m-1}\sqrt{8em}.$$

The result is proved. □

In next theorem, we find another lower bound and an upper bound of variable sum exdeg energy.

Theorem 5.2. *Let $\mathcal{G}(m, e, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ be a connected graph with A_r-eigenvalues $\theta_1 \geq \cdots \geq \theta_m$. Then*

$$4r\sqrt{e} \;\leq\; E_r(\mathcal{G}) \;\leq\; r^{m-1}\left(2\sqrt{2e-m+1} + \sqrt{\frac{8em^2(m-1) - 4e^2(m-1)r^{4-2m}}{m^2}}\right),$$

where $r > 1$ is a real number.

Proof. Using Part (1) of Lemma 2.1, we obtain $\sum_{j=1}^{m} \theta_j^2 = -2 \sum_{1\leq i<j\leq m} \theta_i\,\theta_j$. Now Part (2) of Lemma 2.1 implies

$$(E_r(\mathcal{G}))^2 = \left(\sum_{j=1}^{m} |\theta_j|\right)^2 = \sum_{j=1}^{m} \theta_j^2 + 2\sum_{1\leq i<j\leq m} |\theta_i\,\theta_j| \;\geq\; 2\,\Omega + 2\,\Big|\sum_{1\leq i<j\leq m} \theta_i\,\theta_j\Big| \;=\; 4\,\Omega.$$

That is,

$$4\,\Omega = 4\sum_{1\leq i<j\leq m} (r^{\,d_{\mathcal{G}}^{(w_i)}} + r^{\,d_{\mathcal{G}}^{(w_j)}})^2 \geq 4\sum_{1\leq i<j\leq m} (r^{\delta_{\mathcal{G}}} + r^{\delta_{\mathcal{G}}})^2 = 4\sum_{1\leq i<j\leq m} 4(r^{\delta_{\mathcal{G}}})^2 \geq 16er^2.$$

Thus $E_r(\mathcal{G}) \geq 4r\sqrt{e}$. This proves the inequality on left side. Next we prove the inequality on right side.

http://dx.doi.org/10.22108/toc.2023.133151.1978

We apply Cauchy-Schwartz inequality to obtain $(\sum_{j=2}^{m}|\theta_j|)^2 \leq (m-1)\sum_{j=2}^{m}\theta_j^2$. By Part (2) of Lemma 2.1, $(E_r(\mathcal{G})-\theta_1)^2 \leq (m-1)\,(2\Omega-\theta_1^2)$. Hence by Theorem 4.5, we get

$$\begin{aligned} E_r(\mathcal{G}) &\leq \theta_1+\sqrt{(m-1)\,(2\Omega-\theta_1^2)} \\ &\leq 2r^{m-1}\sqrt{2e-m+1}+\sqrt{(m-1)\left(8er^{2m-2}-\frac{4e^2r^2}{m^2}\right)} \\ &= 2r^{m-1}\sqrt{2e-m+1}+\sqrt{(m-1)\left(\frac{8em^2r^{2m-2}-4e^2r^2}{m^2}\right)} \\ &= r^{m-1}\left(2\sqrt{2e-m+1}+\sqrt{\frac{8em^2(m-1)-4e^2(m-1)r^{4-2m}}{m^2}}\right) \end{aligned}$$

This gives the required result. □

Example 5.3. *Let $r>1$ be a real number. We can easily calculate that*

$$E_r(\mathcal{K}_{6,5}) = 2\sqrt{30}\,(r^5+r^6).$$

Then

$$\begin{aligned} \frac{60r}{11} &\leq 2\sqrt{30}\,(r^5+r^6) \leq 4\sqrt{165}\,r^{10}, \\ 4\sqrt{30}\,r &\leq 2\sqrt{30}\,(r^5+r^6) \leq 2\sqrt{50}\,r^{10}+\frac{1}{11}\sqrt{290400-(3600/r^{18})}, \end{aligned}$$

which conform to Theorems 5.1 and 5.2.

6. Nordhaus Gaddum-type results for variable sum exdeg spectral radius and energy

Hafeez and Farooq [14] obtained Nordhaus-Gaddum-type results for generalized inverse sum indeg index spectral radius and energy. In this section, we determine some Nordhaus-Gaddum-type results for variable sum exdeg spectral radius and energy.

Let $\mathcal{G}$ be a simple graph and $\overline{\mathcal{G}}$ be its complement. Then it is easily seen that $m_{\mathcal{G}} = m_{\overline{\mathcal{G}}}$, $e_{\overline{\mathcal{G}}} = \frac{m_{\mathcal{G}}^2-m_{\mathcal{G}}}{2} - e_{\mathcal{G}}$, $\nabla_{\overline{\mathcal{G}}} = m_{\mathcal{G}}-1-\delta_{\mathcal{G}}$ and $\delta_{\overline{\mathcal{G}}} = m_{\mathcal{G}}-1-\nabla_{\mathcal{G}}$. Denote the A_r-eigenvalues of $\overline{\mathcal{G}}$ by $\overline{\theta}_j$, $j=1,2,\ldots,m$.

Theorem 6.1. *Suppose $\mathcal{G}(m,e,\nabla_{\mathcal{G}},\delta_{\mathcal{G}})$ be a connected graph. Then*

$$\theta_1+\overline{\theta}_1 \;\geq\; \frac{1}{m}(2e(r-1)+m^2-m),$$

where $r>1$ is a real number.

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Proof. Let $z = (z_1, z_2, \ldots, z_m)^T \in \mathbb{R}^m$. Then

$$\begin{aligned} z^T\left[A_r(\mathcal{G}) + A_r(\overline{\mathcal{G}})\right]z &= \sum_{w_iw_j \in E_{\mathcal{G}}} (r^{d_{\mathcal{G}}^{(w_i)}} + r^{d_{\mathcal{G}}^{(w_j)}})z_iz_j + \sum_{w_iw_j \in E_{\overline{\mathcal{G}}}} (r^{d_{\overline{\mathcal{G}}}^{(w_i)}} + r^{d_{\overline{\mathcal{G}}}^{(w_j)}})z_iz_j \\ &\geq \sum_{w_iw_j \in E_{\mathcal{G}}} (r^{\delta_{\mathcal{G}}} + r^{\delta_{\mathcal{G}}})z_iz_j + \sum_{w_iw_j \in E_{\overline{\mathcal{G}}}} (r^{\delta_{\overline{\mathcal{G}}}} + r^{\delta_{\overline{\mathcal{G}}}})z_iz_j \\ &= \sum_{w_iw_j \in E_{\mathcal{G}}} (2r^{\delta_{\mathcal{G}}})z_iz_j + \sum_{w_iw_j \in E_{\overline{\mathcal{G}}}} (2r^{\delta_{\overline{\mathcal{G}}}})z_iz_j \end{aligned}$$

Using Lemma 4.1, with $z = (\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}, \ldots, \frac{1}{\sqrt{m}})^T$, $\delta_{\overline{\mathcal{G}}} = m - 1 - \Delta_{\mathcal{G}}$ and $\Delta_{\mathcal{G}} \leq m - 1$, we obtain

$$\begin{aligned} \theta_1 + \overline{\theta}_1 &\geq \frac{2re}{m} + \frac{2e_{\overline{\mathcal{G}}}}{m}(r^{m-1-\Delta_{\mathcal{G}}}) \\ &\geq \frac{2}{m}\,(re + e_{\overline{\mathcal{G}}}\, r^{m-1-(m-1)}) \\ &= \frac{2}{m}\left(\frac{2re + m^2 - m - 2e}{2}\right) = \frac{1}{m}\,(2e(r-1) + m^2 - m). \end{aligned}$$

This completes the proof. □

Theorem 6.2. *Suppose $\mathcal{G}(m, e, \delta_{\mathcal{G}}, \nabla_{\mathcal{G}})$ be a connected graph and $\mathcal{G}_1$ be a component of $\overline{\mathcal{G}}$ with largest eigenvalue θ_1 and $\overline{\theta}_1 = \theta_1(\mathcal{G}_1)$. Let $r > 1$ be a positive real number.*

(*a*). *If $\nabla_{\mathcal{G}} = m - 1$ or $\nabla_{\overline{\mathcal{G}}} = m - 1$, then*

$$\theta_1 + \overline{\theta}_1 \leq 2(r^{m-1}\sqrt{2e - m + 1} + r^{m_{\mathcal{G}_1}-1}\sqrt{2e_{\mathcal{G}_1} - m_{\mathcal{G}_1} + 1})$$

(*b*). *If $\nabla_{\mathcal{G}} \leq m - 2$ and $\nabla_{\overline{\mathcal{G}}} \leq m - 2$, then*

$$\theta_1 + \overline{\theta}_1 \leq 2r^{m-1}\,(\sqrt{2e - m + 1} + \sqrt{m^2 - 2m - 2e + 1})$$

Proof. (*a*). If $\nabla_{\mathcal{G}} = m - 1$ or $\nabla_{\overline{\mathcal{G}}} = m - 1$, then Theorem 4.5 gives

$$\theta_1 \leq 2r^{m-1}\sqrt{2e - m + 1}. \tag{6.1}$$

Let the components of $\overline{\mathcal{G}}$ are $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_t$. Without loss of generality, suppose that $\theta_1(\mathcal{G}_1) \geq \theta_1(\mathcal{G}_2) \geq \cdots \geq \theta_1(\mathcal{G}_t)$ and $\overline{\theta}_1 = \theta_1(\mathcal{G}_1)$. Thus using Theorem 4.5, we obtain

$$\overline{\theta}_1 \leq 2r^{(m_{\mathcal{G}_1}-1)}\sqrt{2e_{\mathcal{G}_1} + m_{\mathcal{G}_1} + 1}. \tag{6.2}$$

From Equations (6.1) and (6.2), we have

$$\theta_1 + \overline{\theta}_1 \leq 2(r^{m-1}\sqrt{2e - m + 1} + r^{m_{\mathcal{G}_1}-1}\sqrt{2e_{\mathcal{G}_1} - m_{\mathcal{G}_1} + 1}).$$

(*b*). If $\nabla_{\mathcal{G}} \leq m - 2$ and $\nabla_{\overline{\mathcal{G}}} \leq m - 2$, then $\delta_{\overline{\mathcal{G}}} \geq 1$. From Theorem 4.5, we have

$$\theta_1 \leq 2r^{m-2}\sqrt{2e - m + 1}. \tag{6.3}$$

Now using $\delta_{\overline{\mathcal{G}}} = m - 1 - \nabla_{\mathcal{G}}$ and $\nabla_{\overline{\mathcal{G}}} \leq m - 2$, Theorem 4.4 and proof of Theorem 4.5, we obtain

$$
\begin{aligned}
\overline{\theta}_1 &\leq 2r^{m-2}\sqrt{2\frac{m(m-1)}{2} - 2e - \delta_{\overline{\mathcal{G}}}(m-1) + (\delta_{\overline{\mathcal{G}}} - 1)\nabla_{\overline{\mathcal{G}}}} \\
&= 2r^{m-2}\sqrt{(m^2 - 2m - 2e + 1) + \delta_{\mathcal{G}}(2 + \nabla_{\mathcal{G}} - m)} \\
&\leq 2r^{m-2}\sqrt{(m^2 - 2m - 2e + 1) + \delta_{\mathcal{G}}(2 + (m-2) - m)} \\
&= 2r^{m-2}\sqrt{(m^2 - 2m - 2e + 1)}
\end{aligned} \tag{6.4}
$$

From Equations (6.3) and (6.4), we get

$$
\theta_1 + \overline{\theta}_1 \leq 2r^{m-2}\left(\sqrt{2e - m + 1} + \sqrt{m^2 - 2m - 2e + 1}\right). \tag{6.5}
$$

□

Example 6.3. *Let $r > 1$ be a real number and consider the graph $\mathcal{K}_{11}$. We know that $\nabla_{\mathcal{K}_{11}} = 10$. Then*

$$
\begin{aligned}
\operatorname{spec}_{A_r}(\mathcal{K}_{11}) &= \{(20r^{10}), (-2r^{10})^{10}\}, \quad \theta_1 = 20r^{10} \\
\operatorname{spec}_{A_r}(\overline{\mathcal{K}}_{11}) &= \{(0)^{11}, \}, \quad \overline{\theta}_1 = 0.
\end{aligned}
$$

We can easily show that

$$
\begin{aligned}
\theta_1 + \overline{\theta}_1 &\geq \frac{110r}{11}, \\
\theta_1 + \overline{\theta}_1 &\leq 20r^{10},
\end{aligned}
$$

which is in accordance with Theorems 6.1 and 6.2.

We now give Nordhaus-Gaddum-type results on variable sum exdeg energies of $\mathcal{G}$ and $\overline{\mathcal{G}}$.

Theorem 6.4. *Suppose $\mathcal{G}(m, e, \nabla_{\mathcal{G}}, \delta_{\mathcal{G}})$ be a connected graph, $\mathcal{G}_1$ is a connected component of $\overline{\mathcal{G}}$ with $\overline{\theta}_1 = \theta_1(\mathcal{G}_1)$ and $r > 1$ be a real number. Let*

$$
\begin{aligned}
U_1 &= \sqrt{8em^2(m-1) - 4e^2(m-1)r^{4-2m}}, \\
U_1' &= \sqrt{\frac{4m^2(m-1)}{r^{2\delta_{\mathcal{G}}}} - \frac{(m-1)(m^2 - m - 2e)}{r^{2\nabla_{\mathcal{G}}}}}, \\
U_2 &= \sqrt{\frac{8em^2(m-1) - 4e^2(m-1)r^{6-2m}}{m^2}}, \\
U_2' &= 2\sqrt{1 + \frac{1-m}{m^2 - m - 2e}} + \sqrt{4(m-1) - \frac{r^{6-2m}(m^2 - m - 2e)(m-1)}{m^2}}.
\end{aligned}
$$

(a). *If* $\nabla_{\mathcal{G}} = m-1$ *or* $\nabla_{\overline{\mathcal{G}}} = m-1$, *then*

$$E_r(\mathcal{G}) + E_r(\overline{\mathcal{G}}) \leq r^{m-1}\left(2\sqrt{2e-m+1} + \frac{1}{\sqrt{m}}U_1\right) + 2r^{(m_{\mathcal{G}_1}-1)}\sqrt{2e_{\mathcal{G}_1} - m_{\mathcal{G}_1}+1} + \frac{r^{m-1}}{m}\sqrt{m^2-m-2e}\,U_1'.$$

(b). *If* $\nabla_{\mathcal{G}} \leq m-2$ *and* $\nabla_{\overline{\mathcal{G}}} \leq m-2$, *then*

$$E_r(\mathcal{G}) + E_r(\overline{\mathcal{G}}) \leq r^{m-2}\,(2\sqrt{2e-m+1} + U_2 + \sqrt{m^2-m-2e}\,U_2'),$$

Proof. Using Part (2) of Lemma 2.1 on $\overline{\mathcal{G}}$ with $\nabla_{\overline{\mathcal{G}}} = m-1-\delta_{\mathcal{G}}$ and $\delta_{\overline{\mathcal{G}}} = m-1-\nabla_{\mathcal{G}}$, we have

$$2\,\Omega \;=\; 2\sum_{1\leq i<j\leq m}\left(r^{d_{\overline{\mathcal{G}}}^{(w_i)}} + r^{d_{\overline{\mathcal{G}}}^{(w_j)}}\right)^2 \;\leq\; 2\sum_{1\leq i<j\leq m}(2\,r^{\nabla_{\overline{\mathcal{G}}}})^2 \;=\; 4(m^2-m-2e)r^{2m-2-2\delta_{\mathcal{G}}}.$$

Same as the proof of Theorem 4.5, we get

$$\overline{\theta}_1 \;\geq\; \frac{2\,e_{\overline{\mathcal{G}}}\,r^{\delta_{\overline{\mathcal{G}}}}}{m} \;=\; \frac{(m^2-m-2e)\,r^{m-1-\nabla_{\mathcal{G}}}}{m}.$$

Applying Cauchy-Schwartz inequality to obtain $(\sum_{j=2}^{m}|\overline{\theta}_j|)^2 \leq (m-1)\sum_{j=2}^{m}\overline{\theta}_j^2$. Therefore using Part (2) of Lemma 2.1, $(E_r(\overline{\mathcal{G}}) - \overline{\theta}_1)^2 \leq (m-1)\,(2\Omega - \overline{\theta}_1^2)$.

(a). Theorem 5.2 gives

$$\begin{aligned} E_r(\mathcal{G}) &\leq r^{m-1}\left(2\sqrt{2e-m+1} + \sqrt{\frac{8em(m-1) - 2e(m-1)r^{3-2m}}{m}}\right) \\ &= r^{m-1}\left(2\sqrt{2e-m+1} + \frac{1}{\sqrt{m}}U_1\right). \end{aligned} \tag{6.6}$$

Using inequality (6.2), we get

$$\begin{aligned} E_r(\overline{\mathcal{G}}) &\leq \overline{\theta}_1 + \sqrt{(m-1)\,(2\Omega - \overline{\theta}_1^2)} \\ &\leq 2r^{(m_{\mathcal{G}_1}-1)}\sqrt{2e_{\mathcal{G}_1} - m_{\mathcal{G}_1}+1} \\ &\quad + \sqrt{(m-1)\left(4(m^2-m-2e)\,r^{(2m-2-2\delta_{\mathcal{G}})} - \frac{(m^2-m-2e)^2\,r^{(2m-2-2\nabla_{\mathcal{G}})}}{m^2}\right)} \\ &= 2r^{(m_{\mathcal{G}_1}-1)}\sqrt{2e_{\mathcal{G}_1} - m_{\mathcal{G}_1}+1} + \frac{r^{m-1}}{m}\sqrt{m^2-m-2e}\,U_1' \end{aligned} \tag{6.7}$$

From Equations (6.6) and (6.7), we get

$$E_r(\mathcal{G}) + E_r(\overline{\mathcal{G}}) \leq r^{m-1}\left(2\sqrt{2e-m+1} + \frac{1}{m}U_1\right) + 2r^{(m_{\mathcal{G}_1}-1)}\sqrt{2e_{\mathcal{G}_1} - m_{\mathcal{G}_1}+1} + \frac{r^{m-1}}{m}\sqrt{m^2-m-2e}\,U_1'.$$

(b). From proof of Theorem 5.2, we see that

$$E_r(\mathcal{G}) \leq r^{m-2}\left(2\sqrt{2e-m+1}+\sqrt{\frac{8em^2(m-1)-4\,e^2\,(m-1)\,r^{6-2m}}{m^2}}\right) = r^{m-2}\left(2\sqrt{2e-m+1}+U_2\right). \tag{6.8}$$

Using Lemma 4.4, Equation (6.4) and proof of Theorem 4.5, we have

(6.9)

$$\begin{aligned} E_r(\overline{\mathcal{G}}) &\leq \overline{\theta}_1+\sqrt{(m-1)\,(2\varOmega-\overline{\theta}_1^2)} \\ &\leq 2r^{m-2}\sqrt{m^2-2m-2e+1}+ \\ &\sqrt{(m-1)\left(4(m^2-m-2e)r^{2m-2-2\delta_{\mathcal{G}}}-\frac{(m^2-m-2e)^2\,r^{2m-2-2\nabla_{\mathcal{G}}}}{m^2}\right)} \\ &\leq 2r^{m-2}\sqrt{m^2-2m-2e+1}+\sqrt{(m-1)\left(4(m^2-m-2e)r^{2m-4}-\frac{(m^2-m-2e)^2\,r^2}{m^2}\right)} \\ &= r^{m-2}\sqrt{m^2-m-2e}\left(2\sqrt{1+\frac{1-m}{m^2-m-2e}}+\sqrt{4(m-1)-\frac{(m-1)(m^2-m-2e)r^{6-2m}}{m^2}}\right) \\ &= r^{m-2}\sqrt{m^2-m-2e}\,U_2' \end{aligned}$$

From Equations (6.8) and (6.9), we get

$$E_r(\mathcal{G})+E_r(\overline{\mathcal{G}}) \leq r^{m-2}\,(2\sqrt{2e-m+1}+U_2+\sqrt{m^2-m-2e}\,U_2').$$

The proof is complete. □

Theorem 6.5. *Suppose $\mathcal{G}(m,e,\nabla_{\mathcal{G}},\delta_{\mathcal{G}})$ be a connected graph and $r\neq 1$ is a positive real number. Then*

$$E_r(\mathcal{G})+E_r(\overline{\mathcal{G}}) \;\geq\; 4r\sqrt{e}+8(m^2-m-2e)r^{2m-2-2\nabla_{\mathcal{G}}}.$$

Proof. Theorem 5.2 gives

$$E_r(\mathcal{G}) \;\geq\; 4r\sqrt{e}. \tag{6.10}$$

By Part (1) of Lemma 2.1, we have $\sum\limits_{j=1}^{m}\overline{\theta}_j^2=-2\sum\limits_{1\leq i<j\leq m}\overline{\theta}_i\,\overline{\theta}_j$. Using Part (2) of Lemma 2.1, we obtain

$$(E_r(\overline{\mathcal{G}}))^2=\left(\sum_{j=1}^{m}|\overline{\theta}_j|\right)^2=\sum_{j=1}^{m}\theta_j^2+2\sum_{1\leq i<j\leq m}|\overline{\theta}_i\,\overline{\theta}_j| \;\geq\; 2\,\varOmega+2\,|\sum_{1\leq i<j\leq m}\overline{\theta}_i\,\overline{\theta}_j| \;=\; 4\,\varOmega.$$

We know that $\delta_{\overline{\mathcal{G}}}=m-1-\nabla_{\mathcal{G}}$. Now

$$4\,\varOmega=4\sum_{1\leq i<j\leq m}(r^{d_{\overline{\mathcal{G}}}^{(w_i)}}+r^{d_{\overline{\mathcal{G}}}^{(w_j)}})^2 \;\geq\; 4\sum_{1\leq i<j\leq m}(2\,r^{\delta_{\overline{\mathcal{G}}}})^2 \;=\; 8(m^2-m-2e)\,r^{2m-2-2\nabla_{\mathcal{G}}}.$$

Hence

$$E_r(\overline{\mathcal{G}}) \;\geq\; 8(m^2 - m - 2e)\, r^{2m-2-2\nabla_{\mathcal{G}}}. \tag{6.11}$$

From Equations (6.10) and (6.11), we have

$$E_r(\mathcal{G}) + E_r(\overline{\mathcal{G}}) \;\geq\; 4r\sqrt{e} + 8(m^2 - m - 2e)r^{2m-2-2\nabla_{\mathcal{G}}}.$$

This completes the proof. □

Example 6.6. *Let $r \neq 1$ be a positive real number. It is easy to see that*

$$E_r(\mathcal{K}_{11}) + E_r(\overline{\mathcal{K}}_{11}) = 40r^{10} \geq 4r\sqrt{55},$$

which conform to Theorem 6.5.

7. V-Equienergetic graphs

Graphs with equal A_r-spectrum are known as V-cospectral; otherwise V-noncospectral. The graphs having equal variable sum exdeg energy are knows as V-equienergetic. The isomorphic graphs are always V-cospectral and thus have equal variable sum exdeg energy. In the current section, we find certain classes of V-noncospectral graphs having same variable sum exdeg energy. The proofs are similar to the proofs of [13, Section 5] and thus omitted.

The line graph of $\mathcal{G}$, represented by $\mathcal{L}_{\mathcal{G}}$, is the graph with $V_{\mathcal{L}_{\mathcal{G}}} = E_{\mathcal{G}}$ and two vertices of $\mathcal{L}_{\mathcal{G}}$ are adjacent by an edge if edges incident on it are neighbors in $\mathcal{G}$.

Suppose $\mathcal{G}(m, e)$ be a b-regular graph. Also suppose $\mathcal{L}^1_{\mathcal{G}} = \mathcal{L}_{\mathcal{G}}$, $\mathcal{L}^j_{\mathcal{G}} = \mathcal{L}(\mathcal{L}^{j-1}_{\mathcal{G}})$ be the repeated line graphs of $\mathcal{G}$, where $j = 1, 2, \ldots$. Ramane et al. [23] proved the following formula for $E(\mathcal{L}^2_{\mathcal{G}})$.

$$E(\mathcal{L}^2_{\mathcal{G}}) = 2mb(b-2). \tag{7.1}$$

Theorem 7.1. *The line graphs $\mathcal{L}^2_{\mathcal{G}_1}$ and $\mathcal{L}^2_{\mathcal{G}_2}$ are V-noncospectral with $E_r(\mathcal{L}^2_{\mathcal{G}_1}) = E_r(\mathcal{L}^2_{\mathcal{G}_2})$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are two b-regular m-vertex graphs and A-noncospectral.*

Corollary 7.2. *For any $s \geq 2$, the line graphs $\mathcal{L}^s_{\mathcal{G}_1}$ and $\mathcal{L}^s_{\mathcal{G}_2}$ are V-noncospectral with $E_r(\mathcal{L}^s_{\mathcal{G}_1}) = E_r(\mathcal{L}^s_{\mathcal{G}_2})$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are two m-vertex b-regular and A-noncospectral graphs.*

Theorem 7.3. *The graphs $\mathcal{G}_1 \cup \overline{\mathcal{K}}_t$ and $\mathcal{G}_2 \cup \overline{\mathcal{K}}_t$ are V-noncospectral with $E_r(\mathcal{G}_1 \cup \overline{\mathcal{K}}_t) = E_r(\mathcal{G}_2 \cup \overline{\mathcal{K}}_t)$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are two m-vertex V-noncospectral and V-equienergetic graphs.*

Corollary 7.4. *For any $s \geq 2$, the graphs $\mathcal{L}^s_{\mathcal{G}_1} \cup \overline{\mathcal{K}}_t$ and $\mathcal{L}^s_{\mathcal{G}_2} \cup \overline{\mathcal{K}}_t$ are V-noncospectral with $E_r(\mathcal{L}^s_{\mathcal{G}_1} \cup \overline{\mathcal{K}}_t) = E_r(\mathcal{L}^s_{\mathcal{G}_2} \cup \overline{\mathcal{K}}_t)$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are two m-vertex b-regular A-noncospectral graphs.*

Theorem 7.5. *Suppose $\mathcal{G}$ be an m-vertex graph with least one component to be $\mathcal{C}_p$. Also suppose $\mathcal{G}'$ be another m-vertex graph having equal components as of $\mathcal{G}$ excluding $\mathcal{C}_p$. With respect to each cycle $\mathcal{C}_p$ in $\mathcal{G}$, the graph $\mathcal{G}'$ has two cycles $\mathcal{C}_{\frac{p}{2}}$, $\mathcal{C}_{\frac{p}{2}}$. Then $\mathcal{G}$ and $\mathcal{G}'$ are V-noncospectral graphs having equal variable sum exdeg energy.*

8. Conclusion

We introduced variable sum exdeg energy of graphs. Bounds on spectral radius and spread of variable sum exdeg matrix are determined. We also find bounds on variable sum exdeg energy and Nordhaus-Gaddum-type results for variable sum exdeg spectral radius and energy. Some classes having the same variable sum exdeg energy are also determined.

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