http://ijgt.ui.ac.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 13 No. 1 (2024), pp. 31-45.

www.ui.ac.ir

GENERALIZED ORDER DIVISOR GRAPHS OF FINITE GROUPS

PARINYAWAT CHOOSUWAN, PANNARIN SANGSAWANG, CHEEVARAT MATWANGSANG, SIREETHON THONGSUPOL AND SIRIPONG SIRISUK*

Abstract. Let G be a finite group and k a fixed positive integer. We define the generalized order divisor graph of G to be a graph whose vertex set is the group G and in which two vertices a and b are adjacent if and only if the orders $\mathrm{o}(a^k)$ and $\mathrm{o}(b^k)$ are different and either $\mathrm{o}(a^k)$ divides $\mathrm{o}(b^k)$ or $\mathrm{o}(b^k)$ divides $\mathrm{o}(a^k)$. This generalizes the order divisor graphs of finite groups. Some properties of our graph are introduced, and we investigate the structure of the generalized order divisor graphs of finite cyclic groups.

1. Introduction

The relationship between groups and graphs is an active research area. Several classes of graph, including the Cayley graphs, commuting graphs [3], and generating graphs [11], are associated with finite groups. For an overview of graphs that are defined on groups, see [5].

The power graph is a well-known graph that arises from a finite group. The directed power graph was first introduced by Kelarev and Quinn [10]. Chakrabarty et al. [6] introduced the undirected power graph. Let G be a finite group. The *undirected power graph* (specifically, the *power graph*) of G is a graph having G as a vertex set and two vertices a and b which are adjacent if and only if $a \neq b$ and one is a power of the other. This is equivalent to claiming that $a \neq b$ and that either $\langle a\rangle \subseteq \langle b\rangle$ or

Keywords: Power graph, star graph, cyclic group.

MSC(2010): Primary: 05C25; Secondary: 20D60.

Communicated by Hamid Mousavi.

Manuscript Type: Research Paper.

*Corresponding author.

Received: 16 March 2022, Accepted: 11 June 2022.

Cite this article: P. Choosuwan, P. Sangsawang, C. Matwangsang, S. Thongsupol and S. Sirisuk, Generalized order divisor graphs of finite groups, Int. J. Group Theory, **13** no. 1 (2024) 31–45. http://dx.doi.org/10.22108/IJGT.2022.133091.1787 .

$\langle b\rangle \subseteq \langle a\rangle$. The properties of power graphs are discussed in [4]. A range of graphs related to the power graphs have since been discovered.

Rajkumar and Anitha [13] studied the *reduced power graph* of G, which is a graph that has G as a vertex set and in which two vertices a and b are adjacent if and only if $a \neq b$ and either $\langle a\rangle \subsetneq \langle b\rangle$ or $\langle b\rangle \subsetneq \langle a\rangle$. The reduced power graph of G is therefore a subgraph of the power graph of G. Hamzeh and Ashrafi [8] defined the *order supergraph* of the power graph of G as a graph whose vertex set is G and in which two vertices a and b are adjacent if and only if $a \neq b$ and either $\mathrm{o}(a) \mid \mathrm{o}(b)$ or $\mathrm{o}(b) \mid \mathrm{o}(a)$. In [7], the same authors reported some properties of this graph and its automorphism group. Rehman et al. [14] introduced the concept of order divisor graphs of finite groups. They defined the *order divisor graph* of G to be a graph whose vertex set is the group G and in which two vertices a and b are adjacent if and only if $\mathrm{o}(a) \neq \mathrm{o}(b)$ and either $\mathrm{o}(a) \mid \mathrm{o}(b)$ or $\mathrm{o}(b) \mid \mathrm{o}(a)$. Clearly, the order divisor graph of G is a subgraph of the order supergraph of the power graph of G. Note that $\langle a\rangle \subsetneq \langle b\rangle$ implies $\mathrm{o}(a) \neq \mathrm{o}(b)$ and $\mathrm{o}(a) \mid \mathrm{o}(b)$ for any $a, b \in G$. Hence, the order divisor graph of G is a supergraph of the reduced power graph of G.

Lui and Ma [12] generalized some results from the order divisor graphs of finite groups and proposed a classification of finite groups whose order divisor graphs are multi-partite. Zhai and Ma [17] studied perfect codes and total perfect codes of the *proper order divisor graph* of a finite group; this is obtained by deleting the identity element in the order divisor graph.

In this paper, we introduce a novel class of graphs that generalizes the order divisor graphs of finite groups by using the orders and the powers of elements in the groups. In Section 2, we provide notation and discuss the basic properties of the graphs. In section 3, we show the group-theoretical and graph-theoretical interplay between groups and graphs. The results from [14] are extended to our graphs and we investigate those groups with star generalized order divisor graphs. In the final section, we discuss the generalized order divisor graphs of finite cyclic groups.

2. Preliminaries

In this section, we provide definitions and notation, and introduce the properties on which the main results will be built. We refer the reader to [1], [2], and [16] for more details.

Throughout this paper, the graphs are simple graphs. Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. If any two distinct vertices of Γ are adjacent, Γ is called a *complete graph*. We sometimes write $u \sim v$ to make explicit that u and v are adjacent vertices of Γ. The *degree* of a vertex u is the number of vertices adjacent to u, denoted by $\deg(u)$. A graph Γ is said to be a *connected graph* if, for any two vertices u and v of Γ, there exist vertices $w_0, w_1, \ldots, w_d$ such that $u = w_0 \sim w_1 \sim \cdots \sim w_d = v$. The smallest such d is called the *distance* of u and v. The *diameter* of Γ, denoted by $\mathrm{diam}(\Gamma)$, is the maximum distance between any two vertices of Γ. A set I of vertices of Γ is called an *independent set* of Γ if any two vertices of I are not adjacent. A graph Γ is a *k-partite*

http://dx.doi.org/10.22108/IJGT.2022.133091.1787

graph if and only if its vertex set $V(\Gamma)$ can be partitioned into k nonempty independent sets. A 2-partite graph is often called a *bipartite graph*. A k-partite graph is said to be *complete* if any two vertices from different partition sets are adjacent. We denote $K_{n_1,n_2,\ldots,n_k}$ for a complete k-partite graph with k partition sets of $n_1, n_2, \ldots, n_k$ vertices, respectively. We call a complete bipartite graph $K_{1,n}$ a *star graph*.

Let Γ_1 and Γ_2 be graphs. We say that Γ_1 is a *subgraph* of Γ_2 if $V(\Gamma_1) \subseteq V(\Gamma_2)$ and $E(\Gamma_1) \subseteq E(\Gamma_2)$. The *sequential join* of Γ_1 and Γ_2, denoted by $\Gamma_1 \diamond \Gamma_2$, is a graph whose vertex set is $V(\Gamma_1) \cup V(\Gamma_2)$ and whose edge set is $E(\Gamma_1) \cup E(\Gamma_2) \cup \{uv \colon u \in V(\Gamma_1) \text{ and } v \in V(\Gamma_2)\}$. Moreover, Γ_1 is *isomorphic* to Γ_2, denoted by $\Gamma_1 \cong \Gamma_2$, if there exists a bijection $f \colon V(\Gamma_1) \to V(\Gamma_2)$ such that g_1 is adjacent to g_2 in Γ_1 if and only if $f(g_1)$ is adjacent to $f(g_2)$ in Γ_2 for any vertices g_1 and g_2 in Γ_1.

For a positive integer n, the *Euler's totient* $\phi(n)$ is defined to be the number of integers k such that $0 \leq k < n$ and k is relatively prime to n. Clearly, $\phi(p) = p - 1$ if p is a prime number. Note that $\phi(mn) = \phi(m)\phi(n)$ if m and n are relatively prime. Next, let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be a positive integer where $p_1, p_2, \ldots, p_r$ are pairwise distinct prime numbers and $k_1, k_2, \ldots, k_r$ are positive integers. Then $\phi(n) = (p_1^{k_1} - p_1^{k_1-1})(p_2^{k_2} - p_2^{k_2-1}) \cdots (p_r^{k_r} - p_r^{k_r-1})$. Moreover, $\phi(n)$ is even for all $n \geq 3$. The number of positive divisors of n is then $(k_1+1)(k_2+1)\cdots(k_r+1)$.

Finally, we provide some basic information on groups. Let G be a finite group with the identity e. The smallest positive integer k such that $a^k = e$ is called the *order* of a in G, denoted by $\mathrm{o}(a)$. Note that the identity e is the only element of order 1. In fact, if $a^k = e$, then $\mathrm{o}(a) \mid k$. Moreover, $\mathrm{o}(a^k) = \frac{\mathrm{o}(a)}{\gcd(k,\mathrm{o}(a))}$ and thus $\mathrm{o}(a^k) \mid \mathrm{o}(a)$ for all $a \in G$. The *exponent* $\exp(G)$ of G is defined to be $\exp(G) = \min\{n \in \mathbb{N} : \forall a \in G, a^n = e\}$. Then $\mathrm{o}(a) \mid \exp(G)$ for all $a \in G$. Also, if p is a prime number, $p^m \mid \exp(G)$ implies G has an element of order p^m. Moreover, if G is of order n, then $\exp(G)$ divides n, and if p is a prime number dividing n, G has an element of order p. An abelian group G is called an *elementary abelian p-group* where p is a prime number if $\mathrm{o}(a) = p$ for all nonidentity element $a \in G$. In addition, some of our results deal with cyclic groups. A group G of order n is a *cyclic group* if and only if $\mathrm{o}(a) = n$ for some $a \in G$. A cyclic group of order n is clearly a group of exponent n. Note that a cyclic group is abelian and any subgroup of a cyclic group is also cyclic. Furthermore, in a cyclic group of order n, the number of elements of order d where $d \mid n$ is $\phi(d)$. We also have that $\mathrm{o}(a)$ is the order of the cyclic subgroup $\langle a \rangle$. The additive group $\mathbb{Z}_n$ of integers modulo n is clearly a cyclic group. Moreover, $U(\mathbb{Z}_n) = \{a \in \mathbb{Z}_n : \gcd(a,n) = 1\}$ is a multiplicative group of order $\phi(n)$. Finally, we recall the dihedral group D_n for $n \geq 3$ which is a group generated by two elements a and b where

$$D_n = \left\langle a, b : a^n = b^2 = e, ab = ba^{-1} \right\rangle = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}.$$

This D_n is a group of order $2n$ and represents the symmetries of a regular n-gon under composition. Moreover, $\mathrm{o}(a^ib) = 2$ for all $i \in \{1, 2, \ldots, n\}$.

3. Generalized order divisor graphs of finite groups

Let G be a finite group. Fix a positive integer k. We define the *generalized order divisor graph of G of type k*, denoted by $\mathrm{OD}^k(G)$, to be the graph whose vertex set is G and two vertices a and b are adjacent if and only if $\mathrm{o}(a^k) \neq \mathrm{o}(b^k)$ and either $\mathrm{o}(a^k) \mid \mathrm{o}(b^k)$ or $\mathrm{o}(b^k) \mid \mathrm{o}(a^k)$. By this definition of adjacency, $\mathrm{OD}^k(G)$ has no loops and has no multiple edges. $\mathrm{OD}^k(G)$ is therefore a simple graph. In case $k = 1$, the graph $\mathrm{OD}^1(G)$ is the order divisor graph studied in [14]. Thus, this class of generalized order divisor graphs generalizes order divisor graphs of finite groups.

Example 3.1. *Figures* 1–6 *give examples of the generalized order divisor graphs of* $\mathbb{Z}_6$.

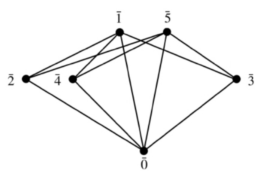


FIGURE 1. $\mathrm{OD}^1(\mathbb{Z}_6)$

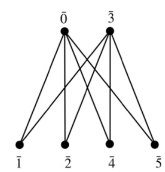


FIGURE 2. $\mathrm{OD}^2(\mathbb{Z}_6)$

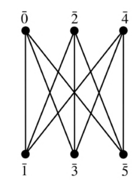


FIGURE 3. $\mathrm{OD}^3(\mathbb{Z}_6)$

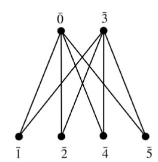


FIGURE 4. $\mathrm{OD}^4(\mathbb{Z}_6)$

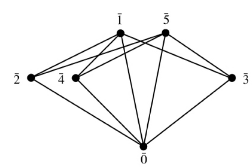


FIGURE 5. $\mathrm{OD}^5(\mathbb{Z}_6)$

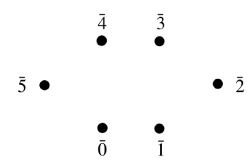


FIGURE 6. $\mathrm{OD}^6(\mathbb{Z}_6)$

For a finite group G of order n and a positive integer k, the division algorithm implies that $k = qn + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < n$. So for each $a \in G$, $a^k = a^{qn+r} = a^r$. Hence, if $r = 0$, $\mathrm{OD}^k(G)$ has no edges. On the other hand, if $r \neq 0$, we can replace k by r and obtain the same graph. Thus, unless otherwise stated, we assume throughout this paper that $k < n$.

We first show that a generalized order divisor graph is either a graph without edges or a connected graph.

Theorem 3.2. *Let G be a finite group of order $n \geq 2$. Then*

(1) $\mathrm{OD}^k(G)$ *has no edges if and only if* $\mathrm{o}(a^k) = 1$ *for any* $a \in G$.
(2) $\mathrm{OD}^k(G)$ *is a connected graph if and only if there exists* $a \in G$ *such that* $\mathrm{o}(a^k) \neq 1$.

Proof. 1. Assume that $\mathrm{OD}^k(G)$ has no edges. Note that $\mathrm{o}(e^k) = 1$. Thus, if there exist $a \in G \setminus \{e\}$ such that $\mathrm{o}(a^k) \neq 1$, then a is adjacent to e, which is a contradiction. On the other hand, if $\mathrm{o}(a^k) = 1$ for all $a \in G$, it is clear that no two vertices are adjacent.

2. Suppose that $\mathrm{OD}^k(G)$ is connected. Then it has an edge. By (1), there exists $a \in G$ such that $\mathrm{o}(a^k) \neq 1$. Conversely, we assume that there exists $a \in G$ such that $\mathrm{o}(a^k) \neq 1$. Let $A_1 = \{a \in G : \mathrm{o}(a^k) = 1\}$ and $A_2 = \{a \in G : \mathrm{o}(a^k) \neq 1\}$. Then A_1 and A_2 are nonempty sets since $e \in A_1$ and $a \in A_2$. To show that $\mathrm{OD}^k(G)$ is connected, let $x, y \in G$ be two distinct vertices. If $x, y \in A_1$, then $\mathrm{o}(x^k) = 1 = \mathrm{o}(y^k)$, so that $x \sim a \sim y$. If $x, y \in A_2$, then $\mathrm{o}(x^k) \neq 1$ and $\mathrm{o}(y^k) \neq 1$, and thus $x \sim e \sim y$. Finally, if $x \in A_1$ and $y \in A_2$ or $x \in A_2$ and $y \in A_1$, then $x \sim y$. Therefore, $\mathrm{OD}^k(G)$ is a connected graph.

□

The proof of Theorem 3.2 (2) gives the following corollary.

Corollary 3.3. *Let G be a finite group. Then* $\mathrm{diam}\left(\mathrm{OD}^k(G)\right) \leq 2$.

Next, we study relationships between the generalized order divisor graph $\mathrm{OD}^k(G)$ and the order divisor graph $\mathrm{OD}^1(G)$ defined in [14]. We first notice that $\mathrm{OD}^k(G)$ is not necessarily a subgraph of $\mathrm{OD}^1(G)$. One can show that $\mathrm{OD}^2(\mathbb{Z}_4)$ is a subgraph of $\mathrm{OD}^1(\mathbb{Z}_4)$. However, $\mathrm{OD}^2(\mathbb{Z}_6)$ is not a subgraph of $\mathrm{OD}^1(\mathbb{Z}_6)$ since vertices $\bar{2}$ and $\bar{3}$ in $\mathbb{Z}_6$ are adjacent in $\mathrm{OD}^2(\mathbb{Z}_6)$ but not in $\mathrm{OD}^1(\mathbb{Z}_6)$.

Theorem 3.4. *Let G be a group of exponent n. Then $\mathrm{OD}^k(G) = \mathrm{OD}^\ell G$ if and only if* $\gcd(k, n) = \gcd(\ell, n)$.

Proof. Assume that $\gcd(k, n) \neq \gcd(\ell, n)$. Then there exists a prime number p and integers s, t with $s \neq t$ such that $p^s \mid \gcd(k, n)$, $p^t \mid \gcd(\ell, n)$ and $p^{s+i} \nmid \gcd(k, n)$ and $p^{t+i} \nmid \gcd(\ell, n)$ for all positive integers i. Without loss of generality, we suppose that $s > t$. Since $\exp(G) = n$ and $p^s \mid n$, there exists $a \in G$ such that $\mathrm{o}(a) = p^s$. Then $\mathrm{o}(a^k) = \frac{p^s}{\gcd(k,p^s)} = \frac{p^s}{p^s} = 1$ and $\mathrm{o}(a^\ell) = \frac{p^s}{\gcd(\ell,p^s)} \neq 1$ since $\gcd(\ell, p^s) < p^s$. Since $\mathrm{o}(e^k) = 1 = \mathrm{o}(e^\ell)$, we conclude that e and a are adjacent in $\mathrm{OD}^\ell(G)$ but not in $\mathrm{OD}^k(G)$. Thus, $\mathrm{OD}^k(G) \neq \mathrm{OD}^\ell(G)$.

Suppose that $\gcd(k, n) = \gcd(\ell, n)$. Let a be any element in G. Then $\mathrm{o}(a^k) = \frac{\mathrm{o}(a)}{\gcd(k,\mathrm{o}(a))}$ and $\mathrm{o}(a^\ell) = \frac{\mathrm{o}(a)}{\gcd(\ell,\mathrm{o}(a))}$. Since $\gcd(k, n) = \gcd(\ell, n)$ and $\mathrm{o}(a) \mid n$, it follows that $\gcd(k, \mathrm{o}(a)) = \gcd(\ell, \mathrm{o}(a))$. Hence, $\mathrm{o}(a^k) = \mathrm{o}(a^\ell)$. Thus, for any $a, b \in G$, $\mathrm{o}(a^k) \mid \mathrm{o}(b^k)$ if and only if $\mathrm{o}(a^\ell) \mid \mathrm{o}(b^\ell)$, so that a and b are adjacent in $\mathrm{OD}^k(G)$ if and only if they are adjacent in $\mathrm{OD}^\ell(G)$. Therefore, $\mathrm{OD}^k(G) = \mathrm{OD}^\ell(G)$. □

If $\gcd(k, n) = d$, then $\gcd(d, n) = d$. Hence, $\mathrm{OD}^k(G) = \mathrm{OD}^d(G)$. We record this in the following corollary.

Corollary 3.5. *Let G be a group of exponent n. Then $\mathrm{OD}^k(G) = \mathrm{OD}^d(G)$ where $d = \gcd(k, n)$.*

Corollary 3.6. *Let G be a group of exponent n. There are then m distinct generalized order divisor graphs of G where m is the number of positive divisors of n.*

Proof. By Corollary 3.5, $\mathrm{OD}^k(G) = \mathrm{OD}^d(G)$ where $d = \gcd(k, n)$. Let $d_1, d_2, \ldots, d_{m-1}$ and $d_m = n$ be all distinct positive divisors of n. Note that $\mathrm{OD}^{d_m}(G)$ has no edges. Let $i \in \{1, 2, \ldots, m-1\}$. Since

$d_i < n$ and $\exp(G) = n$, there exists $a \in G$ such that $a^{d_i} \neq e$, and thus $\mathrm{o}(a^{d_i}) \neq 1$. By Theorem 3.2, $\mathrm{OD}^{d_i}(G)$ is a connected graph. Since $\gcd(d_i, n) = d_i \neq d_j = \gcd(d_j, n)$ for all $i, j \in \{1, 2, \dots, m-1\}$ with $i \neq j$, it follows from Theorem 3.4 that $\mathrm{OD}^{d_1}(G), \mathrm{OD}^{d_2}(G), \dots, \mathrm{OD}^{d_{m-1}}(G)$ are all distinct. □

We next consider the necessary and sufficient conditions for star generalized order divisor graphs. We require the following lemma.

Lemma 3.7. *There is no group G of order $n \geq 3$ such that $y^k \neq e$ for some $y \in G \setminus \{e\}$ and $x^k = e$ for all $x \in G \setminus \{y\}$.*

Proof. Suppose there exists a group G of order $n \geq 3$ such that $y^k \neq e$ for some $y \in G \setminus \{e\}$ and $x^k = e$ for all $x \in G \setminus \{y\}$. Let $x \in G \setminus \{e, y\}$. Then $xy \neq y$. Since $\mathrm{o}(xyx^{-1}) = \mathrm{o}(y)$, it follows that $y = xyx^{-1}$. Thus, $xy = yx$. Then $(xy)^k = x^k y^k = y^k \neq e$ which is a contradiction. □

Theorem 3.8. *Let G be a group of order $n \geq 2$. Then $\mathrm{OD}^k(G)$ is a star graph if and only if $\mathrm{o}(a^k)$ is a prime number for all $a \in G \setminus \{e\}$.*

Proof. It is easy to see that the statement is true for $n = 2$. Let $n \geq 3$. Assume that $\mathrm{OD}^k(G)$ is a star graph. Then $\mathrm{OD}^k(G)$ has two partition sets $V_1 = \{x_0\}$ and $V_2 = G \setminus \{x_0\}$ where x_0 is adjacent to all vertices in V_2 and no two vertices in V_2 are adjacent. Since $\mathrm{OD}^k(G)$ is a star graph which is a connected graph, there exists $y \in G$ such that $\mathrm{o}(y^k) \neq 1$. Hence, $y^k \neq e$.

Suppose that the identity e is in V_2. Since no two vertices in V_2 are adjacent, it follows that $\mathrm{o}(x^k) = 1$ and thus $x^k = e$ for all $x \in V_2$. This requires that $y = x_0 \in V_1$, which contradicts the claim in Lemma 3.7 that there exists no group G such that $y^k \neq e$ and $x^k = e$ for all $x \in G \setminus \{y\}$. Therefore, $e = x_0 \in V_1$. Since e is adjacent to all $a \in V_2$, we have $\mathrm{o}(a^k) \neq 1$ for all $a \in V_2$.

To show that $\mathrm{o}(a^k)$ is a prime number for all $a \in V_2$, we suppose that there exists $a \in V_2$ such that $\mathrm{o}(a^k)$ is not prime. There then exists a prime number p such that $p \mid \mathrm{o}(a^k)$. Since $p \mid n$, there also exists $x \in V_2$ such that $\mathrm{o}(x) = p$. Since $x \in V_2$, we know that $\mathrm{o}(x^k) \neq 1$. Hence, $\mathrm{o}(x^k) = p$. Now, $\mathrm{o}(a^k) \neq \mathrm{o}(x^k)$ and $\mathrm{o}(x^k) \mid \mathrm{o}(a^k)$. Thus, a is adjacent to x, which is a contradiction since a and x are in V_2. Therefore, $\mathrm{o}(a^k)$ is prime for all $a \in G \setminus \{e\}$.

For the reverse implication, we assume that $\mathrm{o}(a^k)$ is prime for all $a \in G \setminus \{e\}$. Clearly, e is adjacent to all $a \in G \setminus \{e\}$. Let x and y be two vertices in $G \setminus \{e\}$. Then $\mathrm{o}(x^k) = p$ and $\mathrm{o}(y^k) = q$ for some prime numbers p and q. Hence, x and y are not adjacent. Thus, $\mathrm{OD}^k(G)$ has two partition sets $\{e\}$ and $G \setminus \{e\}$. Therefore, it is a star graph. □

Theorem 3.9. *Let G be an abelian group of order $n \geq 2$. Then $\mathrm{OD}^k(G)$ is a star graph if and only if G is an elementary abelian p-group for some prime number p with $\gcd(k, p) = 1$.*

Proof. It is easy to see that the statement is true when $n = 2$. Let $n \geq 3$. Assume that $\mathrm{OD}^k(G)$ is a star graph. Suppose that $G = \{e, x_1, x_2, \dots, x_{n-1}\}$. By Theorem 3.8, we have $\mathrm{o}(x_i^k) = p_i$ where p_i is a prime number for all $i \in \{1, 2, \ldots, n-1\}$. Suppose that there exist $i, j \in \{1, 2, \ldots, n-1\}$ such that $\mathrm{o}(x_i^k) \neq \mathrm{o}(x_j^k)$. Since G is abelian, $\mathrm{o}((x_ix_j)^k) = \mathrm{o}(x_i^kx_j^k) = p_ip_j$. Then $x_ix_j \neq e$ and its order is not prime, which is a contradiction. Thus, $\mathrm{o}(x_i^k) = \mathrm{o}(x_j^k)$ for all $i, j \in \{1, 2, \ldots, n-1\}$. That is, $\mathrm{o}(x_i^k) = p$, a prime number, for all $i \in \{1, 2, \ldots, n-1\}$. Since $p \mid n$, there exists $x \in G$ such that $\mathrm{o}(x) = p$. Thus, $p = \mathrm{o}(x^k) = \frac{p}{\gcd(k,p)}$. Therefore, $\gcd(k, p) = 1$.

To show that G is an elementary abelian p-group, we let $x_i \in G \setminus \{e\}$. Note that $p = \mathrm{o}(x_i^k) = \frac{\mathrm{o}(x_i)}{\gcd(k,\mathrm{o}(x_i))}$. Then $\mathrm{o}(x_i) = p\gcd(k, \mathrm{o}(x_i))$. We write $\mathrm{o}(x_i) = p^sm$ where s and m are positive integers such that $p \nmid m$. Suppose that $m \neq 1$. There then exists a prime number q such that $q \mid m$. So $q \mid n$. Hence, there exists $x_j \in G \setminus \{e, x_i\}$ such that $\mathrm{o}(x_j) = q$. So $\mathrm{o}(x_j^k) = \frac{q}{\gcd(k,q)} = q$ or 1. On the other hand, $\mathrm{o}(x_j^k) = p$. This is a contradiction since p is a prime number which is not q. Thus, $m = 1$. Now, $\mathrm{o}(x_i) = p^s$. Since $\gcd(k, p) = 1$, we have $\gcd(k, p^s) = 1$. Therefore, $\mathrm{o}(x_i) = p\gcd(k, \mathrm{o}(x_i)) = p\gcd(k, p^s) = p$. This completes the analysis of G as an elementary abelian p-group.

Next, assume that G is an elementary abelian p-group where p is a prime number such that $\gcd(k, p) = 1$. Let $a \in G \setminus \{e\}$. Then $\mathrm{o}(a) = p$. Hence, $\mathrm{o}(a^k) = \frac{p}{\gcd(k,p)} = p$. Therefore, by Theorem 3.8 $\mathrm{OD}^k(G)$ is a star graph. □

The fact that any cyclic group is abelian allows us to use Theorem 3.9 to prove the following corollary.

Corollary 3.10. *Let G be a cyclic group of order $n \geq 2$. Then $\mathrm{OD}^k(G)$ is a star graph if and only if n is a prime number.*

Proof. Assume that $\mathrm{OD}^k(G)$ is a star graph. Since G is a cyclic group, there exists $a \in G \setminus \{e\}$ such that $\mathrm{o}(a) = n$. By Theorem 3.9, G is an elementary abelian p-group where p is a prime number such that $\gcd(k, p) = 1$. Thus $n = \mathrm{o}(a) = p$.

On the other hand, we assume that n is a prime number. Let $a \in G \setminus \{e\}$. Since $\mathrm{o}(a) \mid n$ and n is prime, $\mathrm{o}(a) = n$. Thus, G is an elementary abelian n-group and $\gcd(k, n) = 1$ since $k < n$. Therefore, Theorem 3.9 entails that $\mathrm{OD}^k(G)$ is a star graph. □

Since $\mathbb{Z}_n$ is a cyclic group, we have

Corollary 3.11. *For $n \geq 2$, $\mathrm{OD}^k(\mathbb{Z}_n)$ is a star graph if and only if n is a prime number.*

Next, we consider the star generalized order divisor graph of the group $U(\mathbb{Z}_n)$.

Lemma 3.12. *For $n \geq 3$, if $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is a star graph, then $U(\mathbb{Z}_n)$ is an elementary abelian 2-group.*

Proof. Assume that $n \geq 3$ and that $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is a star graph. By Theorem 3.9, $U(\mathbb{Z}_n)$ is an elementary abelian p-group for some prime number p. Since $|U(\mathbb{Z}_n)| = \phi(n)$ and $2 \mid \phi(n)$, there exists $a \in U(\mathbb{Z}_n)$ such that $\mathrm{o}(a) = 2$. Thus, $p = 2$. □

Proposition 3.13. *Let $n \geq 3$ and k be an even number. Then $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is not a star graph.*

Proof. Suppose that $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is a star graph. By Lemma 3.12, $U(\mathbb{Z}_n)$ is an elementary abelian 2-group. Since $\gcd(k, 2) = 2$, this contradicts Theorem 3.9. Thus, $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is not a star graph. □

We next consider the graph $\mathrm{OD}^k(U(\mathbb{Z}_n))$ when k is odd.

Theorem 3.14. *Let k be an odd number. Then $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is a star graph if and only if $n \mid 24$.*

Proof. Assume that $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is a star graph. Clearly, 1 and 2 divide 24. Let $n \geq 3$. By Lemma 3.12, $U(\mathbb{Z}_n)$ is an elementary abelian 2-group.

Case 1. n is odd. Then $\bar{2} \in U(\mathbb{Z}_n)$. Hence, $(\bar{2})^2 = \bar{1}$. So $2^2 \equiv 1 \pmod{n}$, that is, $n \mid (2^2 - 1)$. Thus, $n \in \{1, 3\}$. Therefore, $n \mid 24$.

Case 2. $n = 2^t$ where $t \geq 2$. Then $\bar{3} \in U(\mathbb{Z}_n)$. Hence, $(\bar{3})^2 = \bar{1}$. So $3^2 \equiv 1 \pmod{n}$, that is, $n \mid (3^2 - 1)$. Thus, $n \in \{1, 2, 4, 8\}$. Therefore, $n \mid 24$.

Case 3. $n = 2^t m$ where t and m are positive integers such that $2 \nmid m$. Since $\gcd(2^t, m) = 1$, it follows that $U(\mathbb{Z}_n) \cong U(\mathbb{Z}_{2^t}) \times U(\mathbb{Z}_m)$. Therefore, $U(\mathbb{Z}_{2^t})$ and $U(\mathbb{Z}_m)$ are also elementary abelian 2-groups. Then $m \in \{1, 3\}$ by Case 1 and $2^t \in \{1, 2, 4, 8\}$ by Case 2. Hence, $n = 2^t m \in \{3, 4, 6, 8, 12, 24\}$. Therefore, $n \mid 24$.

For the reverse implication, we assume that $n \mid 24$. One can see that all nonidentity elements in $U(\mathbb{Z}_n)$ are of order 2. Thus, $U(\mathbb{Z}_n)$ is an elementary abelian 2-group. Since k is odd, $\gcd(k, 2) = 1$. Therefore, Theorem 3.9 entails that $\mathrm{OD}^k(U(\mathbb{Z}_n))$ is a star graph. □

We have characterised the star generalized order divisors graphs of abelian groups $\mathbb{Z}_n$ and $U(\mathbb{Z}_n)$. For nonabelian groups, we study the star generalized order divisor graph of the dihedral group D_n.

Theorem 3.15. *Let $n \geq 3$. Then $\mathrm{OD}^k(D_n)$ is a star graph if and only if n is a prime number and $\gcd(k, 2n) = 1$.*

Proof. Let $D_n = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$. Assume that the graph $\mathrm{OD}^k(D_n)$ is a star graph. By Theorem 3.8, $\mathrm{o}(x^k)$ is a prime number for all $x \in D_n \setminus \{e\}$. Suppose that $\gcd(k, 2n) \neq 1$. Then there exists a prime number p such that $p \mid \gcd(k, 2n)$. Since p divides $2n = |D_n|$, there exists $y \in D_n \setminus \{e\}$ such that $\mathrm{o}(y) = p$. Hence, $\mathrm{o}(y^k) = \frac{\mathrm{o}(y)}{\gcd(k,\mathrm{o}(y))} = \frac{p}{\gcd(k,p)} = \frac{p}{p} = 1$ which is a contradiction. Thus, $\gcd(k, 2n) = 1$. Also, $\gcd(k, n) = 1$. Since $a \in D_n$ such that $\mathrm{o}(a) = n$, it follows that $\mathrm{o}(a^k) = \frac{\mathrm{o}(a)}{\gcd(k,\mathrm{o}(a))} = \frac{n}{\gcd(k,n)} = n$. Thus, n is a prime number.

On the other hand, assume that n is a prime number and that $\gcd(k, 2n) = 1$. Note that $\mathrm{o}(x^k) = 2$ or n for all $x \in D_n \setminus \{e\}$. Let $i \in \{1, 2, \dots, n-1\}$. Note that $\mathrm{o}(a^i) = \frac{\mathrm{o}(a)}{\gcd(i,\mathrm{o}(a))} = \frac{n}{\gcd(i,n)} = n$.

Hence, $o\left((a^i)^k\right) = \frac{o(a^i)}{\gcd(k,o(a^i))} = \frac{n}{\gcd(k,n)} = n$. We know that $o(b) = 2 = o(a^ib)$ in D_n. Consider $o(b^k) = \frac{2}{\gcd(k,2)} = o((a^ib)^k)$. Since $\gcd(k,2) = 1$, we have $o(b^k) = 2 = o((a^ib)^k)$. Thus, $o(x^k)$ is a prime number for all $x \in D_n \setminus \{e\}$. Therefore, $\mathrm{OD}^k(D_n)$ is a star graph by Theorem 3.8. □

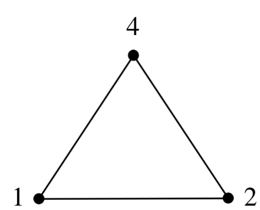
4. Generalized order divisor graphs of finite cyclic groups

In this section, we focus on generalized order divisor graphs of cyclic groups. We first introduce divisor graphs, extended graphs, and reduced graphs.

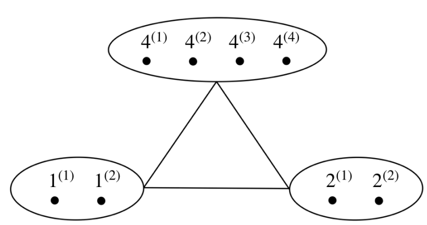
Let m be a positive integer. The *divisor graph* of m, denoted $\mathcal{D}(m)$, is the graph whose vertex set comprises the set of positive divisors of m and in which two vertices d_i and d_j are adjacent if and only if $d_i \neq d_j$ and either $d_i \mid d_j$ or $d_j \mid d_i$. Some properties of this divisor graph are discussed in [9]. Divisor graphs are moreover comparability graphs defined on partially ordered relations (see [15]). Here, it is a divisible relation. Let r be a positive integer. The *r-extended graph* of $\mathcal{D}(m)$, denoted $\mathcal{E}_r(\mathcal{D}(m))$, is the graph obtained by replacing each vertex d of $\mathcal{D}_m$ by $r\phi(d)$ copies of d such that no two copied vertices are adjacent. If $d_1, d_2, \ldots, d_t$ are the divisors of m, then $V\left(\mathcal{E}_r(\mathcal{D}(m))\right) = S_{d_1} \cup S_{d_2} \cup \cdots \cup S_{d_t}$ where $S_{d_i} = \{d_i^{(\ell)} : \ell \in \{1, 2, \ldots, r\phi(d_i)\}\}$, and $d_i^{(\ell)}$ is adjacent to $d_j^{(\ell')}$ in $\mathcal{E}_r(\mathcal{D}(m))$ if and only if $d_i \neq d_j$ and either $d_i \mid d_j$ or $d_j \mid d_i$.

Finally, we define the *reduced graph* of $\mathrm{OD}^k(G)$, denoted $\mathcal{R}(\mathrm{OD}^k(G))$, to be the graph obtained by merging all vertices a in $\mathrm{OD}^k(G)$ such that a^k has the same order.

Example 4.1. *We give examples of a divisor graph and its extended graph in Figure 1 and examples of a generalized order divisor graph and its reduced graph in Figure 2*]. Note that, in Figure 1b of the graph $\mathcal{E}_2(\mathcal{D}(4))$ and Figure 2a of the graph $\mathrm{OD}^2(\mathbb{Z}_8)$, a line joining two independent sets of vertices shows that each vertex in one independent set is adjacent to all vertices in the other independent set.

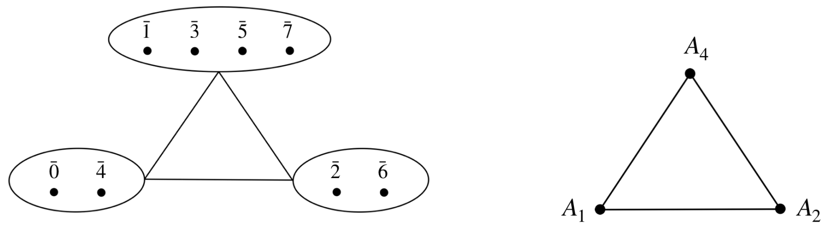


(A) $\mathcal{D}(4)$



(B) $\mathcal{E}(\mathcal{D}(4))$

FIGURE 1. The divisor graph $\mathcal{D}(4)$ and its extended graph $\mathcal{E}_2(\mathcal{D}(4))$



(A) $\mathrm{OD}^2(\mathbb{Z}_8)$

(B) $\mathcal{R}(\mathrm{OD}^2(\mathbb{Z}_8))$

FIGURE 2. The graph $\mathrm{OD}^2(\mathbb{Z}_8)$ and its reduced graph $\mathcal{R}(\mathrm{OD}^2(\mathbb{Z}_8))$

Note that in $\mathcal{R}(\mathrm{OD}^2(\mathbb{Z}_8))$, the vertex A_1 is obtained by merging $\bar{0}$ and $\bar{4}$, A_2 is obtained by merging $\bar{2}$ and $\bar{6}$, and A_4 is obtained by merging $\bar{1}, \bar{3}, \bar{5}$, and $\bar{7}$.

We shall show some nice relationships between the divisor graphs with their extended graphs and the generalized order divisor graphs of finite cyclic groups with their reduced graphs.

We recall that for any element a in a finite group G of order n, $\mathrm{o}(a)$ divides n. One can directly show that $\mathrm{o}(a^k) = \frac{\mathrm{o}(a)}{\gcd(k,\mathrm{o}(a))}$ divides $\frac{n}{\gcd(k,n)}$ for all positive integers k. We first enumerate the number of elements a in G such that $\mathrm{o}(a^k) = d$.

Lemma 4.2. *Let G be a cyclic group of order n and d, k positive integers such that d is a divisor of $\frac{n}{\gcd(k,n)}$. Then the number of elements a in G such that $\mathrm{o}(a^k) = d$ is $\gcd(k,n)\phi(d)$.*

Proof. Let $\varphi\colon G \to G$ be the function $\varphi(a) = a^k$ which is a homomorphism. Then $\ker(\varphi) = \{a \in G : a^k = e\} = \{a \in G\colon a^{\gcd(k,n)} = e\}$. Hence, $|\ker(\varphi)| = \gcd(k,n)$. Also, $\varphi(a) = \varphi(b)$ if and only if $a^k = b^k$ if and only if $(ba^{-1})^k = e$ if and only if $b \in a\ker(\varphi)$. Thus, $\varphi^{-1}(\varphi(a)) = a\ker(\varphi)$. So, φ is a $\gcd(k,n)$ to one function. Also, $\mathrm{Im}(\varphi)$ is a cyclic group of order $\dfrac{n}{\gcd(k,n)}$. Then it has $\phi(d)$ elements of order d. Set $|X| = \{a \in G : \mathrm{o}(a^k) = d\}$. Therefore, $|X| = \gcd(k,n)|\varphi(X)| = \gcd(k,n)\phi(d)$. □

We are now ready to prove the following theorem for the reduction and extension of generalized order divisor graphs of finite cyclic groups.

Theorem 4.3. *Let G be a cyclic group of order n and k be a positive integer. Then*

(1) $\mathcal{R}(\mathrm{OD}^k(G)) \cong \mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$.

(2) $\mathcal{E}_{\gcd(k,n)}\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right) \cong \mathrm{OD}^k(G)$.

Proof. Let $d_1, d_2, \ldots, d_t = \frac{n}{\gcd(k,n)}$ be all distinct positive divisors of $\frac{n}{\gcd(k,n)}$. For each $i \in \{1, 2, \ldots, t\}$, we let $A_{d_i} = \left\{a \in G : \mathrm{o}(a^k) = d_i\right\}$. By Lemma 4.2, $|A_{d_i}| = \gcd(k,n)\phi(d_i) \geq 1$ for all i. Then $V(\mathcal{R}(\mathrm{OD}(G))) = \{A_{d_1}, A_{d_2}, \ldots, A_{d_t}\}$ and $|V(\mathcal{R}(\mathrm{OD}(G)))| = \left|V\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right)\right|$. Hence, a vertex A_{d_i}

in $\mathcal{R}(\mathrm{OD}^k(G))$ is 1-1 corresponding to a vertex d_i in $\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$ for all $i \in \{1, 2, \ldots, t\}$. Let d_i and d_j be two vertices in $\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$ and let A_{d_i} and A_{d_j} be their corresponding vertices in $\mathcal{R}(\mathrm{OD}^k(G))$. Thus,

$$\begin{aligned} A_{d_i} \text{ is adjacent to } A_{d_j} \text{ in } \mathcal{R}(\mathrm{OD}^k(G)) &\Longleftrightarrow d_i \neq d_j \text{ and either } d_i \mid d_j \text{ or } d_j \mid d_i \\ &\Longleftrightarrow d_i \text{ is adjacent to } d_j \text{ in } \mathcal{D}\left(\frac{n}{\gcd(k,n)}\right). \end{aligned}$$

Therefore, $\mathcal{R}(\mathrm{OD}^k(G)) \cong \mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$. This proves (1).

To prove (2), we let $S_{d_i} = \left\{d_i^{(\ell)} : \ell \in \{1, 2, \ldots, \gcd(k,n)\phi(d_i)\}\right\}$ be the set of $\gcd(k,n)\phi(d_i)$ copies of d_i for each $i \in \{1, 2, \ldots, t\}$. Then the vertex set of the extended graph is

$$V\left(\mathcal{E}_{\gcd(k,n)}\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right)\right) = S_{d_1} \cup S_{d_2} \cup \cdots \cup S_{d_t}.$$

Note that $V(\mathrm{OD}^k(G)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_t}$. Since $|S_{d_i}| = \gcd(k,n)\phi(d_i) = |A_{d_i}|$ for all i, we have $\left|V\left(\mathcal{E}_{\gcd(k,n)}\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right)\right)\right| = \left|V(\mathrm{OD}^k(G))\right|$. There is also a 1-1 correspondence between vertices of $\mathrm{OD}^k(G)$ in A_{d_i} and vertices of the extended graph in S_{d_i}. Let $d_i^{(\ell)} \in S_{d_i}$ and $d_j^{(\ell')} \in S_{d_j}$ be two vertices in the extended graph and let $a \in A_{d_i}$ and $b \in A_{d_j}$ be their corresponding vertices in $\mathrm{OD}^k(G)$. Thus,

$$\begin{aligned} & d_i^{(\ell)} \text{ is adjacent to } d_j^{(\ell')} \text{ in } \mathcal{E}_{\gcd(k,n)}\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right) \\ &\Longleftrightarrow d_i \neq d_j \text{ and either } d_i \mid d_j \text{ or } d_j \mid d_i \\ &\Longleftrightarrow \mathrm{o}(a^k) \neq \mathrm{o}(b^k) \text{ and either } \mathrm{o}(a^k) \mid \mathrm{o}(b^k) \text{ or } \mathrm{o}(b^k) \mid \mathrm{o}(a^k) \\ &\Longleftrightarrow a \text{ is adjacent to } b \text{ in } \mathrm{OD}^k(G). \end{aligned}$$

Therefore, $\mathcal{E}_{\gcd(k,n)}\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right) \cong \mathrm{OD}^k(G)$. □

Example 4.4. *Figures 3 and 4 show relationships among the generalized order divisor graph* $\mathrm{OD}^2(\mathbb{Z}_8))$*, the divisor graph* $\mathcal{D}\left(\frac{8}{\gcd(2,8)}\right) = \mathcal{D}(4)$*, the reduced graph* $\mathcal{R}(\mathrm{OD}^2(\mathbb{Z}_8))$ *and the extended graph* $\mathcal{E}_{\gcd(2,8)}\left(\mathcal{D}\left(\frac{8}{\gcd(2,8)}\right)\right) = \mathcal{E}_{\gcd(2,8)}(\mathcal{D}(4))$.

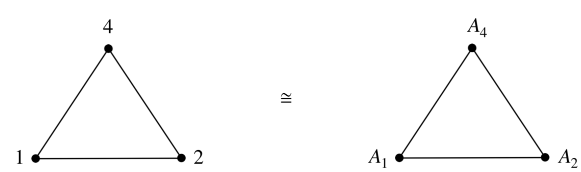


FIGURE 3. $\mathcal{D}\left(\frac{8}{\gcd(2,8)}\right) \cong \mathcal{R}(\mathrm{OD}^2(\mathbb{Z}_8))$

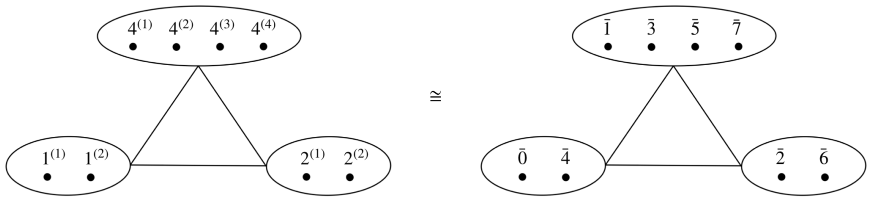


FIGURE 4. $\mathcal{E}_{\gcd(2,8)}\left(\mathcal{D}\left(\frac{8}{\gcd(2,8)}\right)\right) \cong \mathrm{OD}^2(\mathbb{Z}_8)$

Remark 4.5. (1) *From* $\mathcal{R}(\mathrm{OD}^k(G)) \cong \mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$, *a vertex* d *in* $\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$ *corresponds to the vertex* $A_d = \{a \in V(\mathrm{OD}^k(G)) : \mathrm{o}(a^k) = d\}$ *in the reduced graph* $\mathcal{R}(\mathrm{OD}^k(G))$.

(2) *From* $\mathcal{E}_{\gcd(k,n)}\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right) \cong \mathrm{OD}^k(G)$, *each vertex* d *of* $\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$ *is extended to an independent set* S_d *of* $\gcd(k,n)\phi(d)$ *vertices in the extended graph, and this independent set* S_d *corresponds to the independent set* $\{a \in V(\mathrm{OD}^k(G)) : \mathrm{o}(a^k) = d\}$.

Theorem 4.3 is a useful mechanism for investigating generalized order divisor graphs of cyclic groups. We see that the reduced graph of $\mathrm{OD}^k(G)$ is isomorphic to the divisor graph. Some results from the divisor graph can therefore be applied to the graph $\mathrm{OD}^k(G)$. For instance, the following lemma can be applied to generalized order divisor graphs of finite cyclic groups.

Lemma 4.6. [9] *Let* m *be a positive integer. Then the divisor graph* $D(m)$ *is a complete graph if and only if any two vertices* $d_i \neq 1$ *and* $d_j \neq 1$ *in* $D(m)$ *are not relatively prime.*

Corollary 4.7. *Let* G *be a cyclic group of order* n *and suppose that* $d_1 = 1, d_2, \ldots, d_t$ *are all distinct positive divisors of* $\frac{n}{\gcd(k,n)}$. *Then* $\mathrm{OD}^k(G)$ *is a complete* t*-partite graph*

$$K_{\gcd(k,n)\phi(d_1),\gcd(k,n)\phi(d_2),\ldots,\gcd(k,n)\phi(d_t)}$$

if and only if $\gcd(d_i, d_j) \neq 1$ *for all* $i, j \in \{2, 3, \ldots, t\}$.

Proof. Assume that $\mathrm{OD}^k(G)$ is a complete t-partite graph

$$K_{\gcd(k,n)\phi(d_1),\gcd(k,n)\phi(d_2),\ldots,\gcd(k,n)\phi(d_t)}.$$

Then the reduced graph $\mathcal{R}(\mathrm{OD}^k(G))$ which is isomorphic to the divisor graph $\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$ is a complete graph K_t. By Lemma 4.6, $\gcd(d_i, d_j) \neq 1$ for all $i, j \in \{2, 3, \ldots, t\}$.

On the other hand, assume that $\gcd(d_i, d_j) \neq 1$ for all $i, j \in \{2, 3, \ldots, t\}$. Then the divisor graph $\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)$ is a complete graph K_t by Lemma 4.6. Thus, the extended graph $\mathcal{E}_{\gcd(k,n)}\left(\mathcal{D}\left(\frac{n}{\gcd(k,n)}\right)\right)$ is a complete t-partite graph

$$K_{\gcd(k,n)\phi(d_1),\gcd(k,n)\phi(d_2),\ldots,\gcd(k,n)\phi(d_t)}.$$

By Theorem 4.3 (2), therefore, $\mathrm{OD}^k(G)$ is also a complete t-partite graph. □

http://dx.doi.org/10.22108/IJGT.2022.133091.1787

Finally, we show some applications of Corollary 4.7.

Proposition 4.8. *Let G be a cyclic group of order $n = p^m$ where p is a prime number and k is a positive integer such that $\gcd(k, n) = p^t$ for some $0 \le t \le m$. Then $\mathrm{OD}^k(G)$ is a complete $(m-t)$-partite graph*

$$K_{p^t, p^t(p-1), p^{t+1}(p-1), p^{t+2}(p-1), \ldots, p^{m-1}(p-1)}.$$

Moreover, if $a \in G$ such that $\mathrm{o}(a^k) = p^s$ where $0 \le s \le m-t$, then the degree of a is

$$\deg(a) = \begin{cases} p^m - p^t & \text{if } s = 0 \\ p^m - p^{s+t} + p^{s+t-1} & \text{if } s \neq 0. \end{cases}$$

Proof. Since no two divisors other than 1 of $\frac{n}{\gcd(k,n)} = p^{m-t}$ are relatively prime, Corollary 4.7 implies that $\mathrm{OD}^k(G)$ is a complete $(m-t)$-partite graph

$$K_{p^t\phi(1), p^t\phi(p), p^t\phi(p^2), p^t\phi(p^3), \ldots, p^t\phi(p^{m-t})} = K_{p^t, p^t(p-1), p^{t+1}(p-1), p^{t+2}(p-1), \ldots, p^{m-1}(p-1)}.$$

Next, let $a \in G$ be such that $\mathrm{o}(a^k) = p^s$ where $0 \le s \le m-t$. Then a is in a partite set of $p^t\phi(p^s)$ vertices. If $s = 0$, then the degree of a is $p^t\phi(p) + p^t\phi(p^2) + \cdots + p^t\phi(p^{m-t}) = p^m - p^t$. If $s \neq 0$, the degree of a is $p^t + p^t\phi(p) + p^t\phi(p^2) + \cdots + p^{t+s-2}(p-1) + p^{t+s}(p-1) + \cdots + p^t\phi(p^{m-t}) = p^m - p^{s-t} + p^{s+t-1}$. □

Proposition 4.9. *Let G be a cyclic group of order pq where p, q are distinct prime numbers. The following statements hold.*

(1) *If $p \nmid k$ and $q \nmid k$, then*

$$\mathrm{OD}^k(G) \cong \big((p-1)K_1 \diamond (p-1)(q-1)K_1 \diamond (q-1)K_1\big) \diamond K_1.$$

(2) *If $p \mid k$ but $q \nmid k$, then $\mathrm{OD}^k(G) \cong K_{p,p(q-1)}$.*

(3) *If $p \nmid k$ but $q \mid k$, then $\mathrm{OD}^k(G) \cong K_{q,q(p-1)}$.*

(4) *If $p \mid k$ but $q \mid k$, then $\mathrm{OD}^k(G)$ has no edges.*

Proof.

1. Suppose $p \nmid k$ and $q \nmid k$. Then $\gcd(k, pq) = 1 = \gcd(1, pq)$. By Theorem 3.4, $\mathrm{OD}^k(G) = \mathrm{OD}^1(G)$. Thus, the required result follows directly from [14, Theorem 15].
2. Assume that $p \mid k$ but $q \nmid k$. Then $\gcd(k, pq) = p$. Hence, all positive divisors of $\frac{pq}{\gcd(k,pq)} = q$ are 1 and q. By Corollary 4.7, $\mathrm{OD}^k(G)$ is a complete 2-partite graph $K_{p,p(q-1)}$.
3. The proof is similar to (2).
4. Assume that $p \mid k$ and $q \mid k$. Then $pq \mid k$. Hence, $\mathrm{o}(a^k) = 1$ for all $a \in G$. By Theorem 3.2 (1), $\mathrm{OD}^k(G)$ has no edges.

□

Acknowledgments

The authors gratefully acknowledge the financial support provided by Faculty of Science and Technology, Contract No. SciGR 18/2565. They would like to thank the referees for the valuable and helpful comments which have significantly improved the paper. They also thank Mr. John Winward for editorial suggestions.

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Parinyawat Choosuwan

Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), 12110, Pathum Thani, Thailand

Email: parinyawat_c@rmutt.ac.th

Pannarin Sangsawang

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, 12120, Pathum Thani, Thailand

Email: ssw.pannarin@gmail.com

Cheevarat Matwangsang

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, 12120, Pathum Thani, Thailand

Email: cheevarat.mws@gmail.com

Sireethon Thongsupol

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, 12120, Pathum Thani, Thailand

Email: sireethon.tho@gmail.com

Siripong Sirisuk

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, 12120, Pathum Thani, Thailand

Email: siripong@mathstat.sci.tu.ac.th, siripong.srs@gmail.com

http://dx.doi.org/10.22108/IJGT.2022.133091.1787