https://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 13 No. 1 (2024), pp. 41-66.

www.ui.ac.ir

A SPANNING UNION OF CYCLES IN RECTANGULAR GRID GRAPHS, THICK GRID CYLINDERS AND MOEBIUS STRIPS

JELENA ĐOKIĆ, OLGA BODROŽA-PANTIĆ* AND KSENIJA DOROSLOVAČKI

Abstract. Motivated to find the answers to some of the questions that have occurred in recent papers dealing with Hamiltonian cycles (abbreviated HCs) in some special classes of grid graphs we started the investigation of spanning unions of cycles, the so-called 2-factors, in these graphs (as a generalizations of HCs). For all the three types of graphs from the title and for any integer $m \geq 2$ we propose an algorithm for obtaining a specially designed (transfer) digraph $\mathcal{D}^*_m$. The problem of enumeration of 2-factors is reduced to the problem of enumerating oriented walks in this digraph. Computational results we gathered for $m \leq 17$ reveal some interesting properties both for the digraphs $\mathcal{D}^*_m$ and for the sequences of numbers of 2-factors. We prove some of them for arbitrary $m \geq 2$.

1. Introduction

We consider the following (labeled) graphs: Rectangular grid graph $RG_m(n) = P_m \times P_n$, Thick grid cylinder $TkC_m(n) = P_m \times C_n$ and Moebius strip (of fixed width) $MS_m(n)$ (see Figures 1- 3) where P_n and C_n denote the path and cycle with n vertices, respectively. *Thick grid cylinder* $TkC_m(n)$ (*Moebius strip* $MS_m(n)$) can be obtained from the rectangular grid graph $RG_m(n+1)$ by contraction of vertices $A \equiv B_1, B_2, \ldots, B_{m-1}, B_m \equiv B$ with vertices $D \equiv D_1, \ldots, D_{m-1}, D_m \equiv C$ ($C \equiv D_m, D_{m-1}, \cdots, D_2, D_1 \equiv D$), respectively, which does not produce multiple overlapping vertical edges. Note that all observed graphs have $m \cdot n$ vertices.

Keywords: Hamiltonian cycles, generating functions, Transfer matrix method, 2-factor.

MSC(2010): Primary: 05C38; Secondary: 05C50, 05A15, 05C30, 05C85.

Article Type: Research Paper.

*Corresponding author.

Received: 24 November 2021, Accepted: 10 December 2022.

Cite this article: J. Đokić, O. Bodroža-Pantić and K. Doroslovački, A spanning union of cycles in rectangular grid graphs, thick grid cylinders and Moebius strips, Trans. Comb., **13** no. 1 (2024) 41–66. http://dx.doi.org/10.22108/toc.2022.131614.1940 .

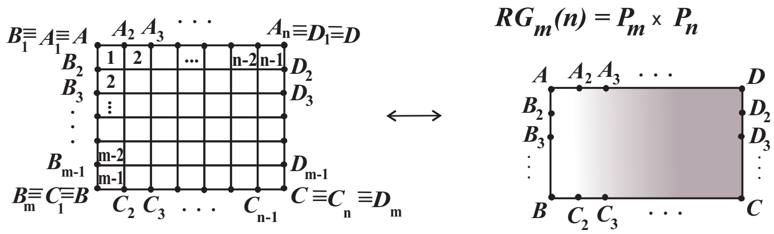

FIGURE 1. Rectangular Grid Graph $P_m \times P_n$.

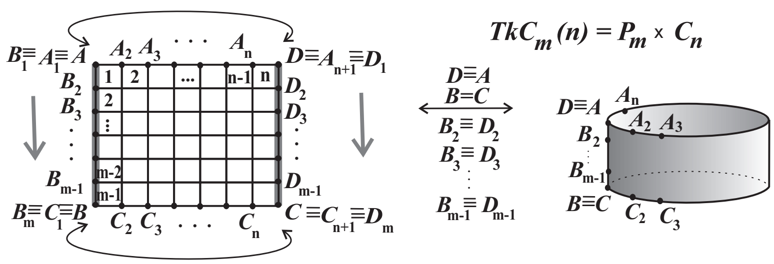

FIGURE 2. Identification of vertices B_i with vertices D_i in constructing the tick grid cylinder $TkC_m(n) = P_m \times C_n$.

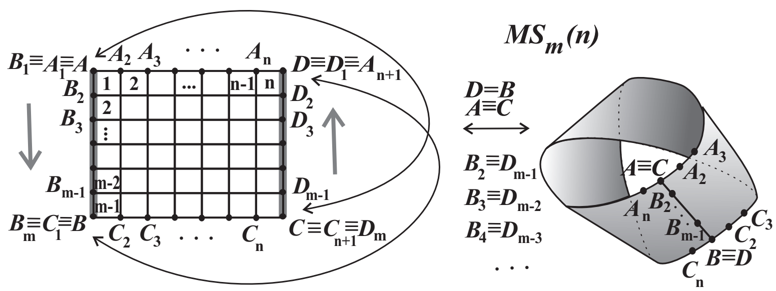

FIGURE 3. Identification of vertices B_i with vertices D_{m-i+1} in constructing the Moebius strip $MS_m(n)$.

A spanning r-regular subgraph of a graph is called an *r-factor*. For $r = 2$, it represents a spanning union of (disjoint) cycles. Hence, Hamiltonian cycles are connected 2-factors. All possible 2-factors of $RG_4(3) = P_4 \times P_3$ and $TkC_2(2) = P_2 \times C_2$ are shown in Figure 4. With the exception of the last case, for both of the aforementioned graphs, all 2-factors are Hamiltonian cycles.

The problems of enumerating and generating Hamiltonian paths in different classes of graphs arise in chemistry, biophysics (polymer melting and protein folding), theoretical physics (study of magnetic systems with O(n) symmetry)[13], engineering (path planning problems for robots and machine tools)[21] and bioinformatics (security and intellectual property protection by using the microelectrode dot array (MEDA) biochips) [14], as well as in the theory of algorithms [20]. They might be useful for the development of statistical algorithms that provide unbiased sampling of such paths [18].

A brief overview of the chronology of research on counting Hamiltonian cycles in different graph families can be found in [25]. The enumeration of HCs on specific grid graphs has been studied extensively in

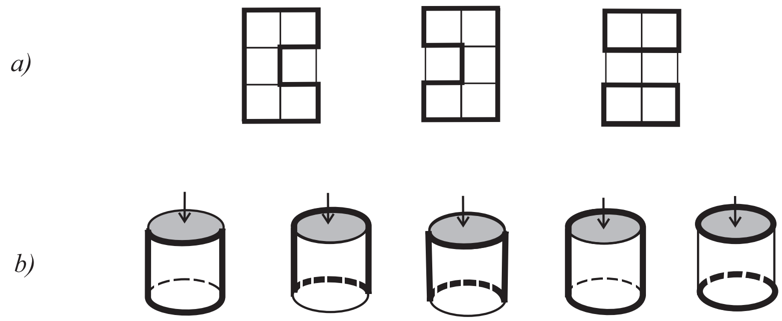

FIGURE 4. 2-factors in a) $P_4 \times P_3$ b) $P_2 \times C_2$.

[1]-[6], [15], [16] and [22]. The intrinsic properties of these grids naturally impose the transfer matrix approach on the problem of enumeration of HCs and related topics[11, 17]. From the computational data obtained from some recent papers a few interesting phenomena (concerning the rectangular grid graphs, thin and thick grid cylinders and their triangular variants) have arisen and were formulated as conjectures. More precisely, the numbers of the so-called contractible and non-contractible HCs (see Figure 5a-b) for thin cylinder graph ($C_m \times P_n$) are asymptotically equal (when $n \to \infty$) [4] and the same is valid for its triangular variant [2].

For the thick grid cylinder $P_m \times C_n$ the contractible HCs are more numerous than the non-contractible ones if and only if m is even. Indeed, the total number of HCs is

$$h_m(n) \sim \begin{cases} a_{m,c} n \theta_{m,c}^n, & \text{if } m \text{ is even,} \\ a_{m,nc} \theta_{m,nc}^n, & \text{if } m \text{ is odd ,} \end{cases}$$

where $\theta_{m,c}$, $\theta_{m,nc}$, $a_{m,c}$ and $a_{m,nc}$ are the positive dominant characteristic roots and their coefficients for the two types of HCs, respectively [3]. Additionally, the coefficient $a_{m,nc}$ for non-contractible HCs is equal to 1 (computational data for $m \leq 10$) [1]. Also, positive dominant characteristic root $\theta_{m,c}$ for contractible HCs in a thick grid cylinder is equal to the same one associated with rectangular grid graph $P_m \times P_n$ (computational data for $m \leq 10$) [1, 5].

The aim of this paper is to find the generating functions for the number of 2-factors in the considered graphs. We were wondering if the same or similar properties related to HCs would remain valid for 2-factors or not. We wanted to see if some conclusions for 2-factors could help proving the mentioned conjectures for HCs. Additionally, we expand our research to the new class of grid graphs - Moebius strips $MS_m(n)$.

We distinguish two types of cycles of a 2-factor on the cylindrical surface of $TkC_m(n)$ (viewed as closed Jordan curves): the *contractible* (abbr. *c-type*) and the *non-contractible* (abbr. *nc-type*) ones (see Figure 5 a-b). The first ones divide the surface into one finite (called the *interior*) and one infinite region (called the *exterior*). One could imagine them being pasted onto the cylindrical surface of $TkC_m(n)$. The latter ones divide the cylindrical surface into two infinite regions resembling a bracelet around an arm. In Figure 5)d), the shown 2-factor consists of two contractible cycles and one non-contractible cycle.

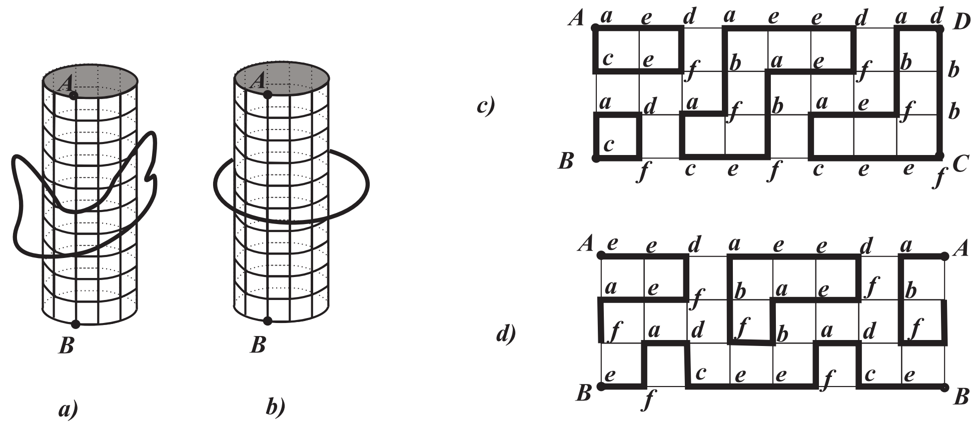

FIGURE 5. Types of cycles in $TkC_m(n)$: a) contractible; b) non-contractible; c) four (contractible) cycles in $RG_4(9)$; d) one non-contractible and two contractible cycles in $TkC_4(8)$

For graphs $TkC_m(n)$ and $MS_m(n)$ it is useful to observe the so-called *Rolling imprints* (RI) (introduced in [1]): Imagine that we at first "cut" the surface of the observed graph (with a given 2-factor) along the line AB (producing on this way the vertices D_i, i=1, 2,...,m on the right side, again). Afterwards, we unroll (unwind) and flatten it. Then, we produce infinitely many copies $R_k \equiv A^{(k)}B^{(k)}C^{(k)}D^{(k)}(k \in Z)$ of this rectangle picture (with adding the superscript "(k)" on all vertex labels), and line them up to the left and to the right of the first one ($R_0 \equiv A^0B^0C^0D^0$) using translation and glide-reflection, taking care that corresponding vertices ($B_i^{(k)}$ and $D_i^{(k-1)}$ for $TkC_m(n)$, or $B_i^{(k)}$ and $D_{m-i+1}^{(k-1)}$ for $MS_m(n)$) of adjacent copies are contracted (see Figure 6c)-d)).

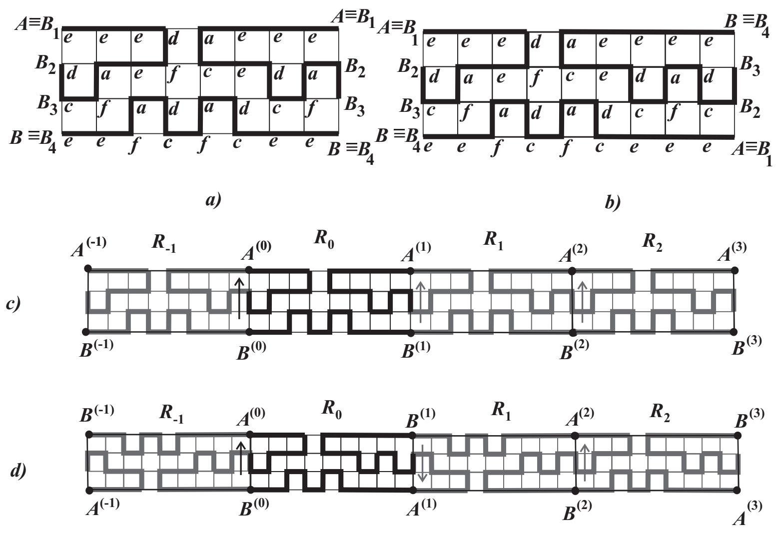

FIGURE 6. a) The spanning union of one c-type and one nc-type cycle in $TkC_4(8)$; b) Hamiltonian (short) cycle in $MS_4(9)$; c) Rolling imprints for the example a) related to $TkC_4(8)$; d) Rolling imprints for the example b) related to $MS_4(9)$

http://dx.doi.org/10.22108/toc.2022.131614.1940

Note that, for each contractible cycle for any type of observed graphs, there exist an infinite number of congruent cycles (polygons) in the infinite grid graph (RI). For a non-contractible cycle in $TkC_m(n)$ there exists a unique infinite path which crosses lines $A^{(k)}B^{(k)}$ an odd number of times.

In case of $MS_m(n)$, due to topological reasons, there are three possible types of cycles: c-type and two nc-types shown in Figure 7a) and b). The first of these nc-cycles is called the *long* nc-cycle. Its image in RI is the union of two infinite paths that cross the line $A^{(k)}B^{(k)}$ an even number of times. The second one is called the *short non-contractible* cycle (abbr. *short nc-type*). Its image in Rolling imprints is the unique infinite path which crosses line $A^{(k)}B^{(k)}$ an odd number of times. A 2-factor of $MS_m(n)$ can have at most one short nc-type cycle (due to topological reasons, too). Additionally, the long nc-cycle divides the surface of $MS_m(n)$ into two parts, while the short one cannot divide that surface. For example, the 2-factor shown in Figure 7d) is the union of one c-cycle, two long nc-cycles and one short nc-cycle.

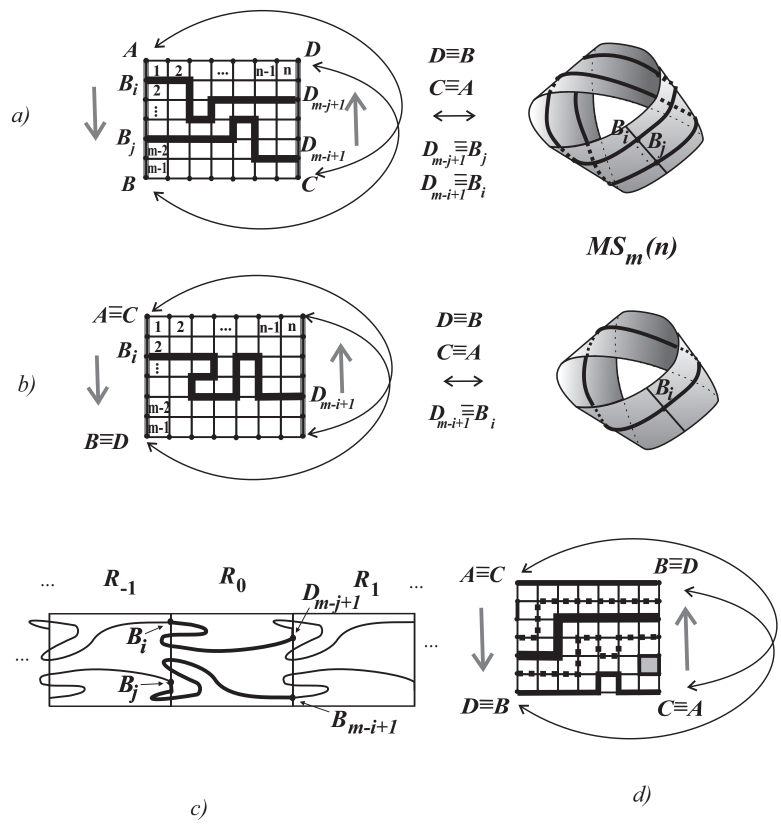

FIGURE 7. Type of cycles in Moebius strip $MS_m(n)$: a) long nc-type cycle; b) short nc-type cycle; c) a long nc-cycle can cross the vertical side [AB] more than 2 times; d) an example of 2-factor with one short and two long nc-type cycles and one contractible cycle.

The rest of the paper is organized as follows. In Section 2, we derive necessary and sufficient conditions for the existence of a 2-factor of a certain type in the grid graphs under consideration. Then, we present a characterization of a 2-factor for each graph $G = G_m(n) \in \mathcal{G} \stackrel{\text{def}}{=} \{RG_m(n), TkC_m(n), MS_m(n)\}$ obtained by the vertex-coding approach. In Section 3, using this we propose applying the transfer matrix method in order to obtain the numbers of 2-factors, labeled by $f_m^G(n)$, for the considered graph $G_m(n) \in \mathcal{G}$. Actually,

http://dx.doi.org/10.22108/toc.2022.131614.1940

this enumeration problem is reduced to the problem of enumerating oriented walks in a specially designed digraph $\mathcal{D}_m^*$ (so-called *transfer digraph*). Computational results we gathered for $m \leq 17$ (partially given in Section 4, the rest of them in the extended version of this paper [9]) reveal some interesting properties of the digraphs $\mathcal{D}_m^*$. We prove some of them for arbitrary $m \geq 2$ in Section 3. The properties referring to the asymptotic behaviour of the numbers of 2-factors $f_m^G(n)$ were observed to be similar to the ones that appeared while studying Hamiltonian cycles. In Section 4, we propose a few conjectures.

2. Code matrix

Definition 2.1.

- *We orient a (contractible) cycle C in $RG_m(n)$ clockwise and call it the* **base figure***.*
- *If a contractible cycle C in $TkC_m(n)$ or $MS_m(n)$ is disjoint with segment $[AB]$, then we orient the corresponding cycle in RI that lies entirely in the rectangle $R_0 \backslash [A^{(1)}B^{(1)}]$ clockwise and call it the* **base figure***. Otherwise, let B_i $(1 \leq i \leq m)$ be the vertex from the intersection $C \cap [AB]$ with minimal index i. The* **base figure** *is the corresponding cycle in rolling imprints that contains the vertex $B_i^{(0)}$ oriented clockwise, too.*
- *For a non-contractible cycle C in $TkC_m(n)$ (or short nc-cycle C in $MS_m(n)$), let B_i $(1 \leq i \leq m)$ be the vertex from the intersection $C \cap [AB]$ with minimal index i $(1 \leq i \leq m)$. We orient the part of image of C in RI from $B_i^{(0)}$ to $D_i^{(0)} \equiv B_i^{(1)}$ $(D_{m-i+1}^{(0)} \equiv B_i^{(1)})$ in this direction and call it the* **base figure***.*
- *Finally, let B_i $(1 \leq i \leq m)$ denote the vertex of segment $[AB]$ with minimal index i $(1 \leq i \leq m)$ which belong to a long nc-cycle C in $MS_m(n)$. The* **base figure** *is the part of the infinite path (in RI) containing the vertex $B_i^{(0)}$ which is determined by vertices $B_i^{(0)}$ and $D_{m-i+1}^{(1)} \equiv B_i^{(2)}$ and oriented from $B_i^{(0)}$ to $D_{m-i+1}^{(1)}$.*

In this way, we establish a bijection between the set of all edges of the union of base figures for the considered 2-factor and the set of all of its edges. The edges of base figures, considered as oriented segments in rolling imprints, can be treated as unit vectors of 4 possible directions ($\uparrow$, $\downarrow$, $\rightarrow$ and $\leftarrow$). Let $\sharp_C(\uparrow)$, $\sharp_C(\downarrow)$, $\sharp_C(\rightarrow)$ and $\sharp_C(\leftarrow)$ denote the number of edges of corresponding direction which we pass when walking through the base figure of a cycle C.

Proposition 2.2. *If C is a c-cycle (for all three graphs $RG_m(n)$, $TkC_m(n)$ and $MS_m(n)$), then*

$$\sharp_C(\uparrow) = \sharp_C(\downarrow) \text{ and } \sharp_C(\rightarrow) = \sharp_C(\leftarrow). \tag{2.1}$$

If C is an nc - cycle for $TkC_m(n)$, then

$$\sharp_C(\uparrow) = \sharp_C(\downarrow) \text{ and } \sharp_C(\rightarrow) - \sharp_C(\leftarrow) = n. \tag{2.2}$$

If C is a short nc - cycle for $MS_m(n)$, then

$$\sharp_C(\downarrow) - \sharp_C(\uparrow) = m + 1 - 2i \text{ and } \sharp_C(\rightarrow) - \sharp_C(\leftarrow) = n. \tag{2.3}$$

If C is a long nc - cycle for $MS_m(n)$, then

$$\sharp_C(\uparrow) = \sharp_C(\downarrow) \text{ and } \sharp_C(\rightarrow) - \sharp_C(\leftarrow) = 2n. \tag{2.4}$$

Proof. Straightforward. □

Let us denote the total numbers of 2-factors in $RG_m(n)$, $TkC_m(n)$ and $MS_m(n)$ by $f_m^{RG}(n)$, $f_m^{TkC}(n)$ and $f_m^{MS}(n)$, respectively. The total number of 2-factors in $TkC_m(n)$ which contain odd (even) number of non-contractible cycles is labeled by $f_{1,m}^{TkC}(n)$ ($f_{0,m}^{TkC}(n)$). Similarly, label $f_{1,m}^{MS}(n)$ ($f_{0,m}^{MS}(n)$) represents the total numbers of 2-factors in $MS_m(n)$ which contain (do not contain) a short cycle. Thus, we have

$$f_m^{TkC}(n) = f_{1,m}^{TkC}(n) + f_{0,m}^{TkC}(n) \quad \text{and} \quad f_m^{MS}(n) = f_{1,m}^{MS}(n) + f_{0,m}^{MS}(n).$$

Theorem 2.3.

a) $f_m^{RG}(n) = 0$ *if and only if both m and n ($m, n \geq 2$) are odd;*

b) $f_{0,m}^{TkC}(n) = 0$ *if and only if both m and n ($m, n \geq 1$) are odd;*

c) $f_{1,m}^{TkC}(n) = 0$ *if and only if m is even and n is odd ($m, n \geq 1$);*

d) $f_{0,m}^{MS}(n) = 0$ *if and only if both m and n ($m, n \geq 1$) are odd;*

e) $f_{1,m}^{MS}(n) = 0$ *if and only if both m and n ($m, n \geq 1$) are even.*

Proof. We derive sufficiency of the corresponding condition (relative to parity of m and n) by contraposition.

a) If there exists a 2-factor in $RG_m(n)$, then by using (2.1) we conclude that the number of its edges must be even. Consequently, it is not possible for both m and n to be odd.

b) If there exists a 2-factor in $TkC_m(n)$ with an even number of nc-cycles, then by using (2.1) and (2.2) we obtain the same conclusion as in case a).

c) If there exists a 2-factor in $TkC_m(n)$ with an odd number of nc-cycles, then by using (2.1) and (2.2) we conclude that the number of its edges $m \cdot n$ must be of the same parity as n. It further implies that m is odd or n is even, which is the negation of the given condition.

d) Suppose that there exists a 2-factor in $MS_m(n)$ without a short nc-cycle. By using (2.1) and (2.4) we arrive at the same conclusion as in cases a) and b).

e) If there exists a 2-factor in $MS_m(n)$ with a short nc-cycle, then from (2.1), (2.3) and (2.4) the number of edges ($m \cdot n$) must be of the same parity as $m + n + 1$. This implies that m and n can only not both be even.

The proofs of necessity of these conditions go by construction of 2-factors for each of the three remaining combinations of m and n for each item separately. For example, one of the possible 2-factors for each of the three admitted combinations in case e) (2-factors with a short nc-cycle on $MS_m(n)$) are show in Figures 8. The rest of the proof is left to the readers as an exercise. □

Let us consider all graphs: $RG_m(n)$, $TkC_m(n)$ and $MS_m(n)$ simultaneously. First, "cut and develop in the plane" the surfaces of given graphs as shown in Figures 1 - 3. Observe the corresponding rectangular grid graphs, whereby the described identifications of vertices have been performed. In this way, we can use the words: *left, right, upper* and *lower* to mark the positions of adjacent vertices in the considered graph G, with respect to each other.

We label each vertex of $G \equiv G_m(n)$ by an ordered pair $(i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ where i represents the ordinal number of the row viewed from top to down, while j represents the ordinal

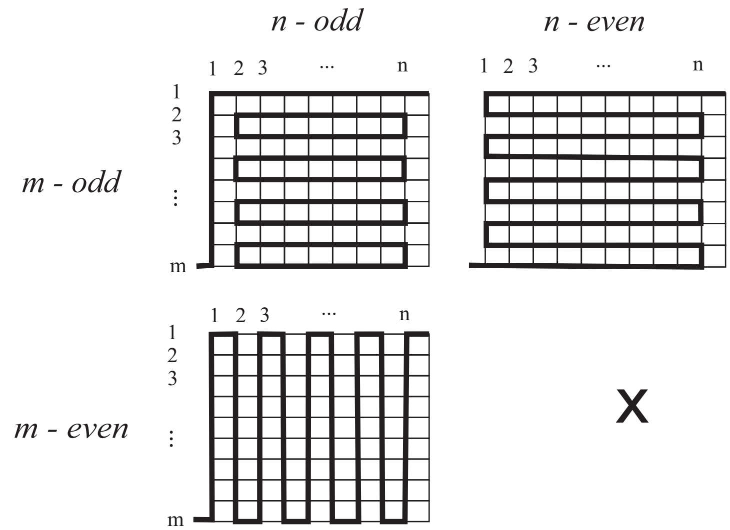

FIGURE 8. The existence of 2-factors in $MS_m(n)$ with a short nc-cycle.

number of the column, viewed from left to right. The vertices labeled by $A_1, A_2, \ldots, A_n$ in Figures 1 - 3 belong to the first row, and the vertices labeled by $B_1, B_2, \ldots, B_m$ belong to the first column.

Let us observe an arbitrary 2-factor of G. One of six possible labels shown in Figure 9 is assigned to each vertex $(i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$. We call this label an *alpha-letter* of the vertex and denote it by $\alpha_{i,j}$. By reading alpha-letters for vertices from the same column in the grid graph under consideration, from top to down, we obtain an *alpha word*. For instance, in the first 2-factor shown in Figures 8, for the first three and for the last (nth) column the corresponding alpha words are $ab^{m-2}f$, $e(ac)^{(m-1)/2}$, e^m and $e(df)^{(m-1)/2}$, respectively.

For each alpha-letter $\alpha_{i,j}$, denote by $\overline{\alpha}_{i,j}$ ($\alpha'_{i,j}$) the alpha label shown in Figure 9 obtained by applying reflection symmetry with the horizontal (vertical) axis as its line of symmetry onto $\alpha_{i,j}$. Thus, $\overline{a} = c, \overline{b} = b, \overline{c} = a$, $\overline{d} = f, \overline{e} = e$ and $\overline{f} = d$ and $a' = d, b' = b, c' = f, d' = a, e' = e$ and $f' = c$.

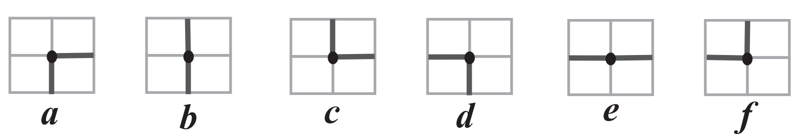

FIGURE 9. Six possible situations for given 2-factor in any vertex

Definition 2.4. *For any alpha-word* $\alpha \equiv \alpha_1\alpha_2\cdots\alpha_{k-1}\alpha_k$ $(k \in N)$*, the alpha word* $\overline{\alpha} \equiv \overline{\alpha}_k\overline{\alpha}_{k-1}\cdots\overline{\alpha}_2\overline{\alpha}_1$ *(obtained by reflection over horizontal axis) is called* **horizontal conversion** *of the word* α*.*
The alpha word $\alpha' \equiv \alpha'_1\alpha'_2\cdots\alpha'_{k-1}\alpha'_k$ *(obtained by reflection over vertical axis) is called* **vertical conversion** *of the word* α*.*

If we know the alpha-letter of a vertex (i, j) in $G_m(n)$, then the alpha-letter of its adjacent vertex can not be just about any letter from the set $\{a, b, c, d, e, f\}$. Instead, it is determined by the digraphs $\mathcal{D}_{lr}$ and $\mathcal{D}_{ud}$, as shown in Figure 10, depending on the mutual position of the two adjacent vertices.

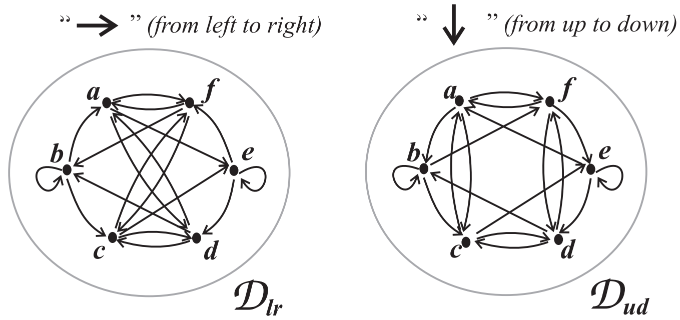

FIGURE 10. Digraphs $\mathcal{D}_{ud}$ and $\mathcal{D}_{lr}$ and

For example, for $1 \leq j \leq n-1$ and $1 \leq i \leq m-1$, if $\alpha_{i,j} = a$, then $\alpha_{i,j+1} \in \{d, e, f\}$ and $\alpha_{i+1,j} \in \{b, c, f\}$. But, if $j = n$ and $G = MS_m(n)$, then $\alpha_{i,n} = a$ implies $\alpha_{m-i+1,1} \in \{d, e, f\}$ because of $D_{m-i+1} \equiv B_i$ and $\overline{\alpha}_{m-i+1,n} = c$. Note that for graph $RG_m(n)$ alpha-letters for the corner vertices (A,B,C and D) have to be $\alpha_{1,1} = a$, $\alpha_{m,1} = c$, $\alpha_{1,n} = d$ and $\alpha_{m,n} = f$, respectively, which is not valid for the other graphs.

Now, with each 2-factor of the observed graph G we associate the *code matrix* $[\alpha_{i,j}]_{m\times n}$ where $\alpha_{i,j}$ is the alpha-letter of the vertex (i, j) $(1 \leq i \leq m,\ 1 \leq j \leq n)$.

Lemma 2.5. *(Characterisation of 2-factors) The code matrix $[\alpha_{i,j}]_{m\times n}$ associated with a 2-factor of a grid graph $G \in \mathcal{G}$ satisfies the following properties:*

(1) ***Column conditions:*** *For every fixed j $(1 \leq j \leq n)$,*
  (a) *the ordered pairs $(\alpha_{i,j}, \alpha_{i+1,j})$, where $1 \leq i \leq m-1$, must be arcs in the digraph $\mathcal{D}_{ud}$.*
  (b) $\alpha_{1,j} \in \{a, d, e\}$ *and* $\alpha_{m,j} \in \{c, e, f\}$.
(2) ***Adjacency of column condition:*** *For every fixed j, where $1 \leq j \leq n-1$, the ordered pairs $(\alpha_{i,j}, \alpha_{i,j+1})$, where $1 \leq i \leq m$, must be arcs in the digraph $\mathcal{D}_{lr}$.*
(3) ***First and Last Column conditions:***
  (a) *If $G = RG_m(n)$, then the alpha-word of the first column consists of the letters from the set $\{a, b, c\}$ and of the last column of the letters from the set $\{b, d, f\}$.*
  (b) *If $G = TkC_m(n)$, then the ordered pairs $(\alpha_{i,n}, \alpha_{i,1})$, where $1 \leq i \leq m$, must be arcs in the digraph $\mathcal{D}_{lr}$.*
  (c) *If $G = MS_m(n)$, then the ordered pairs $(\overline{\alpha}_{i,n}, \alpha_{m-i+1,1})$, where $1 \leq i \leq m$, must be arcs in the digraph $\mathcal{D}_{lr}$.*

The converse is also true. Every matrix $[\alpha_{i,j}]_{m\times n}$ with entries from $\{a, b, c, d, e, f\}$ that satisfies conditions 1–3 determines a unique 2-factor on the grid graph G.

Proof. The properties above can easily be proved directly, by checking all the possible edge arrangements (their compatibility) for adjacent vertices of G and constraints imposed by the structure of the considered graph G. Vice versa, alpha-letters and the possibility of their contact, expressed by digraphs $\mathcal{D}_{ud}$ and $\mathcal{D}_{lr}$, make sure that the subgraph of the graph G determined by the matrix (marked with bold lines) is a spanning 2-regular graph, i.e. a union of cycles (a 2-factor). □

3. Enumeration of 2-factors

Now, we can create for each integer m $(m \in N)$ a digraph $\mathcal{D}_m \overset{\text{def}}{=} (V(\mathcal{D}_m), E(\mathcal{D}_m))$ (common for all graphs from $\mathcal{G}$), in the following way: the set of vertices $V(\mathcal{D}_m)$ consists of all possible words $\alpha_{1,j}\alpha_{2,j}\cdots\alpha_{m,j}$ over alphabet $\{a,b,c,d,e,f\}$ (called *alpha-words*) which fulfill Condition 1 (*Column conditions*) from Lemma 2.5; an arc joins the vertex $v = \alpha_{1,j}\alpha_{2,j}\cdots\alpha_{m,j}$ and the vertex $u = \alpha_{1,j+1}\alpha_{2,j+1}\cdots\alpha_{m,j+1}$, i.e. $(v,u) \in E(\mathcal{D}_m)$, or $v \to u$ if and only if for the vertex v and u Condition 2 (*Adjacency of column condition*) from Lemma 2.5 is satisfied (vertex v might be the previous column for vertex u in a code matrix).

The subset of $V(\mathcal{D}_m)$ which consists of all the possible first (last) columns in a code matrix $[\alpha_{i,j}]_{m\times n}$ for $RG_m(n)$ (Condition 3a) is denoted by $\mathcal{F}_m$ $(\mathcal{L}_m)$. From Condition 1b, the first column of $[\alpha_{i,j}]_{m\times n}$ is an alpha-word from $\{a,b,c\}^m$, with $\alpha_{1,1} = a$ and $\alpha_{m,1} = c$. Similarly, the last column of $[\alpha_{i,j}]_{m\times n}$ is an alpha-word from $\{b,d,f\}^m$, with $\alpha_{1,n} = d$ and $\alpha_{m,n} = f$.

Lemma 3.1. *The cardinality of sets $\mathcal{F}_m$ and $\mathcal{L}_m$ $(m \in N)$ are equal to the $(m-1)$th member of the Fibonacci sequence F_{m-1}.*

Proof. The cardinal number of the set $\mathcal{F}_m$ is equal to the number of all oriented walks of length $m-1$ starting with vertex a and ending with vertex c in the subdigraph of $\mathcal{D}_{ud}$ induced by the set $\{a,b,c\}$. Note that characteristic polynomial of the adjacency matrix of this digraph is $\lambda(\lambda^2-\lambda-1)$, i.e. the required sequence $\mid \mathcal{F}_m \mid$ $(m \in N)$ obeys the same recurrence relation as the Fibonacci sequence. Since $\mid \mathcal{F}_1 \mid = 0 = F_0$ and $\mid \mathcal{F}_2 \mid = 1 = F_1$ $(\mid \mathcal{F}_3 \mid = 1 = F_2$; $\mathcal{F}_2 = \{ac\}$, $\mathcal{F}_3 = \{abc\})$, by induction, we conclude our assertion. The proof for the set $\mathcal{L}_m$ can be carried out analogously. □

Let $\mathcal{P}_m = [p_{ij}]$ be the square binary matrix of order $\mid V(\mathcal{D}_m) \mid$ for which $p_{i,j} = 1$ if and only if the i-th and j-th vertex of the digraph $\mathcal{D}_m$ can be obtained from each other by horizontal conversion; otherwise $p_{i,j} = 0$. Note that matrix $\mathcal{P}_m$ is a symmetric one. The following lemma is essential for the enumeration of 2-factors.

Lemma 3.2. *If $f_m^G(n)$ $(m \geq 2)$ denotes the number of 2-factors of $G \in \mathcal{G}$, then*

$$f_m^G(n) = \begin{cases} \sum\limits_{v_i \in \mathcal{F}_m} \sum\limits_{v_j \in \mathcal{L}_m} a_{i,j}^{(n-1)} = \sum\limits_{v_i \in \mathcal{F}_m} a_{i,i}^{(n)} = a_{1,2}^{(n+1)}, & \text{if } G = RG, \\ tr(\mathcal{T}_m^n) = \sum\limits_{v_i \in V(\mathcal{D}_m)} a_{i,i}^{(n)}, & \text{if } G = TkC, \\ tr(\mathcal{P}_m \cdot \mathcal{T}_m^n) = \sum\limits_{\substack{v_i, v_j \in V(\mathcal{D}_m) \\ \overline{v}_i = v_j}} a_{i,j}^{(n)}, & \text{if } G = MS, \end{cases}$$

where $\mathcal{T}_m = [a_{ij}]$ is the adjacency matrix of the digraph $\mathcal{D}_m$ (transfer matrix) and vertices $v_1, v_2 \in V(\mathcal{D}_m)$ are words $db^{m-2}f$ and $ab^{m-2}c$, respectively.

Proof. Using Lemma 2.5 the problem of enumeration of 2-factors in $G \in \mathcal{G}$ (with $m \times n$ vertices) reduces to the enumeration of all the possible code matrices for $G_m(n)$, i.e. all oriented walks of the length $n-1$ in the digraph $\mathcal{D}_m$ for which the initial and final vertices satisfy The First and Last conditions from Lemma 2.5. Note that the set $\mathcal{F}_m$ consists of all the direct successors of all the vertices from $\mathcal{L}_m$ (including vertex $db^{m-2}f$), and the set $\mathcal{L}_m$ consists of all the direct predecessors of all the vertices from $\mathcal{F}_m$ (including vertex $ab^{m-2}c$. So, in this way, for $G = RG_m(n)$ our problem of enumeration of all oriented walks of length $n-1$ in the digraph $\mathcal{D}_m$ with initial vertices in $\mathcal{F}_m$ and the final vertices in $\mathcal{L}_m$ can be reduced to enumeration of all the closed oriented walks of length n in the digraph $\mathcal{D}_m$ with initial vertex from the set $\mathcal{F}_m$ or the number of all oriented walks of length $n+1$ with initial vertex $db^{m-2}f$ and the final vertex $ab^{m-2}c$. Recall that the (i,j)-entry $a_{i,j}^{(k)}$ of k-th power of the adjacency matrix $\mathcal{T}_m$ of the digraph $\mathcal{D}_m$ represents the number of all oriented walks of length k $(k \in N)$ which start with vertex v_i and end with vertex v_j $(v_i, v_j \in V(\mathcal{D}_m))$[19]. This implies the assertion of Lemma 3.2. □

Proposition 3.3. *For vertical conversion the following applies:*
a) $v \to v'$, *where* $v \in V(\mathcal{D}_m)$,
b) If $v \to w$, *then* $w' \to v'$, *where* $v, w \in V(\mathcal{D}_m)$.

Proof. Straightforward. □

From Definition 2.4 we have $(v')' = v$ for any vertex $v \in V(\mathcal{D}_m)$. For example, vertical conversion pairs for $\mathcal{D}_3$ are: abf and dbc, edf and eac, dfe and ace, abc and dbf, afe and dce, edc and eaf; while the vertical conversion of $eee \in V(\mathcal{D}_3)$ is this vertex itself (see Figure 11).

Lemma 3.4. *The number of vertices in* $\mathcal{D}_m$ *is* $\mid V(\mathcal{D}_m) \mid = \frac{1}{2}(3^m + (-1)^m)$.

Proof. The number of all the words of length m over alphabet $\{a, b, c, d, e, f\}$ that satisfy Column conditions of Lemma 2.5) is equal to the number of all the oriented walks of length $m-1$ in $\mathcal{D}_{ud}$ starting with a vertex from $\{a, d, e\}$ and finishing with a vertex from $\{c, e, f\}$. Characteristic polynomial of the adjacency matrix of this digraph is $P(\lambda) = \lambda^4(1+\lambda)(\lambda-3)$. Thus, the required sequence, labeled with c_m $(m \in N)$, obeys the recurrence relation $c_m = 2c_{m-1} + 3c_{m-2}$, with initial conditions $c_1 = 1$ (the word e), $c_2 = 5$ (the words ac, bd, ee, dc and af). Using standard procedure for solving recurrence relations we obtain $c_m = \dfrac{(-1)^m + 3^m}{2}$, which implies $\mid V(\mathcal{D}_m) \mid \leq \dfrac{(-1)^m + 3^m}{2}$. In order to prove strict equality note that for any word w (with properties described above), from Proposition 3.3 a) we have $w \to w'$ and $w' \to w$. This implies that the word w appears as a column in the code matrix of some 2-factor of the graph $TkC_m(2)$. □

For each vertex $\alpha_1\alpha_2\cdots\alpha_m \in V(\mathcal{D}_m)$ we introduce a binary word called an *outlet (inlet) word* depending on whether the situations shown in Figure 9 matched to its letters have an edge "on the right" ("on the left") or not.

Definition 3.5. *The* outlet word *(*inlet word*) of the word* $\alpha \equiv \alpha_1\alpha_2\cdots\alpha_m \in V(\mathcal{D}_m)$ *is the binary word* $o(\alpha) \equiv o_1o_2\cdots o_m$ $(i(\alpha) \equiv i_1i_2\cdots i_m)$ *where*

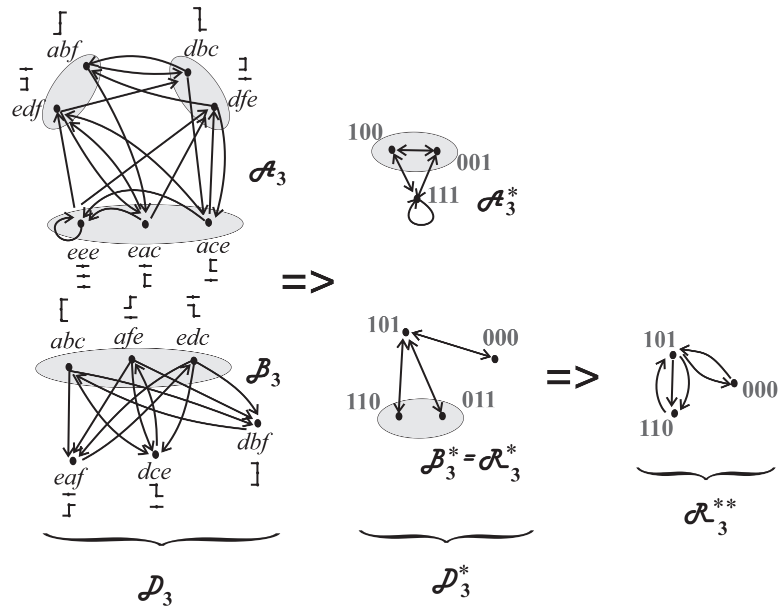

FIGURE 11. Digraphs $\mathcal{D}_3$, $\mathcal{D}_3^*$ and $\mathcal{R}_3^{**}$.

$$o_j \stackrel{\text{def}}{=} \begin{cases} 0, & if \;\; \alpha_j \in \{b,d,f\} \\ 1, & if \;\; \alpha_j \in \{a,c,e\} \end{cases} \quad and \;\; i_j \stackrel{\text{def}}{=} \begin{cases} 0, & if \;\; \alpha_j \in \{a,b,c\} \\ 1, & if \;\; \alpha_j \in \{d,e,f\} \end{cases} , \quad 1 \leq j \leq m$$

Lemma 3.6. *Digraph $\mathcal{D}_m$ for $m \geq 2$ is disconnected. Each of its components is a strongly connected digraph.*

Proof. Let $v \to w$, where $v, w \in \mathcal{D}_m$ and $w \equiv \alpha_1\alpha_2\cdots\alpha_m$. Notice that $o(v) \equiv i(w)$ and the total number of bold edges in all situations shown in Figure 9 for all $\alpha_1, \alpha_2, \ldots, \alpha_m$ is equal to $2m$. Considering that each vertical edge is calculated twice in that sum, we conclude that the numbers of 1s in $o(v)$ and in $o(w)$ have the same parity. Consequently, any two vertices of $\mathcal{D}_m$ having different parity of the number of 1's in their outlet words (for example, the words $ab^{m-2}c$ and $ab^{m-2}f$) can not belong to the same component of $\mathcal{D}_m$, i.e. $\mathcal{D}_m$ is disconnected.

To prove the second assertion of the lemma, observe an arbitrary oriented walk $w_0w_1\cdots w_{k-1}w_k$ (of length $k \in N$). Then using Proposition 3.3 conclude that there exists an oriented walk $w_kw'_kw'_{k-1}\cdots w'_1w'_0w_0$ which starts and finishes with w_k and w_0, respectively. Consequently, all components are strongly connected digraphs. □

Let $\mathcal{D}_m = \mathcal{A}_m \cup \mathcal{B}_m$, $m \geq 2$ where $\mathcal{A}_m$ is the component of the digraph $\mathcal{D}_m$ which contains the vertex e^m ($m \geq 2$), i.e. with the outlet word $11\cdots1$ - the unique vertex with loop. The fact that the word $db^{m-2}f$ ($m \geq 2$) (with the outlet word $00\cdots0$) belongs to $\mathcal{A}_m$ depends on the parity of m. Now, let $\mathcal{R}_m$ denote the component of $\mathcal{D}_m$ which contains the vertex $db^{m-2}f$ ($m \geq 2$). Its vertices are all the possible columns of the code matrices for $RG_m(n)$, $n \in N$.

Lemma 3.7. *$\mathcal{A}_m \equiv \mathcal{R}_m$ if and only if m is even.*

Proof. For $m = 2k$ $(k \in N)$ $e^m \to (df)^k \to ab^{m-2}c \to db^{m-2}f$ which implies $\mathcal{A}_m \equiv \mathcal{R}_m$. For m odd, the outlet words of vertices in $V(\mathcal{A}_m)$ have odd, while outlet words of vertices in $V(\mathcal{R}_m)$ have even numbers of 1s. Consequently, $\mathcal{A}_m \not\equiv \mathcal{R}_m$. □

In order to reduce the transfer matrix $\mathcal{T}_m$, note that two vertices from $\mathcal{D}_m$ with the same outlet word have the same set of direct successors. For all vertices from $V(\mathcal{D}_m)$ having the same corresponding outlet word we replace them with just one vertex, labeled by their common outlet word. Arcs from $E(\mathcal{D}_m)$ starting from these contracted vertices and finishing with the same vertex are substituted with only one arc. This way, we obtain the digraph $\mathcal{D}_m^*$, with adjacency (transfer) matrix $\mathcal{T}_m^*$. For example, in Figure 11 vertices $abc, afe, edc \in V(\mathcal{D}_3)$ with the common outlet word 101 are contracted at $101 \in V(\mathcal{D}_3^*)$. (Here the double-headed arrow represents two different edges, one for each direction.)

Note that two different vertices from $V(\mathcal{D}_m)$ with the same outlet word can not have the same direct predecessor. This implies that there exist no multiple edges in $\mathcal{D}_m^*$, i.e. entries of $\mathcal{T}_m^*$ are from the set $\{0, 1\}$.

Theorem 3.8. *Adjacency matrix $\mathcal{T}_m^*$ of the digraph $\mathcal{D}_m^*$ is a symmetric binary matrix, i.e. $\mathcal{T}_m^* = (\mathcal{T}_m^*)^T$.*

Proof. Let us prove that $v \to w$ if and only if $w \to v$, for any two vertices $v, w \in V(\mathcal{D}_m^*)$.

Suppose that v is a direct predecessor of w, i.e. $v \to w$. This implies that there exist vertices $x, y \in V(\mathcal{D}_m)$ such that $x \to y$, where $o(x) = v$ and $o(y) = w$. Using reflection symmetry (with the vertical axis) we have $v = o(x) = i(y) = o(y')$ and $y \to y'$. Consequently, $w \to v$. □

Since the vertices from $V(\mathcal{D}_m)$ which are contracted belong to the same component, Lemma 3.6 implies

Theorem 3.9. *Digraph $\mathcal{D}_m^*$ for $m \geq 2$ is disconnected. Each of its components is a strongly connected digraph.*

The component which contains the vertex $0^m \equiv 00\cdots0$ is denoted by $\mathcal{R}_m^*$. The component containing the unique loop $(1^m \to 1^m)$ is labeled by $\mathcal{A}_m^*$ and the union of the remaining components by $\mathcal{B}_m^*$. Digraphs $\mathcal{D}_m^*$ for $m = 4$ and $m = 5$ are shown in Figure 12 and 13, respectively.

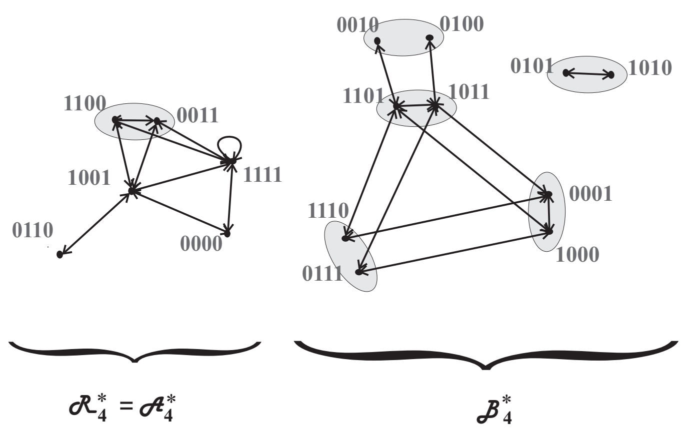

FIGURE 12. Digraph $\mathcal{D}_4^*$

http://dx.doi.org/10.22108/toc.2022.131614.1940

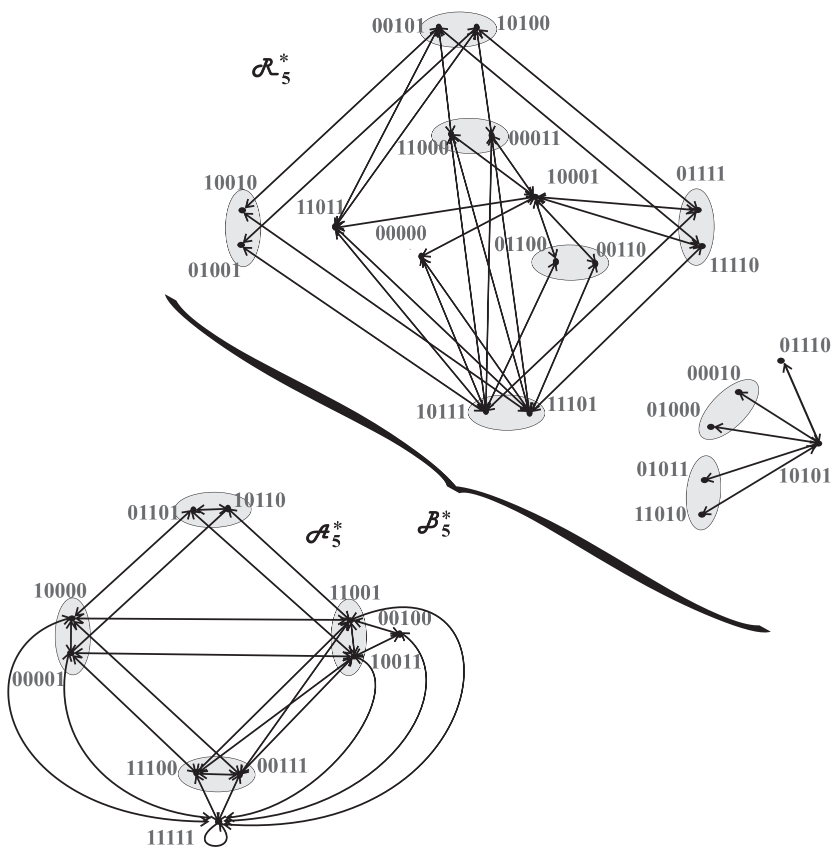

FIGURE 13. Digraph $\mathcal{D}_5^*$

Theorem 3.10. *For the number of vertices in digraph $\mathcal{D}_m^*$ we have*

$$| V(\mathcal{D}_m^*) |= \begin{cases} 2^m, & \text{if } m \text{ is even} \\ 2^m - 1, & \text{if } m \text{ is odd} \end{cases}$$

Proof. At first, note that any word $x \in V(\mathcal{D}_m)$ whose outlet word has the prefix $(01)^k$ $(1 \leq k \leq \lfloor \frac{m}{2} \rfloor)$ must have the prefix $(dc)^k$. Similarly, any word $x \in V(\mathcal{D}_m)$ whose outlet word has the suffix $(10)^s$ $(1 \leq s \leq \lfloor \frac{m}{2} \rfloor)$ must have the suffix $(af)^s$. If this suffix is of length $m-1$, then the first letter must be e, i.e. the word $010101...010 \notin V(\mathcal{D}_m^*)$.

To prove that there exists at least one vertex $x \in V(\mathcal{D}_m)$ with $o(x) = v$, where v is a binary word of length m different from 010101...010 (in case m-odd), we start from the word v and demonstrate the construction of x.

Let k and s be the maximum non-negative integers for which $(01)^k$ is the prefix and $(10)^s$ is the suffix of v. Then $v \equiv (01)^k w (10)^s$, where the subword w is different from 0 and it has neither prefix 01 nor suffix 10. We replace the prefix $(01)^k$ with $(dc)^k$, and the suffix $(10)^s$ with $(af)^s$. If the word $w \equiv 0^{m-2(k+s)}$ $(m - 2(k+s) > 1)$, then the word $db^{m-2(k+s+1)}f$ can be inserted instead of w. Another case is that the word w has at least one letter 1. Let t and p be the maximum non-negative integers for which 0^t is a prefix and 0^p is a suffix of w. Clearly, $t \neq 1$, $p \neq 1$ and $w \equiv 0^t \alpha 0^p$ where the subword α has the first and the last letter 1. In case $t \geq 2$ ($p \geq 2$), then the subword 0^t (0^p) is substituted with $db^{t-2}f$ ($db^{p-2}f$). Consider now all the maximal zero-subwords of α. Every one of them, of length $q \geq 2$, we replace by word $db^{m-2}f$. If $q = 1$, then the subword 01, obtained by extending this letter 0 with the first letter 1 below, is replaced by word dc. Finally, replacing the remaining letters 1 with letters e we obtain the vertex $x \in \mathcal{D}_m$. □

Theorem 3.11. *The number of edges in $\mathcal{D}^*_m$ is* $\mid E(\mathcal{D}^*_m) \mid = \frac{1}{2}(3^m + (-1)^m)$.

Proof. Consider all the vertices from $V(\mathcal{D}_m)$ with the same outlet word $v \in V(\mathcal{D}^*_m)$. Observe that they have different inlet words, which represent all possible direct predecessors for v in $\mathcal{D}^*_m$. In this way, the bijection between the set $E(\mathcal{D}^*_m)$ and $V(\mathcal{D}_m)$ is established. Lemma 3.4 implies our assertion. □

For the binary word $v \equiv b_1 b_2 \cdots b_{m-1} b_m \in \{0,1\}^m$, we introduce the label $\overline{v} \stackrel{\text{def}}{=} b_m b_{m-1} \cdots b_2 b_1$.

Proposition 3.12.
For arbitrary $x, y \in V(\mathcal{D}_m)$, $o(\overline{x}) = \overline{o(x)}$. Consequently, if $o(x) = o(y)$, then $o(\overline{x}) = o(\overline{y})$.

Proof. Straightforward. □

Let $\mathcal{P}^*_m = [p_{ij}]$ be the square binary matrix of order $\mid V(\mathcal{D}^*_m) \mid$ whose entry $p_{i,j} = 1$ if and only if the i-th and j-th vertices of the digraph $\mathcal{D}^*_m$ satisfy $v_i = \overline{v_j}$ (and $v_j = \overline{v_i}$); otherwise $p_{i,j} = 0$. Note that the matrix $\mathcal{P}^*_m$ is a symmetric one. We can improve the process of enumeration of 2-factors using the new transfer-matrix $\mathcal{T}^*_m$.

Theorem 3.13.

$$f_m^G(n) = \begin{cases} a_{1,1}^{(n)}, & \textit{if} \quad G = RG, \\ tr((\mathcal{T}^*_m)^n) = \displaystyle\sum_{v_i \in V(\mathcal{D}^*_m)} a_{i,i}^{(n)}, & \textit{if} \quad G = TkC, \\ tr(\mathcal{P}^*_m \cdot (\mathcal{T}^*_m)^n) = \displaystyle\sum_{\substack{v_i, v_j \in V(\mathcal{D}^*_m) \\ \overline{v}_i = v_j}} a_{i,j}^{(n)}, & \textit{if} \quad G = MS, \end{cases}$$

where $\mathcal{T}^*_m = [a_{ij}]$ is the adjacency matrix of the digraph $\mathcal{D}^*_m$ and $v_1 \equiv 00 \cdots 0$.

Proof. Let $\mathcal{W}_x^y(n)$ denote the number of oriented walks of length n in the observed digraph ($\mathcal{D}_m$ or $\mathcal{D}^*_m$) which start with vertex x and finish with vertex y. The (i,j)-entry $a_{i,j}^{(n)}$ of n-th power of $\mathcal{T}^*_m$ represents the number of all the oriented walks of the length n in $\mathcal{D}^*_m$ which start with $v_i \in V(\mathcal{D}^*_m)$ and finish with $v_j \in V(\mathcal{D}^*_m)$, i.e. $\mathcal{W}_{v_i}^{v_j}(n)$. Note that for arbitrary $x_1, x_2, y \in V(\mathcal{D}_m)$,

$$\text{if } o(x_1) = o(x_2), \text{ then } \mathcal{W}_{x_1}^y(n) = \mathcal{W}_{x_2}^y(n). \tag{3.1}$$

Consequently, the number $\mathcal{W}_{v_i}^{v_j}(n)$ is equal to the number of all the oriented walks of the length n in $\mathcal{D}_m$ which start with a vertex $x \in V(\mathcal{D}_m)$ which is assigned to the vertex $v_i \in V(\mathcal{D}_m^*)$ $(o(x) = v_i)$ and finish with vertices $y \in V(\mathcal{D}_m)$ which are assigned to $v_j \in V(\mathcal{D}_m^*)$ $(o(y) = v_j)$, i.e.

$$\mathcal{W}_{v_i}^{v_j}(n) = \sum_{\substack{y \in V(\mathcal{D}_m) \\ o(y) = v_j}} \mathcal{W}_x^y(n), \text{ where } x \in V(\mathcal{D}_m) \text{ and } o(x) = v_i. \tag{3.2}$$

Now, from Lemma 3.2, having in mind that the set of direct predecessors of $ab^{m-2}c$ is $\mathcal{L}_m$, and using (3.2) we have

$$f_m^{RG}(n) = \mathcal{W}_{db^{m-2}f}^{ab^{m-2}c}(n+1) = \sum_{y \in \mathcal{L}_m} \mathcal{W}_{db^{m-2}f}^{y}(n) = \mathcal{W}_{0^m}^{0^m}(n) = a_{1,1}^{(n)}.$$

By using (3.2),(3.1) and Proposition 3.12 we have

$$\mathcal{W}_{v_i}^{v_i}(n) = \sum_{\substack{x \in V(\mathcal{D}_m) \\ o(x) = o(x_1) = v_i}} \mathcal{W}_{x_1}^{x}(n) = \sum_{\substack{x \in V(\mathcal{D}_m) \\ o(x) = v_i}} \mathcal{W}_x^x(n). \tag{3.3}$$

and

$$\mathcal{W}_{v_i}^{\overline{v}_i}(n) = \sum_{\substack{x \in V(\mathcal{D}_m) \\ o(x) = o(x_1) = v_i}} \mathcal{W}_{x_1}^{\overline{x}}(n) = \sum_{\substack{x \in V(\mathcal{D}_m) \\ o(x) = v_i}} \mathcal{W}_x^{\overline{x}}(n). \tag{3.4}$$

Applying Lemma 3.2 again and (3.3) to $TkC_m(n)$ we obtain

$$f_m^{TkC}(n) = \sum_{x \in V(\mathcal{D}_m)} \mathcal{W}_x^x(n) = \sum_{v_i \in V(\mathcal{D}_m^*)} \sum_{\substack{x \in V(\mathcal{D}_m) \\ o(x) = v_i}} \mathcal{W}_x^x(n) = \sum_{v_i \in V(\mathcal{D}_m^*)} a_{i,i}^{(n)} = tr((\mathcal{T}_m^*)^n).$$

For $MS_m(n)$ Lemma 3.2 and (3.4) yield that

$$f_m^{MS}(n) = \sum_{x \in V(\mathcal{D}_m)} \mathcal{W}_x^{\overline{x}}(n) = \sum_{v_i \in V(\mathcal{D}_m^*)} \sum_{\substack{x \in V(\mathcal{D}_m) \\ o(x) = v_i}} \mathcal{W}_x^{\overline{x}}(n) = \sum_{v_i \in V(\mathcal{D}_m^*)} \mathcal{W}_{v_i}^{\overline{v}_i}(n), \quad \text{i.e.}$$

$f_m^{MS}(n) = tr(\mathcal{P}_m^* \cdot (\mathcal{T}_m^*)^n)$. □

Theorem 3.14. *The subdigraph of $\mathcal{D}_m^*$ induced by the set of vertices having odd numbers of 0's is a bipartite digraph.*

http://dx.doi.org/10.22108/toc.2022.131614.1940

Proof. Having in mind Theorem 3.8, it is sufficient to prove that this subdigraph does not contain any odd-length oriented cycle. Assume the opposite: that there exist an odd integer n and an oriented cycle of length n in it. This implies the existence of a 2-factor in $TkC_m(n)$.

Case I: m-even Every vertex of the considered oriented cycle observed as a binary word has an odd number of 1's. It implies that corresponding 2-factor in $TkC_m(n)$ contains odd number of non-contractible cycles. Using Proposition 2.2 we conclude that the number of all the edges in the observed 2-factor has the same parity as n, i.e. is odd. On the other side, this number must be $m \cdot n$, i.e. even. Contradiction.

Case II: m-odd Now, the corresponding 2-factor in $TkC_m(n)$ contains an even number of non-contractible cycles (related binary words have even number of 1's). Applying Proposition 2.2 again, we conclude that the number of edges of the considered 2-factor is even, while $m \cdot n$ is odd which leads to a contradiction. □

Corollary 3.15. *The subdigraph of $\mathcal{D}_m$ induced by the set of vertices whose outlet words have an odd number of 0's is a bipartite digraph.*

Proof. It is sufficient to prove that two vertices connected with an arc do not have the same outlet word. Suppose the opposite, that there exist two vertices v and w with $v \to w$ and $o(v) = o(w)$. Then, $o(w) = o(v) = i(w)$ holds. Note that the only vertex in $\mathcal{D}_m$ with the same inlet and outlet word is e^m, which does not belong to the considered subdigraph. Contradiction. □

Corollary 3.16. *For odd m, both $\mathcal{R}_m$ and $\mathcal{R}^*_m$ are bipartite digraphs.*

When m is even, vertex 0^m is both a direct successor and a direct predecessor for 1^m in $\mathcal{D}^*_m$ (for example see Figure 12). The following theorem is a consequence of Lemma 3.7.

Theorem 3.17. *$\mathcal{A}^*_m \equiv \mathcal{R}^*_m$ if and only if m is even.*

Theorem 3.18. *For even m, all palindromes from $V(\mathcal{D}^*_m)$ belong to the component $\mathcal{A}^*_m$ (i.e. $\mathcal{R}^*_m$).*

Proof. We prove this theorem, by strong induction, for all $k \in N$ where $m = 2k$. The base case is obviously correct ($00 \leftrightarrow 11$). Assume that the statement holds for all palindromes $w\overline{w}$ of length less then $2k$. Consider a palindrome $v\overline{v} \in V(\mathcal{D}^*_m)$, where $v\overline{v} \neq 0^m$.

<u>Case 1:</u> $v\overline{v} = 1w\overline{w}1$.

Let $x\overline{x}$ be one of the alpha words from $V(\mathcal{D}_{m-2})$ whose outlet word is $w\overline{w}$, i.e. $o(x\overline{x}) = w\overline{w}$. Then there exist such palindromes $w_j\overline{w}_j \in V(\mathcal{D}^*_{m-2})$ and alpha words $x_j\overline{x}_j \in V(\mathcal{D}_{m-2})$ with $o(x_j\overline{x_j}) = w_j\overline{w_j}$, $1 \leq j \leq t$ ($t \in N$) for which there exist walks $w\overline{w} \to w_1\overline{w_1} \to w_2\overline{w_2} \to \cdots \to w_t\overline{w_t}$ and $x\overline{x} \to x_1\overline{x_1} \to x_2\overline{x_2} \to \cdots \to x_t\overline{x_t}$ in $\mathcal{D}^*_{m-2}$ and $\mathcal{D}_{m-2}$, respectively, and $w_t\overline{w}_t = 0^{m-2}$ (inductive hypothesis). Now, the walk $ex\overline{x}e \to ex_1\overline{x_1}e \to ex_2\overline{x_2}e \to \cdots \to ex_t\overline{x_t}e \to db^{2(k-1)}f$ in $\mathcal{D}_m$ ($x_t\overline{x_t} \in \mathcal{L}_{m-2}$) justifies the existence of the walk $v\overline{v} = 1w\overline{w}1 \to 1w_1\overline{w_1}1 \to 1w_2\overline{w_2}1 \to \cdots \to 1w_t\overline{w_t}1 \to 0^m$ in $\mathcal{R}^*_m$ (see Figure 14a).

<u>Case 2:</u> $v\overline{v} = 0^s1w\overline{w}10^s$ where $s \geq 1$ (including the possibility that w is an empty word).

If w is not an empty word, then there exist such walks $w\overline{w} \to w_1\overline{w_1} \to w_2\overline{w_2} \to \cdots \to w_t\overline{w_t}$ and $x\overline{x} \to x_1\overline{x_1} \to x_2\overline{x_2} \to \cdots \to x_t\overline{x_t}$ (both of length $t \in N$) in $\mathcal{D}^*_{2(k-s-1)}$ and $\mathcal{D}_{2(k-s-1)}$, respectively, where $o(x\overline{x}) = w\overline{w}$, $o(x_j\overline{x_j}) = w_j\overline{w_j}$ for all $1 \leq j \leq t$ and $w_t\overline{w}_t = 0^{2(k-s-1)}$ (inductive hypothesis).

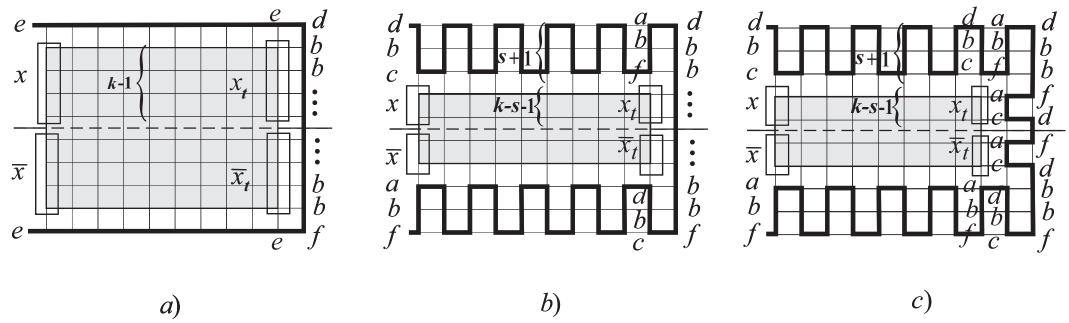

FIGURE 14. Constructions walks to vertex 0^{2k} in digraph $\mathcal{D}_{2k}^{*}$.

Case 2.1: t is odd.
Required walk (of length $t+1$) in $\mathcal{D}_m$ is $db^{s-1}cx\overline{x}ab^{s-1}f \to ab^{s-1}fx_1\overline{x_1}db^{s-1}c \to db^{s-1}cx_2\overline{x_2}ab^{s-1}f \to ab^{s-1}fx_3\overline{x_3}db^{s-1}c \to db^{s-1}cx_4\overline{x_4}ab^{s-1}f \to \cdots \to ab^{s-1}fx_t\overline{x_t}db^{s-1}c \to db^{m-2}f$. The fact $db^{m-2}f \in \mathcal{L}_m$ implies the statement (see Figure 14b).

Case 2.2: t is even.
Required walk (of length $t+2$) in $\mathcal{D}_m$ is $db^{s-1}cx\overline{x}ab^{s-1}f \to ab^{s-1}fx_1\overline{x_1}db^{s-1}c \to db^{s-1}cx_2\overline{x_2}ab^{s-1}f \to ab^{s-1}fx_3\overline{x_3}db^{s-1}c \to db^{s-1}cx_4\overline{x_4}ab^{s-1}f \to \cdots \to db^{s-1}cx_t\overline{x_t}ab^{s-1}f \to ab^{s-1}f(ac)^{k-s-1}db^{s-1}c \to db^sf(df)^{k-s-2}db^sf$. Since $db^s(fd)^{k-s-1}b^sf \in \mathcal{L}_m$, the statement holds (see Figure 14c).

Case 2.3: $v\overline{v} = 0^{k-1}110^{k-1}$.
The walk (of length 2) $db^{k-2}cab^{k-2}f \to ab^{k-2}fdb^{k-2}c \to db^{m-2}f$ in $\mathcal{R}_m$ implies the existence of the walk $0^{k-1}110^{k-1} \to 10^{m-2}1 \to 0^m$ in $\mathcal{R}_m^*$. Consequently, $0^{k-1}110^{k-1} \in \mathcal{R}_m^*$. □

Further reduction of the transfer matrices is possible just in the case $G = RG_m(n)$ using the following

Theorem 3.19. *If $v \in V(\mathcal{R}_m^*)$, then $\overline{v} \in V(\mathcal{R}_m^*)$.*

Proof. If $o(x) = v$ $(x \in V(\mathcal{R}_m))$, then there exists an integer $n \geq 1$ for which $\mathcal{W}_x^{db^{m-2}f}(n) \neq 0$. By using the property of reflection symmetry $(\overline{db^{m-2}f} = db^{m-2}f)$, we obtain that $\mathcal{W}_{\overline{x}}^{db^{m-2}f}(n) = \mathcal{W}_x^{db^{m-2}f}(n) \neq 0$, which implies $\overline{x} \in V(\mathcal{R}_m)$. Since $\overline{v} = o(\overline{x})$, we conclude that $\overline{v} \in V(\mathcal{R}_m^*)$. □

Now, we can contract the vertices v and $\overline{v}$ into one vertex for all $v, \overline{v} \in V(\mathcal{R}^*)$ resulting in a new digraph $\mathcal{R}_m^{**}$. For example, for $m = 5$ digraph $\mathcal{R}_5^*$ is bipartite and has 6 (unordered) pairs of different vertices $\{v, \overline{v}\}$ which are rounded in Figure 13. During contraction of vertices v and $\overline{v}$ we retain arcs starting from just one of these two vertices and delete the ones starting from another vertex. Multiple (double) arcs appear when v and $\overline{v}$ have a common direct predecessor (see Figure 11). The symmetry of the rectangular grid $RG_m(n)$ leads us to the conclusion that $\mathcal{W}_{\overline{v}}^{0^m}(n) = \mathcal{W}_v^{0^m}(n)$ for any vertex $v \in V(\mathcal{R}^*)$. Consequently, the number $\mathcal{W}_{0^m}^{0^m}(n)$ remains the same in both digraph $\mathcal{R}^*$ and $\mathcal{R}^{**}$, i.e.

Theorem 3.20. *The number $f_m^{RG}(n)$ is equal to entry $a_{1,1}^{(n)}$ of the n-th power of the adjacency matrix for $\mathcal{R}^{**}$ where $v_1 \equiv 0^m$.*

This way we obtain the recurrence relations for $f_m^{RG}(n)$ of lower order then the ones we got using the digraph $\mathcal{R}^*$(see Tabular 2). Note that for odd m, $\mathcal{R}^*$ is a bipartite digraph and vertices v and $\overline{v}$ are at an even distance (of the same colour), so $\mathcal{R}^{**}$ has no loops. On the contrary, when m is even new loops can appear. For example, for $m = 4$ (see Figure 12) one more loop appears when the vertices 1100 and 0011 are contracted.

Computing the generating functions $\mathcal{F}_m^G(x) \stackrel{\text{def}}{=} \sum_{n\geq 1}^{\infty} f_m^G(n)x^n$ is a matter of routine [7, 8, 23]. We wrote computer programs for the computation of the adjacency matrices of these digraphs $\mathcal{D}_m$, $\mathcal{D}_m^*$, $\mathcal{R}_m$, $\mathcal{R}_m^*$, $\mathcal{R}_m^{**}$ and initial members of required sequences $f_m^G(n)$.

4. Computational results

4.1. Cardinality of sets of vertices for components of $\mathcal{D}_m^*$.

Some properties of the digraphs $\mathcal{D}_m$, $\mathcal{D}_m^*$ and $\mathcal{R}_m^*$, and their related sequences $f_m^G(n)$, spotted upon analyzing the computational data for $m \leq 12$ (in case of $RG_m(n)$ for $m \leq 17$) have been discussed and proved in the previous section for arbitrary $m \in N$. Beside numerical results we further present a few more properties concerning the cardinality of sets of vertices for components of $\mathcal{D}_m^*$. Due to limited space these properties are here formulated as conjectures and will be the subjects of separate papers [10].

Conjecture 4.1. *For each $m \geq 2$, digraph $\mathcal{D}_m^*$ has exactly $\left\lfloor \frac{m}{2} \right\rfloor + 1$ components, i.e. $\mathcal{D}_m^* = \mathcal{A}_m^* \cup \mathcal{B}_m^*$, where $\mathcal{B}_m^*$ consists of exactly $\left\lfloor \frac{m}{2} \right\rfloor$ components $\mathcal{B}_m^{*(1)}, \mathcal{B}_m^{*(2)}, \ldots, \mathcal{B}_m^{*(\lfloor m/2 \rfloor)}$ $(\mid V(\mathcal{B}_m^{*(1)}) \mid \geq \mid V(\mathcal{B}_m^{*(2)}) \mid \geq \cdots \geq \mid V(\mathcal{B}_m^{*(\lfloor m/2 \rfloor)}) \mid)$. All the components $\mathcal{B}_m^{*(k)}$ $(1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor)$ are bipartite digraphs and*

$$\mid V(\mathcal{B}_m^{*(k)}) \mid \;=\; \begin{cases} \dbinom{m+1}{(m+1)/2 - k}, & \text{if } m \text{ is odd,} \\ \\ 2 \cdot \dbinom{m}{m/2 - k}, & \text{if } m \text{ is even,} \end{cases}$$

and

$$\mid V(\mathcal{A}_m^*) \mid \;=\; \begin{cases} \dbinom{m}{(m-1)/2}, & \text{if } m \text{ is odd,} \\ \dbinom{m}{m/2}, & \text{if } m \text{ is even.} \end{cases}$$

The vertices v and $\overline{v}$ belong to the same component. If $v \in \mathcal{B}_m^{*(s)}$, $1 \leq s \leq \left\lfloor \frac{m}{2} \right\rfloor$, then $\overline{v}$ is placed in the same class if and only if m is odd.

Conjecture 4.2. *For m – odd, the word 0^m belongs to $V(\mathcal{B}_m^{*(1)})$, i.e. $\mathcal{R}_m^* \equiv \mathcal{B}_m^{*(1)}$. The number of all palindromes from $V(\mathcal{B}_m^{*(1)})$ is equal to $\dbinom{(m+1)/2}{\lfloor (m+1)/4 \rfloor}$.*

Conjecture 4.3. *For m – odd, the number of vertices in $\mathcal{R}_m^*$ is equal to the binomial coefficients (in OEIS A001791):*

$$\mid V(\mathcal{R}_m^*) \mid = \binom{m+1}{(m-1)/2} \tag{4.1}$$

TABLE 1. The numbers of vertices of $\mathcal{D}_m$, $\mathcal{D}^*_m$, components of $\mathcal{D}^*_m$ and the order of the recurrence relations (the same) for both thick cylinder graphs $TkC_m(n)$ and Moebius strips $MS_m(n)$.

m	2	3	4	5	6	7	8	9	10	11	12
$\mid V(\mathcal{D}_m) \mid$	5	13	41	121	365	1093	3281	9841	29525	88573	265721
$\mid V(\mathcal{D}^*_m) \mid$	4	7	16	31	64	127	256	511	1024	2047	4096
$\mid V(\mathcal{A}^*_m) \mid$	2	3	6	10	20	35	70	126	252	462	924
$\mid V(\mathcal{B}^{*(1)}_m) \mid$	2	4	8	15	30	56	112	210	420	792	1584
$\mid V(\mathcal{B}^{*(2)}_m) \mid$	-	-	2	6	12	28	56	120	240	495	990
$\mid V(\mathcal{B}^{*(3)}_m) \mid$	-	-	-	-	2	8	16	45	90	220	440
$\mid V(\mathcal{B}^{*(4)}_m) \mid$	-	-	-	-	-	-	2	10	20	66	132
$\mid V(\mathcal{B}^{*(5)}_m) \mid$	-	-	-	-	-	-	-	-	2	12	24
$\mid V(\mathcal{B}^{*(6)}_m) \mid$	-	-	-	-	-	-	-	-	-	-	2
order	4	5	13	19	49	69	178	249	649	-	-

while the number of vertices in digraph $\mathcal{R}^{**}_m$ is equal to

$$\mid V(\mathcal{R}^{**}_m) \mid = \frac{1}{2}\left[\binom{m+1}{(m-1)/2} + \binom{(m+1)/2}{\lfloor (m+1)/4 \rfloor}\right]. \tag{4.2}$$

For m – even, the number of vertices in $\mathcal{R}^*_m$ is equal to the Central binomial coefficients (in OEIS A000984):

$$\mid V(\mathcal{R}^*_m) \mid = \binom{m}{m/2} \tag{4.3}$$

while the number of vertices in digraph $\mathcal{R}^{**}_m$ is equal to A005317 in OEIS, i.e.

$$\mid V(\mathcal{R}^{**}_m) \mid = 2^{(m-2)/2} + \frac{1}{2}\binom{m}{m/2}. \tag{4.4}$$

For m-odd, equations (4.1) and (4.2) are trivial consequences of Conjecture 4.1 and Conjecture 4.2. Equation (4.3) is a consequence of Theorem 3.17 and Conjecture 4.1. Using Theorem 3.18 the equality (4.4) is easy to prove from (4.3). Also, we noticed that the members of the sequence $\mid V(\mathcal{R}_m) \mid$ for even $m \leq 10$ coincide with the initial members of the sequence A082758 in OEIS [24].

4.2. Asymptotic behaviour of the numbers of 2-factors $f^G_m(n)$.

The properties referring to asymptotic behaviour of numbers of 2-factors $f^{RG}_m(n)$ and $f^{TkC}_m(n)$ (when $n \to \infty$) were observed to be similar to the ones that appeared while studying Hamiltonian cycles.

Since the adjacency matrix $\mathcal{T}^*_m$ of $\mathcal{D}^*_m$ ($m \geq 2$) is symmetric (Theorem 3.8), i.e. Hermitian, the spectrum of $\mathcal{D}^*_m$ contains only real numbers. Each of the components of $\mathcal{D}^*_m$ is a strongly connected digraph (Theorem 3.9) and, therefore, has an irreducible adjacency matrix [8] (which is a block in the diagonal block matrix $\mathcal{T}^*_m$). From Perron-Frobenius theorems [12] the maximum modulus eigenvalues for these nonnegative and irreducible matrices are algebraically simple eigenvalues. If the set of all maximum modulus eigenvalues for a nonnegative and irreducible matrix has exactly $k \geq 2$ distinct elements, they are

TABLE 2. The numbers of vertices of $\mathcal{F}_m, \mathcal{L}_m, \mathcal{R}_m, \mathcal{R}^*_m$ and $\mathcal{R}^{**}_m$ and the order of the recurrence relations for $RG_m(n)$.

m	2	3	4	5	6	7	8	9	10	11	12	13
$\mid \mathcal{F}_m \mid = \mid \mathcal{L}_m \mid$	1	1	2	3	5	8	13	21	34	55	89	144
$\mid V(\mathcal{R}_m) \mid$	3	6	19	60	141	532	1107	4608	8953	$\ll$	$\ll$	$\ll$
$\mid V(\mathcal{R}^*_m) \mid$	2	4	6	15	20	56	70	210	252	792	924	3003
$\mid V(\mathcal{R}^{**}_m) \mid$	2	3	5	9	14	31	43	110	142	406	494	1519
order	2	1	5	3	13	9	35	25	96	-	-	-

m	14	15	16	17
$\mid \mathcal{F}_m \mid = \mid \mathcal{L}_m \mid$	233	377	610	987
$\mid V(\mathcal{R}^*_m) \mid$	3433	$\ll$	$\ll$	$\ll$
$\mid V(\mathcal{R}^{**}_m) \mid$	1780	5755	6563	21942

precisely the kth roots of 1 times the maximum eigenvalue θ [12]. Since all eigenvalues of our considered matrices are real numbers, for bipartite digraphs there exist exactly two simple eigenvalues of maximal modulus (θ and $-\theta$), i.e. k must be two.

Let θ_m (we also use the labels θ^{TkC}_m and θ^{MS}_m to emphasize corresponding grid graph) and θ^{RG}_m be the maximum eigenvalues of the adjacency matrices of $\mathcal{D}^*_m$ and $\mathcal{R}^*_m$, respectively. Computational results for $m \leq 12$ show that the maximum eigenvalue θ_m of $\mathcal{T}^*_m$ is simple and unique maximum modulus eigenvalue. Additionally, θ_m is attached to the component $\mathcal{A}^*_m$ for all $m \geq 2$, i.e.

Conjecture 4.4. *The maximum eigenvalue of $\mathcal{A}^*_m$ is the unique maximum modula eigenvalue of $\mathcal{D}^*_m$.*

According to the foregoing we have

$$f^{TkC}_m(n) \sim a^{TkC}_m \theta^n_m, \text{ where } a^{TkC}_m = 1 \ .$$

(Note that the property for the coefficients of maximum eigenvalue being equal to 1 appeared by Hamiltonian cycles when m was odd [1] and [3].) For instance,

$f^{TkC}_9(99) =$ **175073846218165**27713388082077727019550307034420282580173180880933611360786760679564966706639273723674798766385930557092858331879012953635968195685205,

$f^{TkC}_9(100) =$ **5503**48885165019283285755151853301860827173003486081734884093077979833998668505674221837749197473620246193874089192224295399960428521091684471910843826,

$f^{TkC}_{10}(99) =$ **547294669589**5734348165268778293176272799246831355358749477747452490650121461570806401953439141822755252536835769696328386335929233345742122626919959848159690280754707 and

$f^{TkC}_{10}(100) =$ **2645**316310319933683496095009841718024759437947764204153951092048286901777725903134922353487293196697135565035974993061484188198332687554807453750119675251682976586688605

http://dx.doi.org/10.22108/toc.2022.131614.1940

while
$\theta_9^{99} = \underline{\mathbf{1.75073846218165}}923653 \cdots \cdot 10^{148}$,
$\theta_9^{100} = \underline{\ \mathbf{5.503}}355253 \cdots \cdot 10^{149}$,
$\theta_{10}^{99} = \underline{\mathbf{5.47}}\mathbf{294669589}622318 \cdots \cdot 10^{166}$, and
$\theta_{10}^{100} = \underline{\ \mathbf{2.645}121801666}474501 \cdots \cdot 10^{168}$.
The coefficient a_m^{TkC} is equal to 1 because of Theorem 3.13 and the fact that the trace of n-th power of matrix $\mathcal{T}_m^*$ is equal to the sum of n-th powers of all of its eigenvalues (the value $f_m^{TkC}(n)$ has the unique representation as a linear combination of n standard solutions of the recurrence relation for $\mathcal{T}_m^*$ corresponding to all its eigenvalues).

Recall, that for Hamiltonian cycles in $TkC_m(n)$, contractible HCs are more numerous than non-contractible ones if and only if m is even [3]. The similar assertion can be formulated for 2-factors dividing them into 2-factors with even and odd number of nc-cycles. Note that in case m is even, the digraph $\mathcal{A}_m^*$ determines 2-factors in $TkC_m(n)$ with an even number of nc-cycles (however not all of them). For odd m $\mathcal{A}_m^*$ determines 2-factors with odd numbers of nc-cycles and this kind of 2-factors are then dominant assuming Conjecture 4.4. More precisely,

$$f_m^{TkC}(n) \sim \begin{cases} f_{1,m}^{TkC}(n) \ , & \text{for } m \text{ odd} \\ \\ f_{0,m}^{TkC}(n) \ , & \text{for } m \text{ even} \end{cases} \qquad (n \to +\infty).$$

Assuming that all components $\mathcal{B}_m^{*(k)}$ are bipartite (Conjecture 4.1) we have that, for n odd, the only 2-factors obtained from component $\mathcal{A}_m^*$ are counted in $f_m^{TkC}(n)$. (Therefore, more digits coincide in $f_9^{TkC}(99)$ and θ_9^{99}, or $f_{10}^{TkC}(99)$ and θ_{10}^{99}, than in $f_9^{TkC}(100)$ and θ_9^{100}, or $f_{10}^{TkC}(100)$ and θ_{10}^{100}.)

Table 3. The approximate values of $\theta_m = \theta_m^{TkC} = \theta_m^{MS}$ and $a_m^{TkC} = a_m^{MS} = 1$ for $1 \leq m \leq 12$, where $\approx_{(n)}$ means the estimate based on the first n entries of the sequence.

m	$\theta_m^{TkC} = \theta_m^{MS}$	$a_m^{TkC} = a_m^{MS}$
2	1.6180339887498948482045868344	1
3	2.4142135623730950488016887242	1
4	3.6941816601239106665999753656	1
5	5.6532020378824433814716902315	1
6	8.6709538972300632454385724873	1
7	13.312178239997254208159205016	1
8	20.451693229411496623118690839	1
9	31.434479637181596582999666842 9	1
10	48.330852621858437394324274600 7	1
11	$\approx_{(100)}$ 74.32697213	1
12	$\approx_{(50)}$ 114.326	1

TABLE 4. The approximate values of θ_m^{RG} and a_m^{RG} for $2 \leq m \leq 17$, where $\approx_{(n)}$ means the estimate based on the first n entries of the sequence.

m	θ_m^{RG}	a_m^{RG}
2	$(1+\sqrt{5})/2$	$\sqrt{5}/5$
3	1.73205080756887729352744634151	0.288675134594812882254574390
4	3.69418166012391066659997536564	0.311853777156519857011382468
5	4.62518160134423951692596223359	0.268966073785024485542699862
6	8.67095389723006324543857248731	0.252057339976282862165401091
7	11.5193830042298614862975296130	0.242040240108164179761287858
8	20.4516932294114966231186908391	0.214968661101422992565401329
9	28.0703410924057870863760633239	0.218559873860749395413375924
10	48.3308526218584373943242746007	0.188566846109428479683989429
11	$\approx_{(600)}$ 67.7256340927618460544544369622	$\approx_{(600)}$ 0.198719011769436403820671988
12	$\approx_{(600)}$ 114.326554075137475903315037896	$\approx_{(600)}$ 0.168332193334906639461183213
13	$\approx_{(200)}$ 162.525641651709590009538707518	$\approx_{(200)}$ 0.181832548137559030499896532
14	$\approx_{(200)}$ 270.594404874261731	$\approx_{(200)}$ 0.152084575433189642
15	$\approx_{(200)}$ 388.759158231636826603830485900	$\approx_{(200)}$ 0.167278718198176374172092348
16	$\approx_{(200)}$ 640.690454998007	$\approx_{(200)}$ 0.1386133711863155
17	$\approx_{(100)}$ 927.945466754283	$\approx_{(100)}$ 0.154581709489037

Recall, that contractible Hamiltonian cycles in $TkC_m(n)$ and $RG_m(n)$ have the same positive dominant eigenvalue when m is even [3]. In case of 2-factors this property is more obvious if we assume Conjecture 4.4 and apply Theorem 3.17 (see Tabular 3 and Tabular 4), i.e.

$$f_m^{RG}(n) \sim \begin{cases} a_m^{RG}\theta_m^n\ , & \text{for } m \text{ even} \\ a_m^{RG}(\theta_m^{RG})^n + a_m^{RG}\left(-\theta_m^{RG}\right)^n\ , & \text{for } m \text{ odd} \end{cases} \qquad (n \to +\infty),$$

where a_m^{RG} are positive numbers.

The novelty appears with the Moebius strip $MS_m(n)$. According to the Conjecture 4.4 we have

$$f_m^{MS}(n) \sim a_m^{MS}\theta_m^n.$$

Numerical data show that the coefficient of the maximal eigenvalue is again one, i.e. $a_m^{MS} = 1$.

For example,

$$
\begin{aligned}
f_9^{MS}(99) = {} & \underline{1750738462181 6}6570172308214692733819358151508674784471234125368540 6319\\
& 1553795793184785069123799306157361712365337734920832932311990627148209\\
& 033567981,\\
f_9^{MS}(100) = {} & \underline{5503}4888516501637769311167637142930764279700321809994978478737169093 88\\
& 7001537742285391720378803648896975631451618649151832275727205508702975\\
& 7535019482,\\
f_{10}^{MS}(99) = {} & \underline{547}3392413551435904097524137222257556522154835755962205364841172527434\\
& 5587572680168475798796527038604871940909874270974422734365157886590851\\
& 5074153809504410916247858 81, \quad \text{and}\\
f_{10}^{MS}(100) = {} & \underline{264512180166 6}6489130484908423420654675472254920269952448762897195796 63\\
& 8721230465776899400708104186158155469514422006233945602999944685195281\\
& 334386733211511040015864815 23.
\end{aligned}
$$

Here, if we assume Conjecture 4.4 is true, then in case m-even, digraph $\mathcal{A}_m^*$ determines the majority of the 2-factors without a short nc-cycle (though generally not all of them). For m-odd, $\mathcal{A}_m^*$ determines the 2-factors which contain a short nc-cycle and this kind of 2-factors are then dominant. More precisely,

$$
f_m^{MS}(n) \sim \begin{cases} f_{1,m}^{MS}(n) \ , & \text{if } m \text{ is odd} \\ f_{0,m}^{MS}(n) \ , & \text{if } m \text{ is even} \end{cases} \ , \text{ when } n \to +\infty \ .
$$

Assuming Conjecture 4.1 we have that for both m and n odd the only 2-factors obtained from component $\mathcal{A}_m^*$ (containing a short nc-cycle) are counted in $f_m^{MS}(n)$. (Therefore more digits coincide in numbers $f_9^{MS}(99)$ and θ_9^{99} than in $f_9^{MS}(100)$ and θ_9^{100}. On the other hand, for both m and n even, the only the 2-factors obtained from component $\mathcal{A}_m^*$ (without a short nc-cycle) are counted in $f_m^{MS}(n)$. (Therefore, more digits coincide in $f_{10}^{MS}(100)$ and θ_{10}^{100} than in $f_{10}^{MS}(99)$ and θ_{10}^{99}.

4.3. Generating functions.

Our results for the rectangular grid graph $RG_m(n) \equiv P_m \times P_n$ for $m \leq 7$ confirm the data previously obtained in another way (by coding cells) in 1994 [6]. We got the generating functions $\mathcal{F}_m^{RG}(x)$, $\mathcal{F}_m^{TkC}(x)$ and $\mathcal{F}_m^{MS}(x)$ for $2 \leq m \leq 10$. These generating functions and the first 30 members of the sequences $f_m^{RG}(n)$ $(2 \leq m \leq 17)$, $f_m^{TkC}(n)$ $(2 \leq m \leq 12)$ and $f_m^{MS}(n)$ $(2 \leq m \leq 12)$ are exposed in the extended version of this paper[9].

By observing the denominators of the generating functions for the components $\mathcal{B}_m^{*(k)}$, where $k \geq 2$, one can notice that each of them consists of the factors of the denominator of the generating functions for $\mathcal{B}_m^{*(1)}$ or $\mathcal{A}_m^*$.

Acknowledgments

The authors are indebted to the anonymous referee for helpful comments which improved the clarity of the presentation. The authors would like to express their gratitude to Bojana Pantić for her meticulous reading of the first draft of the manuscript and on many useful suggestions and comments.

This work was supported by the Ministry of Science,Technological Development and Innovation of the Republic of Serbia. (Grants No. 451-03-9/2022-14/200125, 451-03-68/2022-14/200156) and the Project of the Department for fundamental disciplines in technology, Faculty of Technical Sciences, University of Novi Sad "Application of general disciplines in technical and IT sciences".

References

[1] O. Bodroža-Pantić, H. Kwong, R. Doroslovački and M. Pantić, Enumeration of Hamiltonian cycles on a thick grid cylinder—Part I: Non-contractible Hamiltonian cycles, *Appl. Anal. Discrete Math.*, **13** (2019) 28–60.

[2] O. Bodroža-Pantić, H. Kwong, R. Doroslovački and M. Pantić, A limit conjecture on the number of Hamiltonian cycles on thin triangular grid cylinder graphs, *Discuss. Math. Graph Theory*, **38** (2018) 405–427.

[3] O. Bodroža-Pantić, H. Kwong, J.Đokić, R. Doroslovački and M. Pantić, Enumeration of Hamiltonian cycles on a thick grid cylinder—Part II: Contractible Hamiltonian cycles, *Appl. Anal. Discrete Math.*, **16** (2022) 246–287.

[4] O. Bodroža-Pantić, H. Kwong and M. Pantić, A conjecture on the number of Hamiltonian cycles on thin grid cylinder graphs, *Discrete Math. Theor. Comput. Sci.*, **17** (2015) 219–240.

[5] O. Bodroža-Pantić, B. Pantić, I. Pantić and M. Bodroža Solarov, Enumeration of Hamiltonian cycles in some grid graphs, *MATCH Commun. Math. Comput. Chem.*, **70** (2013) 181–204.

[6] O. Bodroža-Pantić and R. Tošić, On the number of 2-factors in rectangular lattice graphs, *Publ. Inst. Math. (Beograd) (N.S.)*, **56** (1994) 23–33.

[7] R. A. Brualdi and D. M. Cvetković, *A combinatorial approach to matrix theory and its applications*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2009.

[8] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs. Theory and application*, Second edition, VEB Deutscher Verlag der Wissenschaften, Berlin, 1982.

[9] J. Đokić, O. Bodroža-Pantić and K. Doroslovački, A spanning union of cycles in rectangular grid graphs, thick grid cylinders and Moebius strips (with Appendix), arXiv:2109.12432[math.co] (2021) 1–26.

[10] J. Đokić, K. Doroslovački and O. Bodroža-Pantić, The structure of the 2-factor transfer digraph common for rectangular, thick cylinder and Moebius strip grid graphs, *Appl. Anal. Discrete Math.*, arXiv:2212.00317[math.co] (2022) 1–16.

[11] I. G. Enting and I. Jensen, *Exact enumerations*, Polygons, polyominoes and polycubes, , Lecture Notes in Phys., **775**, Springer, Dordrecht, 2009 143–179.

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[13] J. L. Jacobsen, Exact enumeration of Hamiltonian circuits, walks and chains in two and three dimensions, *J. Phys. A: Math. Theor.*, **40** (2007) 14667–14678.

[14] T. C. Liang, K. Chakrabarty and R. Karri, Programmable daisychaining of microelectrodes to secure bioassay IP in MEDA biochips, *IEEE Transactions on Very Large Scale Integration (VLSI) Systems*, **25** (2020) 1269–1282.

[15] A. M. Karavaev, Kodirovanie sostoyaniĭ v metode matricy perenosa dlya podscheta gamil′tonovyh ciklov na pryamougol′nyh reshetkah, cilindrah i torah, *Informacionnye Processy*, **11** (2011) 476–499.

[16] A. Karavaev and S. Perepechko, Counting Hamiltonian cycles on triangular grid graphs, *IV International Conference, SIMULATION-2012*, https://web.archive.org/web/20161015205252/http://flowproblem.ru/references, Kiev, 2012.

[17] A. Kloczkowski and R. L. Jernigan, Transfer matrix method for enumeration and generation of compact self-avoiding walks. I., Square lattices., *J. Chem. Phys.*, **109** (1998) 5134–5146.

[18] E. S. Krasko, I. N. Labutin and A. V. Omelchenko, The enumeration of labeled and unlabeled Hamiltonian cycles in complete k-partite graphs, *J. Math. Sci. (N.Y.)*, **255** (2021) 71–87.

[19] W. Kocay and D. L. Kreher, *Discrete mathematics and its applications - graphs, algorithms, and optimization*, Second Edition-CRC Press LLC-Chapman and Hall-CRC, 2017

[20] J. A. Montoya, On the counting complexity of mathematical nanosciences, *MATCH Commun. Math. Comput. Chem.*, **86** (2021) 453–488.

[21] R. I. Nishat and S. Whitesides, *Reconfiguring Hamiltonian cycles in L-shaped grid graphs*, Graph-theoretic concepts in computer science, Lecture Notes in Comput. Sci., Springer, Cham, (2019) 325–337.

[22] V. H. Pettersson, Enumerating Hamiltonian cycles, *Electron. J. Combin.*, **21** (2014) 1–15.

[23] R. P. Stanley, *Enumerative Combinatorics*, **I**, Cambridge University Press, Wadsworth, Monterey, 2002.

[24] N. J. A. Sloane, *he on-line encyclopedia of integer sequences*, (OEIS), https://oeis.org/A082758.

[25] A. Vegi Kalamar, T. Žerak, D. Bokal, Counting hamiltonian cycles in 2-tiled graphs, *Mathematics*, **9** (2021) 1–27.

Jelena Đokić

Faculty of Technical Sciences, University of Novi Sad, P.O.Box 21000, Novi Sad, Serbia

Email: jelenadjokic@uns.ac.rs

Olga Bodroža-Pantić

Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, P.O.Box 21000, Novi Sad, Serbia

Email: olga.bodroza-pantic@dmi.uns.ac.rs

Ksenija Doroslovački

Faculty of Technical Sciences, University of Novi Sad, P.O.Box 21000, Novi Sad, Serbia

Email: ksenija@uns.ac.rs