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COUNTEREXAMPLES TO A CONJECTURE ON MATCHING KNESER GRAPHS

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Abstract. Let G be a graph and $r \in \mathbb{N}$. The matching Kneser graph $\mathsf{KG}(G, rK_2)$ is a graph whose vertex set is the set of r-matchings in G and two vertices are adjacent if their corresponding matchings are edge-disjoint. In [M. Alishahi and H. Hajiabolhassan, On the Chromatic Number of Matching Kneser Graphs, *Combin. Probab. and Comput.*, **29**, No. 1 (2020), 1–21] it was conjectured that for any connected graph G and positive integer $r \geq 2$, the chromatic number of $\mathsf{KG}(G, rK_2)$ is equal to $|E(G)| - \mathsf{ex}(G, rK_2)$, where $\mathsf{ex}(G, rK_2)$ denotes the largest number of edges in G avoiding a matching of size r. In this note, we show that the conjecture is not true for snarks.

Given two positive integers n and r, the Kneser graph $\mathsf{KG}(n, r)$, is the graph whose vertices represent the r-subsets of $\{1, \ldots, n\}$, and where two vertices are adjacent if and only if they correspond to disjoint subsets. This graph introduced by Lovász in 1978 [3]. The matching Kneser graph was introduced as a generalization of Kneser graph [1]. Precisely, $\mathsf{KG}(nK_2, rK_2) \cong \mathsf{KG}(n, r)$. Also, by a simple greedy coloring, it was proved that $\chi(\mathsf{KG}(G, rK_2)) \leq |E(G)| - \mathsf{ex}(G, rK_2)$ for any graph G and positive integer r. This upper bound is tight when G satisfies some certain properties as is claimed in the main result of [1] and is shown throughout the paper. Finally the following conjecture was introduced according to the results of the paper.

Conjecture 0.1. *For any connected graph G and positive integer $r \geq 2$,*

$$\chi(\mathsf{KG}(G, rK_2)) = |E(G)| - \mathsf{ex}(G, rK_2).$$

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We show that the conjecture is false when G is a snark. A snark is a simple, connected, bridgeless cubic graph with chromatic index equal to four [2]. Let G be a snark of order $2r$. Petersen's Theorem from 1891 [4], states that every bridgeless cubic graph has a perfect matching. Further, Schönberger in 1934 has proved that every bridgeless cubic graph has a perfect matching not containing two arbitrarily prescribed edges [5]. Therefore, $\mathsf{ex}(G, rK_2) \leq |E(G)| - 3$. In addition, by removing three edges incident with an arbitrary vertex, the resulting graph has no matching of size r. Therefore $\mathsf{ex}(G, rK_2) = |E(G)| - 3$. In addition, since $\chi'(G) = 4$, the intersection of any two perfect matchings of G is nonempty. Therefore, $\mathsf{KG}(G, rK_2)$ has no edge and so $\chi(\mathsf{KG}(G, rK_2)) = 1 \neq |E(G)| - \mathsf{ex}(G, rK_2) = 3$.

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