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HADAMARD MATRICES OF COMPOSITE ORDERS

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Abstract. In this paper, we give a method for the constructions of Hadamard matrices of composite orders by using suitable T-matrices and known Hadamard matrices. We establish a formula for T-matrices and Hadamard matrices and discuss under what condition we can get T-matrices from the known Hadamard matrices.

1. Introduction

A $(1, -1)$ matrix H of order n is called a Hadamard matrix if

$$HH^T = nI_n,$$

where H^T is the transpose of H, I_n is the identity matrix of order n. It is well known that when $n > 2$, if H is a Hadamard matrix, then n must be divisible by 4. It is conjectured that the condition $n \equiv 0 \pmod 4$ is also sufficient for the existence of a Hadamar matrix of order n. But this problem is still open (see [3]).

Hadamard matrices can be constructed from Abelian Hadamard difference sets (see [1]), or Williamson matrices (see [4, 7]), or supplementary difference sets (see [5]).

Let E be a subset of an abelian group G of order t. Write

$$\chi_E(g) = 1 \text{ or } 0$$

according to $g \in E$ or not. Denote $I(E) = (e_{ij})_{1<=i,j<=t}$, where

$$e_{i,j} = \chi_E(g_j g_i^{-1})$$

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and the elements of G are arranged for any order: $g_1, \ldots, g_t$. We call $I(E)$ is a matrix of type 1. If $G = Z_t$ and the elements of G are arranged for natural order: $0, 1, \ldots, t-1$, then $I(E)$ is circulant. When $G = GF(t)$, for $a, b \in G$, $a \neq 0$, put

$$F = aE + b = \{ag + b : g \in E\} \text{ and } I(F) = (f_{ij})_{1<=i,j<=t},$$

where

$$f_{ij} = \chi_F(g_j g_i^{-1}),$$

then $I(F)$ is of type 1 too.

Obviously, the following facts are true:

(1) If a matrix A is of type 1, so is A^T.

(2) If the matrices A and B are of type 1, then $A + B$ and $A - B$ are of type 1.

(3) If matrices A and B are of type 1, then $AB = BA$, i.e., they are commutative.

Four circulant or type 1 $(0, \pm 1)$ matrices T_1, T_2, T_3, T_4 of order t are called T-matrices if the following two conditions are satisfied:

(1) $T_i * T_j = 0$, for $1 \leq i, j \leq 4$, $i \neq j$,

(2) $\sum_{i=1}^{4} T_i T_i^T = tI_t$,

where * denotes Hadamard product.

T-matrices play a significant role in constructing a Hadamard matrix of composite order (See [6, 10, 11, 12]). There is another conjecture: there exist T-matrices of order t for all integer $t > 0$.

Let G be an Abelian group of order t. It is convenient to use the group ring $Z[G]$ of the group G over the ring Z of rational integers with the addition and multiplication. Here the elements of $Z[G]$ are of the form

$$a_1 g_1 + a_2 g_2 + \cdots + a_t g_t,\ a_i \in Z,\ g_i \in G,\ 1 \leq i \leq t.$$

In $Z[G]$, the addition $+$ is defined by

$$(\sum_g a(g)g) + (\sum_g b(g)g) = \sum_g (a(g) + b(g))g.$$

The multiplication in $Z[G]$ is defined by

$$(\sum_g a(g)g)(\sum_h b(h)h) = \sum_k (\sum_{gh=k} a(g)b(h))k.$$

For any subset A of G, we define an element by

$$A = \sum_{g \in A} g \in Z[G].$$

From [6] we know that if subsets $F_1, \ldots, F_8$ of G form a C-partition of G, i.e.,

$$F_i \cap F_j = \phi \text{ for } i \neq j,\ \cup_{i=1}^{8} F_i = G,$$

and

$$\sum_{i=1}^{8} F_i F_i^{-1} - \sum_{j=1}^{4} (F_j F_{j+4}^{-1} + F_{j+4} F_j^{-1}) = t.$$

Then there are T-matrices $T_1, \ldots, T_4$ of order t with

$$T_i = I(F_i) - I(F_{i+4}),\ i = 1, 2, 3, 4.$$

Instead of $F_1, \ldots, F_8$, we consider

$$Q_i = F_i - F_{i+4},\ i = 1, 2, 3, 4.$$

T_i is called the incidence matrix of Q_i, $i = 1, 2, 3, 4$. Thus $T_1, \ldots, T_4$ are T-matrices of order t if and only if $\sum_{i=1}^{4} Q_i Q_i^{-1} = t$. From this point of view the problem "looking for T-matrices of order t" becomes another equivalent problem "looking for $Q_1, \ldots, Q_4$ such that $\sum_{i=1}^{4} Q_i Q_i^{-1} = t$". If G is a finite field of order t, then $\sum_{i=1}^{4} Q_i Q_i^{-1} = t$ if and only if $\sum_{i=1}^{4} R_i R_i^{-1} = t$ for any $a,\ b \in G$ and $a \neq 0$, where $R_i = aQ_i + b = \{ag + b : g \in Q_i\}$, $i = 1, 2, 3, 4$.

2. Main results

Definition 2.1. *Let T_1, T_2, T_3, T_4 be T-matrices, if*

$$(T_1 + T_2) * (T_3 + T_4)^T = 0,$$

we call the matrices are strongly disjoint.

Let T_i be the incidence matrix of Q_i. Write

$$S(Q_i) = \{g : g \in Q_i\},\ i = 1, 2, 3, 4.$$

Then $T_1, \ldots, T_4$ are strongly disjoint if and only if

$$S(Q_1) \cup S(Q_2) = S(Q_1)^{-1} \cup S(Q_2)^{-1}.$$

In other words, $T_1, \ldots, T_4$ are strongly disjoint if and only if $S(Q_1) \cup S(Q_2)$ is symmetric. If $G = Z_t$, then $T_1, \ldots, T_4$ are strongly disjoint if and only if $k(S(Q_1) \cup S(Q_2))$ is symmetric for every $k \in G$ and $k \neq 0$, if $G = Z_t$ provided the greatest common divisor $(k, t) = 1$, where $kS = \{kg : g \in S\}$. In [11] we proved that if there exist base sequences of lengths $m + p$, $m + p$, m, m (p odd), then there exist strongly disjoint T-matrices of order $2m + p$. For $t = 73, 79, 113$, using cyclotomic classes we constructed strongly disjoint T-matrices of order t. Thus, there exist strongly disjoint T-matrices of order t for $t \in S_1$, where

$$\begin{aligned} S_1 \quad = \quad & \{2m + 1 : 0 < m < 36\} \cup \{6m - 1 : 12 < m < 19\} \\ & \cup \{2^i 10^j 26^k + 1 : i, j, k \geq 0\} \cup \{73, 79, 113\}. \end{aligned}$$

(see [2] for the details of base sequences.)

Example 2.2. *In $G = Z_{19}$ let $Q_1, \ldots, Q_4$ be as follows:*

$$0,\ \ \{0\},\ \ \{1, 2, 3, 7, 11, 14\} - \{4, 6, 9\},\ \ \{8, 10, 12, 13, 15, 18\} - \{5, 16, 17\}.$$

It is easy to see that they are strongly disjoint matrices of order 19.

Definition 2.3. *If T-matrices T_1, T_2, T_3, T_4 satisfy*

$$T_1T_4 - T_2T_3 = (T_1T_4 - T_2T_3)^T,$$

we call they are suitable T-matrices.

Suitable T-matrices can be used to construct weighing matrices [8], orthogonal designs [9] and Hadamard matrices [11]. Similarly, if $G = Z_t$, we have $T_1, \ldots, T_4$ are suitable if and only if $k(Q_1Q_4 - Q_2Q_3)$ is symmetric for every k provided the greatest common divisor $(k, t) = 1$.

Example 2.4. *In $G = Z_{11}$, let $Q_1, \ldots, Q_4$ be as follows:*

$$\{0,1\} - \{3,7\},\ \ \{2\},\ \ \{4,5,8,10\} - \{9\},\ \ \{6\}.$$

It is easy to verify that the corresponding incidence matrices are suitable [10]. But clearly, they are not strongly disjoint. Now take $R_i = Q_i + 4$, $i = 1, 2, 3, 4$, we have

$$\{4,5\} - \{0,7\},\ \ \{6\},\ \ \{1,3,8,9\} - \{2\},\ \ \{10\}.$$

The quadruple above is strongly disjoint.

Example 2.4 shows that the "strongly disjoint" may be obtained from "suitable" by adding some appropriate element of G.

Example 2.5. *In $G = Z_{11}$, let $Q_1, \ldots, Q_4$ be the follows:*

$$\{0,1,2\},\ \ \{3,9\} - \{8\},\ \ \{6\},\ \ \{4,7\} - \{5,10\}.$$

It is easy to verify that they are suitable but not strongly disjoint by adding any number to Q's.

From [10, 11] we know that there are suitable T-matrices of order t for $t \in S_2$, where

$$S_2 = \{2m + 1 : 0 < m < 13, m \neq 11\}.$$

(see appendix for suitable T-matrices). Without proof, we quote the following theorem from [11].

Theorem 2.6. *Let T-matrices A_1, A_2, A_3, A_4 of order t_1 be strongly disjoint and B_1, B_2, B_3, B_4 be suitable T-matrices of order t_2, set*

$$\begin{aligned} C_1 &= A_1 \otimes B_1 + A_2^T \otimes B_2 + A_3 \otimes B_3^T + A_4^T \otimes B_4^T, \\ C_2 &= A_1 \otimes B_3 + A_2^T \otimes B_4 - A_3 \otimes B_1^T - A_4^T \otimes B_2^T, \\ C_3 &= A_2 \otimes B_1 - A_1^T \otimes B_2 - A_4 \otimes B_3^T + A_3^T \otimes B_4^T, \\ C_4 &= A_2 \otimes B_3 - A_1^T \otimes B_4 + A_4 \otimes B_1^T - A_3^T \otimes B_2^T, \end{aligned} \tag{2.1}$$

where $\otimes$ denotes Kronecker product. Then C_1, C_2, C_3, C_4 are T-matrices of order t_1t_2.

Example 2.7. *Let $t_1 = 3$, $t_2 = 5$.*

$$0,\ \ \{0\},\ \ \{1\},\ \ \{2\}$$

and

$$0,\ \ \{0,1\},\ \ \{2\},\ \ \{3\} - \{4\}$$

are Q's corresponding strongly disjoint T-matrices of order 3 *and the corresponding suitable T-matrices of order* 5 *respectively* [10]*. From (2.1) we get Q's in* Z_{15} *as follows:*

$$\{0,6,7,13\}-\{1\},\ \ \{3\}-\{4,9,10\},\ \ \{2\}-\{8,11\},\ \ \{12\}-\{5,14\}.$$

The resulting Q's will give us T-matrices of order 15*.*

Remark 2.8. *Let us recall the construction of composite Hadamard matrices. If there are T-matrices* T_1, T_2, T_3, T_4 *of order* t *and* $(1,-1)$ *amicable set of matrices* $A_1,\ldots,A_4$ *of order* m *(type* 1*), i.e.,*

$$A_1A_2^T - A_3A_4^T = (A_1A_2^T - A_3A_4^T)^T,$$

such that

$$\sum_{i=1}^{4} A_iA_i^T = 4mI_m,$$

then there is a Hadamard matrix of order $4tm$ *with the components*

$$\begin{aligned}
B_1 &= T_1\otimes A_1 + T_2\otimes A_2 + T_3\otimes A_3^T + T_4\otimes A_4^T,\\
B_2 &= T_1\otimes A_2 - T_2\otimes A_1 + T_3\otimes A_4^T - T_4\otimes A_3^T,\\
B_3 &= T_1\otimes A_3 - T_2\otimes A_4 - T_3\otimes A_1^T + T_4\otimes A_2^T,\\
B_4 &= T_1\otimes A_4 + T_2\otimes A_3 - T_3\otimes A_2^T - T_4\otimes A_1^T,
\end{aligned}$$

suitable for use in Goethals-Seidel array. (see Seberry and Yamada [3]*).*

Theorem 2.9. *Suppose A, B, C, D are commutative and symmetric* $(1,-1)$ *matrices of order* n *such that*

$$A^2+B^2+C^2+D^2 = 4nI_n,\ \ AB+CD=0. \tag{2.2}$$

Let T_1, T_2, T_3, T_4 *be suitable T-matrices of order* t*.Then there is a Hadamard matrix of order* $4ntr$*,where* r *is an order of T-matrices.*

Proof. Put

$$\begin{aligned}
W_1 &= T_1\otimes A + T_2\otimes C - T_3\otimes D + T_4\otimes B,\\
W_2 &= T_1\otimes B + T_2\otimes D - T_3\otimes C + T_4\otimes A,\\
W_3 &= T_1\otimes C - T_2\otimes A + T_3\otimes B + T_4\otimes D,\\
W_4 &= T_1\otimes D - T_2\otimes B + T_3\otimes A + T_4\otimes C.
\end{aligned}$$

It is easy to see that W_1, W_2, W_3, W_4 are commuting $(1,-1)$ matrices and satisfy

$$\sum_{i=1}^{4} W_iW_i^T = 4ntI_{nt}.$$

Moreover,

$$\begin{aligned}
W_1W_2 + W_3W_4 &= (T_1^2+T_2^2+T_3^2+T_4^2)\otimes(AB+CD) + 4n(T_1T_4 - T_2T_3)\otimes I_n\\
&= 4n(T_1T_4 - T_2T_3)\otimes I_n.
\end{aligned}$$

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Set

$$H = (H_{ij})_{1\leq i,j\leq 4} = \begin{pmatrix} W_1 & W_2^T & W_3 & W_4^T \\ -W_2^T & W_1 & -W_4^T & W_3 \\ -W_3^T & W_4 & W_1^T & -W_2 \\ -W_4 & W_3^T & W_2 & W_1^T \end{pmatrix}. \tag{2.3}$$

We have

$$\sum_{k=1}^{4} H_{ik}H_{jk}^T = \delta_{ij}4ntI_{nt},$$

where $\delta_{ij} = 0$ or 1 according to $i \neq j$ or $i = j$. Hence H is a Hadamard matrix of order $4nt$ with 16 commuting blocks of size $nt \otimes nt$. Let K_1, K_2, K_3, K_4 be T-matrices of order r. Take

$$M_i = \sum_{j=1}^{4} K_j \otimes H_{ij}, \quad i = 1, 2, 3, 4.$$

Then

$$\sum_{i=1}^{4} M_i M_i^T = 4ntrI_{ntr}.$$

Note that $M_1, \ldots, M_4$ are all of type 1. Using Goethals-Seidel array, we can form a Hadamard matrix of order $4ntr$. The proof is complete. □

In (2.3) we write

$$K_1 = \begin{pmatrix} W_1 & W_2^T \\ -W_2^T & W_1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} W_3 & W_4^T \\ -W_4^T & W_3 \end{pmatrix},$$

then the Hadamard matrix of order $4nt$ has the following form:

$$K = \begin{pmatrix} K_1 & K_2 \\ -K_2^T & K_1^T \end{pmatrix}. \tag{2.4}$$

Corollary 2.10. *There is a Hadamard matrix of order $4tn$ with form (2.4) for $n \in S$ and $t \in S_2$, where*

$$S = \{9^a m^4 : \ a = 0 \text{ or } 1, \ m \text{ integer}\},$$

(see [1, 4, 7] for the details of S).

Corollary 2.11. *There exists a Hadamard matrix of order $4nqt^2$ for $n \in S$, $q \in S_1$, $t \in S_2$.*

In Theorem 2.9 we deal with t and qt instead of t and r, respectively.

Corollary 2.12. *If there exist both strongly disjoint T-matrices of order t, and suitable T-matrices of order t, then there exists a Hadamard matrix of order $4nt^b$ for $n \in S$ and integer $b \geq 0$. Especially, there is a Hadamard matrix of order $4n3^at^b$ for $a \geq 0$, $b \geq 0$, $n \in S$, and $t = 5, 7, 11, 13, 17, 19$.*

Proof. Since $n3^{2m}t^{4k} \in S$, for any integer $m \geq 0$, $k \geq 0$, it is enough to consider the case $a = 0$, 1 and $b = 1$, 2, 3. From Theorem 2.9 the conclusion of the first part of the corollary follows. Since there are suitable and strongly disjoint T-matrices of corresponding orders, the statement of the rest of the Corollary holds too. □

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Corollary 2.13. *There exists a Hadamard matrix of order* $4ntv$ *for* $n \in S$, $t \in S_2$, $v \in S_3$, *where*

$$S_3 = \{q^2 : \ q \ prime\ power\ \equiv 3 \ (mod\ 8)\}.$$

(see [6] for the details of S_3*).*

Note that if a quadruple of T-matrices of order t for t odd is suitable, it is not strongly disjoint. Conversely, if it is strongly disjoint, it can not be suitable. (see [12] for the proof).

Remark 2.14. *For any odd integer* v *there is a factorization:*

$$\begin{aligned} v &= pq^2u, \\ p &= p_1p_2\cdots p_k, \\ q &= q_1q_2\cdots q_l, \\ u &= 9^a m^4, \end{aligned}$$

where p_i, q_j *are odd primes,* $q_j > 3$, $a = 0$ *or* 1, m *integer. If we have strongly disjoint* T*-matrices of order* p *and suitable* T*-matrices of order* q*, from theorem above we can construct a Hadamard matrix of order* v*. So, it is enough to deal with the case that the order* t *is a product of finite distinct primes.*

It is known there are special Williamson matrices of order n for $n \in S$ that satisfying

$$AB + CD = AC + BD = AD + BC = 0. \tag{2.5}$$

Naturally, there is a question: Are there exist 4 symmetric $(1, -1)$ matrices of order n, satisfying (2.2) only, but not (2.5)?

Let T_1, T_2, T_3, T_4 be T-matrices of order t. Put

$$\begin{aligned} W_1 &= T_1 + T_2 + T_3 + T_4, \\ W_2 &= T_1 - T_2 - T_3 + T_4, \\ W_3 &= T_1 - T_2 + T_3 - T_4, \\ W_4 &= -T_1 - T_2 + T_3 + T_4. \end{aligned} \tag{2.6}$$

W_1, W_2, W_3 and W_4 are commuting $(1, -1)$ matrices and satisfy

$$\sum_{i=1}^{4} W_i W_i^T = 4tI_t. \tag{2.7}$$

Moreover,

$$W_1W_2 + W_3W_4 = 4(T_1T_4 - T_2T_3), \tag{2.8}$$

$$W_1 * W_2 * W_3 * W_4 = -J, \tag{2.9}$$

where J is a matrix of order t with all entries 1. From (2.6) it follows that

$$\begin{aligned} T_1 &= \tfrac{W_1+W_2+W_3-W_4}{4}, \\ T_2 &= \tfrac{W_1-W_2-W_3-W_4}{4}, \\ T_3 &= \tfrac{W_1-W_2+W_3+W_4}{4}, \\ T_4 &= \tfrac{W_1+W_2-W_3+W_4}{4}. \end{aligned} \tag{2.10}$$

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Theorem 2.15. *Suppose W_1, W_2, W_3, W_4 are circulant or type 1 $(1, -1)$ matrices of order t, satisfying (2.7) and (2.9). Then T_1, T_2, T_3, T_4 obtained from (2.10) are T-matrices of order t.*

Proof. It is easy to know that T_1, T_2, T_3, T_4 are $(0, \pm 1)$ matrices of order t. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We have

$$(W_i * W_j + W_k * W_l) * (W_i * W_j + W_k * W_l) = J - J - J + J = 0,$$

so,

$$W_i * W_j + W_k * W_l = 0.$$

Since

$$T_1 * T_2 = -\frac{W_1 * W_4 + W_2 * W_3}{8} = 0 = -T_3 * T_4,$$
$$T_1 * T_3 = \frac{W_1 * W_3 + W_2 * W_4}{8} = 0 = -T_2 * T_4,$$
$$T_1 * T_4 = \frac{W_1 * W_2 + W_3 * W_4}{8} = 0 = -T_2 * T_3.$$

and

$$\sum_{i=1}^{4} T_i T_i^T = tI_t.$$

The proof is complete. □

Obviously, (2.8) holds by (2.10). Thus we establish the relationship between T-matrices and some components that satisfying (2.9) of Goethals-Seidel type Hadamard matrix by (2.10).

Corollary 2.16. *The assumptions are as in Theorem 2.15. $T_1T_4 - T_2T_3$ is symmetric if and only if $W_1W_2 + W_3W_4$ is symmetric too.*

Example 2.17. *In Z_7, the first rows of W's are as follows:*

$$- - + + + + +, \quad - - - + - - +, \quad - - - - - + +, \quad + + - + - + -,$$

where $+$, $-$ denote 1, -1, respectively. It is easy to verify that

$$\sum_{i=1}^{4} W_i W_i^T = 28 I_7 \text{ and } W_1 * W_2 * W_3 * W_4 = -J.$$

From (2.10) we get 4 matrices, say T_1, T_2, T_3 and T_4, their first rows are as follows:

$$- - 0000+, \quad 00 + 0 + 00, \quad 00000 + 0, \quad 000 + 000.$$

It is easy to verify that they are suitable T-matrices of order 7.

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Appendix

A list of suitable T-matrices are listed below. We write Q's in the order: Q_1, Q_2, Q_3, Q_4, and assume that G is the set of all residues modulo t, except 25.

n	Q_1, Q_2, Q_3, Q_4	References
$3 = 1^2 + 1^2 + 1^2 + 0^2$	{0}, {1}, {2}, 0.	[10]
$5 = 2^2 + 1^2 + 0^2 + 0^2$	0, {0, 1}, {2}, {3}−{4}.	[10]
$7 = 2^2 + 1^2 + 1^2 + 1^2$	{6}−{0, 1}, {2, 4}, {5}, {3}.	[10]
$9 = 3^2 + 0^2 + 0^2 + 0^2$	0, {0}−{1}, {2}−{6}, {3, 4, 5, 8}− {7}.	[11]
$9 = 2^2 + 2^2 + 1^2 + 0^2$	0, {0, 1}, {2, 6}, {3, 5}−{4, 7, 8}.	[10]
$11 = 3^2 + 1^2 + 1^2 + 0^2$	{0, 1, 2}, {3, 9}−{8}, {6}, {4, 7}−{5, 10}.	
	{0, 1}−{2, 7}, {6, 9}−{3}, {4, 8, 10}, {5}.	
	{0, 1}−{3, 7}, {2}, {4, 5, 8, 10}−{9}, {6}.	[10]
$13 = 3^2 + 2^2 + 0^2 + 0^2$	0, {0}−E_0, E_3, $E_1 - E_2$, where $E_i = \{2^{4j+i} : j = 0, 1, 2\}$, $i = 0, 1, 2, 3$.	[10]
	0, {0, 1, 2, 12}−{3, 11}, {4, 6, 8}, {5, 9}−{7, 10}.	
	{0}−{1}, {2, 9}, {5, 6, 7, 10}−{12}, {3, 8}−{4, 11}.	
	{0, 1, 6}, {3, 10}−{9, 12}, {4, 8}, {5, 7} − {2, 11}.	
$13 = 2^2 + 2^2 + 2^2 + 1^2$	{0, 1}, {2, 7}, {8, 9}−{12}, {4, 11}−{3, 5, 6, 10}.	[11]
	{0, 1}, {4, 9}−{11}, {6, 10, 12}−{5}, {2, 3, 8}−{7}.	
	{0, 1}−{ 3}, {5, 9, 11}−{10}, {7, 8}, {2, 4, 12}−{6}.	
	{0, 1, 4}−{6}, {2, 12}−{9}, {5, 11}, {3, 8, 10}−{7}.	
	{0, 1, 5}−{7, 10}, {2, 8}, {4, 6, 12}−{11}, {3, 9}.	
	{0, 1, 6}−{2}, {3}−{7, 9, 11}, {5, 12}, {4, 8}−{10}.	
	{0, 1, 12}−{2}, {5, 9}, {6, 11}−{10}, {4, 7, 8}−{3}.	
	{0, 1, 3, 9}−{10, 11}, {2, 12}, {4, 5}, {7}−{6, 8}.	
	{0, 1, 5, 8}−{4, 6}, {7, 11}−{12}, {9, 10}, {2, 3}.	
$15 = 3^2 + 2^2 + 1^2 + 1^2$	{0, 3}−{4, 9, 10, 12, 13}, {1, 5}, {2, 7}−{6}, {8, 14}−{11}.	
	{0, 1, 2}−{6}, {3, 8, 9, 14}−{10}, {4}−{5, 13}, {7, 11}−{12}.	[10]
	{0, 1, 5}−{10}, {2, 7, 8, 13}−{4, 12, 14}, {3, 6, 9}, {11}.	
	{0, 1, 3, 10}−{6, 9, 13}, {4, 14}−{12}, {5, 7}, {2, 8, 11}.	
	{0, 1, 6, 13}−{5}, {2, 3, 10, 12}−{4, 8}, {9}, {14}−{7, 11}.	
	{0, 1, 6, 13}−{3, 7, 9}, {2, 10}−{12}, − 4, 14}, {5, 8, 11}.	
	{0, 1, 3, 7, 13}−{6, 9}, {2, 10}−{12}, {4, 14}, {5, 11}−{8}.	
$17 = 4^2 + 1^2 + 0^2 + 0^2$	{0, 1}−{3, 7, 11, 13, 14, 16}, {2, 15}−{6, 10}, {4, 9}−{5}, {12}−{8}.	[11]
	{0, 1, 8}−{9, 10, 14}, {2, 14}−{5, 7}, {11}−{13, 16}, {3, 6, 12, 15}.	
	{0, 1, 3, 4, 11}−{8}, {2, 7}−{5, 9}, {12, 13, 15}−{6, 14, 16}, {10}.	
	{0, 1, 3, 4, 11}−{8}, {2, 7, 16}−{5, 6, 10}, {14}−{9, 15}, {12}−{13}.	
$17 = 3^2 + 2^2 + 2^2 + 0^2$	0, {2, 6, 12}−{7}, {5, 8, 9, 13, 14}−{10, 16}, {0, 1, 3, 15}−{4, 11}.	[11]
	{0, 1, 2}−{7}, {3}−{4, 12, 15}, {5, 6, 16}, {8, 11, 13}−{9, 10, 14}.	
	{0, 1, 3}−{5}, {2, 8}−{9, 13}, {4, 11, 12, 16}−{7}, {6, 10, 15}−{14}.	
	{0, 1, 5}−{14}, {2, 6, 7, 8, 11}−{3, 13}, {9}−{10}, {4, 12, 15}−{16}.	
$19 = 3^2 + 3^2 + 1^2 + 0^2$	{0}, E_0, $E_1 - E_3$, $E_2 + E_5 - E_4$, where $E_i = \{2^{6j+i} : j = 0, 1, 2\}$, $i = 0, \ldots, 5$.	[11]
	{0, 1, 2, 9, 16}−{4, 10}, {3, 13}−{15, 17}, {7, 14, 18}−{6, 11}, {5, 8, 12}.	
$19 = 4^2 + 1^2 + 1^2 + 1^2$	{0, 1, 3, 7, 12}−{4}, {5, 11}−{10, 15, 18}, {6, 16}−{14}, {2, 13, 17}−{8, 9}.	[11]
$21 = 4^2 + 2^2 + 1^2 + 0^2$	0, {0, 1, 2, 5}−{19, 20}, {6, 8, 10}−{7, 12}, {3, 9, 13, 15, 16, 17, 18}−{4, 11, 14}.	[11]
$25 = 5^2 + 0^2 + 0^2 + 0^2$	0, {0} − $E_0 - E_1$, $E_3 - E_6$, $E_2 + E_4 - E_5 - E_7$, where	
	$E_i = \{g^{8j+i} : j = 0, 1, 2\}$, $i = 0, \ldots, 7$, $g = x + 1(\mod\ x^2 - 3, \mod 5)$.	[11]

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