International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 13 No. 1 (2024), pp. 1-15.

https://ijgt.ui.ac.ir
www.ui.ac.ir

CONSTRUCTIONS AND INVOLUTORY PROPERTIES IN LATIN QUANDLES

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ABSTRACT. This work studied involutory properties in Latin quandles using methods of quasiqroup theory, and classified latin quandle Q into Left Involutory Property latin Quandle (LIPQ), Right Involutory Property Latin Quandle (RIPQ) and Involutory Property Latin Quandle (IPQ). It investigated a fourth property called the Cross Involutory Property Latin Quandle (CIPQ). The result showed that a latin quandle Q that is a LIPQ and RIPQ is an IPQ. Moreover, it established that the necessary and sufficient conditions for a latin Alexander quandle Q to be a CIPQ is that $b = t^2a + (1-t)(tb+a)$ for all $a, b \in Q$ and $t \in A(Q)$.

1. Introduction

The mappings L_a and R_a on a magma $(Q, \cdot)$ such that $L_a : Q \to Q$ and $R_a : Q \to Q$ defined as $L_a(x) = a \cdot x$ and $R_a(x) = x \cdot a$ for all $x \in Q$ are called left and right translations respectively. For a finite Q, it is possible to evaluate the inverses $L_a^{-1}(x) = a\backslash x$ and $R_a^{-1}(x) = x/a$. These maps will have a lot of work to do in this paper. A magma $(Q, \cdot)$ is called a left quasigroup if, for all $a, b \in Q$, there exists a unique solution $x \in Q$ to the equation $a \cdot x = b$. That is, any left translation is a bijection, and this implies that every row of its multiplication table is a permutation on Q. Moreover, it is called a right quasigroup if, for all $a, b \in Q$, there exists a unique solution $y \in Q$ to the equation $y \cdot a = b$, which implies also that any right translation is a bijection on Q or that every column of Q is a permutation whenever Q is finite and is expressed in a multiplication table. Therefore, a quasigroup is a left and right quasigroup. Thus, the left and right translations are bijections, and for a finite Q

Keywords: Latin quandles, LIPQ, RIPQ, CIPQ, IPQ.
MSC(2010): Primary: 20N05; Secondary: 57M27.
Communicated by Attila Maroti.
Article Type: Research Paper.
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Received: 21 February 2022, Accepted: 13 July 2022.
Cite this article: A. O. Isere and J. U. Ezurike, Constructions and involutory properties in latin quandles, Int. J. Group Theory, **13** no. 1 (2024) 1–15. http://dx.doi.org/10.22108/IJGT.2022.132799.1780 .

the rows and columns of its multiplication table are permutations. From a universal algebra point of view, a quasigroup $(Q, \star, \backslash, /)$ is a set Q equipped with three binary operations, namely multiplication, left division and right division respectively, such that the following identities hold:

(1) $(x/y) \star y = x$
(2) $(x \star y)/y = x$
(3) $x \star (x \backslash y) = y$
(4) $x \backslash (x \star y) = y$

for all $x, y \in Q$. If there is an identity element $e \in Q$ such that $a \star e = a = e \star a$ for all $a \in Q$, then the quasigroup is called a loop. Quasigroups and loops are generally not associative. Instead, associative identity is replaced by weak forms of associativity which largely determine the various types of loops. For detail study of quasigroups and loops, we make recourse to the following references: [5, 6, 7, 8, 9, 12, 15, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44]. In another development, associative identity is replaced by self-distributivity. In this case, the quasigroup is called self - distributive quasigroup or a latin quandle in our modern terminology. In this paper, we shall be focussing on latin quandles which are a special class of quandles. Quandles are generally right quasigroups, that is, they are only right invertible which implies that every column of a finite quandle is a permutation. Therefore, a unique left divisibility is imposible for right quandles, hence, quandles are not division algebras. For detail study of quandle theory (see [1, 2, 3, 4, 14, 16, 18, 23, 26, 27, 28, 29, 36, 39, 45]). On the other hand, latin quandles are division algebras. These are quandles whose multiplication tables are latin squares [8, 17]. They are also left and right quasigroups. Thus, one can study latin quandles either by quasigroup theory or by quandle theory [13]. In this work, we shall be studying latin quandles based on involutory properties, using quasigroup theory, and a specific example of each property shall be given.

The remaining part of this paper is organized this way: Section 2 covers the preliminary information in which some definitons and basic results in literature that are relevant to this work will be discussed. Then sections 3 and 4 present the main results and the conclusion respectively.

2. Preliminary

Definition 2.1. [35] *A quandle is a set X with a binary operation $(a, b) \mapsto a \triangleright b$ such that*

(1): *For any $a \in X$, $a \triangleright a = a$*
(2): *For any $a, b \in X$, there is a unique $x \in X$ such that $a = x \triangleright b$*
(3): *For any $a, b, c \in X$, $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$*

Definition 2.2. [11, 35] *A quandle $(X, \triangleright)$ is commutative if it satisfies the identity*

$$x \triangleright y = y \triangleright x \ \forall \ x, y \in X.$$

Any set X with the operation $x \triangleright y = x$ for any $x, y \in X$ is a quandle called the trivial quandle. The trivial quandle of n elements is denoted by T_n.

http://dx.doi.org/10.22108/IJGT.2022.132799.1780

Definition 2.3. [26] *An abelian quandle is a quandle satisfying the identity:*

$$(w \triangleright x) \triangleright (y \triangleright z) = (w \triangleright y) \triangleright (x \triangleright z)$$

Remark 2.4. *An abelian quandle is also called a medial quandle by some authors.*

Definition 2.5. [26] *An involutory quandle is a quandle which satisfies:*

$$(x \triangleright y) \triangleright y = x$$

Remark 2.6. *An involutory quandle is also called Kei, particularly if the right translations are involutions.* [10]

Definition 2.7. [17] *An algebraic structure $(Q, \triangleright)$ is called a latin quandle if it obeys the following laws simultaneously:*

- **(i):** $x \triangleright x = x$ *[idempotent law]*
- **(ii):** $a \triangleright x = b$ *[left division law]*
- **(iii):** $y \triangleright a = b$ *[right division law]*
- **(iv):** $a \triangleright (x \triangleright y) = (a \triangleright x) \triangleright (a \triangleright y)$ *[left distributive law]*
- **(v):** $(x \triangleright y) \triangleright a = (x \triangleright a) \triangleright (y \triangleright a)$ *[right distributive law]*

If in addition, the structure obeys

- **(vi):** $x \triangleright (x \triangleright y) = y$ *[left involutory law]*

Then the latin quandle shall be called left involutory latin quandle.

A latin quandle that obeys Definition 2.5 is called a right involutory latin quandle.

Definition 2.8. [39] *A latin quandle Q is called a Symmetric Latin Quandle (SLQ) if $xa = ax$ and $x(xa) = a$ for all $a, x \in Q$.*

Remark 2.9. *All symmetric latin quandles are commutative except for the converse.*

Joyce reported a class of quandles in which the symmetries $S(y)$ are all involutions [42]. Such quandle is a symmetric quandle. For this class of quandles the two involutions coincide. We shall discuss more on this class of quandles in the next section.

Definition 2.10. [37] *A groupoid Q is called Jordan if it obeys $((xx)y)x = (xx)(yx)$ for all $x, y \in Q$.*

Remark 2.11. *The Jordan identity is simply another form of flexibility identity. Therefore, all latin quandles obey the Jordan identity*

Definition 2.12. [35] *Given two quandles $(X, \star)$ and $(Y, \bullet)$, a map $f : (X, \star) \to (Y, \bullet)$ is a quandle homomorphism if*

$$f(a \star b) = f(a) \bullet f(b) \ \forall \ a, b \in X$$

If f is a bijection then f is called an isomorphism, and $(X, \star)$ and $(Y, \bullet)$ are said to be isomorphic quandles.

http://dx.doi.org/10.22108/IJGT.2022.132799.1780

Definition 2.13. [35, 29] *The automorphism group of a quandle $(X, \star)$ denoted as $A\ (X)$, is the group of all isomorphisms $\rho : X \to X$. The elements of $A(X)$ act on those of X by right action.*

Definition 2.14. [38] *An abelian group Q with a quandle operation $a \star b = ta + (1-t)b$ for all $a, b \in Q$ and $t \in A(Q)$, where $A(Q)$ denote the automorphism group of Q is called an Alexander quandle.*

The trivial quandle T_n is an Alexander quandle if $t = 1$. Alexander quandles are very useful in knot theory.

Proposition 2.15. [38] *Consider the following:*

(1) *Alexander quandles are abelian.*
(2) *Alexander quandles are left-distributive*
(3) *If M is an Alexander quandle, then for all $a, b \in M$ we have*

$$a \triangleright b + b \triangleright a = a + b$$

The above results were helpful in proving some results in section 3.

Definition 2.16. [41]

(1) *A quasigroup $(Q, \circ)$ has the Left Inverse Property (LIP) if there exists a permutation λ of the set Q such that*

$$\lambda x \circ (x \circ y) = y$$

for all $x, y \in Q$ (By Belousov).

(2) *A quasigroup $(Q, \circ)$ has the Right Inverse Property (RIP) if there exists a permutation ρ of the set Q such that*

$$(x \circ y) \circ \rho y = x$$

for all $x, y \in Q$ (By Belousov).

(3) *A quasigroup $(Q, \circ)$ has the Inverse Property (IP) if it is a LIP and RIP-quasigroup (By Belousov).*

(4) *A quasigroup $(Q, \circ)$ has the Cross Inverse Property (CIP) if there exists a permutation J of the set Q such that*

$$(x \circ y) \circ Jx = y$$

for all $x, y \in Q$ (First by Artzy and later by Keedwell and Shcherbacov) [31, 41].

Motivated by this definition and also by the desire to investigate the application of latin quandles in cryptography Isere et al [17] introduced these inverse properties in latin quandles and stated that these properties induce involutions in latin quandles. This paper is taking a closer look at these induced involutory properties in latin quandles. The definition given in [17] is presented below.

Definition 2.17. [17] *A latin quandle $(Q, \triangleright)$ that obeys properties:*

(1): *$x \triangleright (x \triangleright y) = y$ Left Inverse Property (LIP) is called a LIPQ.*

(2): $(y \triangleright x) \triangleright x = y$ *Right Inverse Property (RIP) is called a RIPQ.*

(3): *(1) and (2) Inverse Property (IP) is called an IPQ.*

(4): $x \triangleright y \triangleright x = y$ *or* $y = x \triangleright y \triangleright x$ *Cross Inverse Property (CIP) is called a CIPQ.*

The authors observed that these inverse-properties in latin quandles are involutions. Rightfully, involutions are involutory mappings. That is, mappings that are their own inverses. In other words, involutions are self-invertible functions. For example, $L_xL_x(y) = I(y) = y = x(xy)$ is the left involution and $R_yR_y(x) = I(x) = x = (xy)y$ is the right involution, where I is identity mapping. Thus, a latin quandle Q with the left involution-property is called a LIPQ. Appropriately, RIPQ is the latin quandle Q with the right involution-property. Using Definition 2.16 as a microscope, one can perceive that λ, ρ and J are identity permutations when compare to Definition 2.17. That is $\lambda_x(x) = x$ etc. Therefore, we shall reintroduce these properties in the next section. The concepts of inverse property quasigroups and loops and the acronyms LIPQ, RIPQ, CIPQ and IPQ are well known in literature in connection with quasigroups (loops) Q. However, it is a novelty in quandle theory. For detail study of quasigroups and loops see the following references: [15, 19, 20, 21, 22, 24, 25, 40]

Definition 2.18. [5] *Let* $(G, \cdot)$ *be a group, or more generally, a Bol loop. The binary algebra* $(G, \star)$*, with*

$$x \star y = xy^{-1}x \ \forall \ x, y \in G.$$

is an involutory quandle called the core of $(G, \cdot)$*.*

The core operation was first introduced by Bruck[7]. He had earlier shown that the core of a Moufang loop, originally defined as

$$x + y = yx^{-1}y$$

is an involutory (Kei) quandle. It is to be noted that these constructed examples were not latin quandles. Following these methods, the example below gives a construction of a latin quandle.

Example 2.1. [17] *Let* $(G, \cdot)$ *be a uniquely 2-divisible group or generally, a CIP Osborn loop of order* n *such that*

$$x + y = xy^{-1} \cdot x \ \forall \ x, y \in G.$$

Then, $(G, +)$ *is an involutory latin quandle of order* n*.*

3. Main Results

Definition 3.1. *A latin quandle* $(Q, \triangleright)$ *that obeys properties:*

(1): $x \triangleright (x \triangleright y) = y$ *Left Involutory Property (LIP) is called a LIPQ.*

(2): $(y \triangleright x) \triangleright x = y$ *Right Involutory Property (RIP) is called a RIPQ.*

(3): *(1) and (2) Involutory Property (IP) is called an IPQ.*

(4): $x \triangleright (y \triangleright x) = y$ *or* $y = (x \triangleright y) \triangleright x$ *Cross Involutory Property (CIP) is called a CIPQ.*

For all $x, y \in Q$*.*

Remark 3.2. *Every latin quandle obeys weak form of associativity: $x(yx) = (xy)x$. This property is called flexibility. Latin quandles are flexible algebra.*

Lemma 3.3. *Consider a finite latin quandle Q:*

(1) *Let $Q(\triangleright)$ be a LIPQ and L_a a left translation on Q, then for all $x \in Q$*
  (i): $L_a(L_a(x)) = x$
  (ii): *L_a is an automorphism such that $L_a : Q \to Q$*

(2) *Let $Q(\triangleright)$ be a RIPQ and R_a a right translation on Q, then for all $x \in Q$*
  (i): $R_a(R_a(x)) = x$
  (ii): *R_a is an automorphism such that $R_a : Q \to Q$*

(3) *Let $Q(\triangleright)$ be a CIPQ and L_x and R_x are left and right translations on Q, then for all $y \in Q$*
  (i): $L_x(R_x(y)) = y = R_x(L_x(y))$
  (ii): *$L_x(R_x)[R_x(L_x)]$ is an automorphism such that $L_x(R_x) : Q \to Q[R_x(L_x) : Q \to Q]$*

Proof. (1) **(i):** $L_a(L_a(x)) = a \triangleright (a \triangleright x) = x$ since Q is a LIPQ.
  (ii): $L_a(x \triangleright y) = (a \triangleright x) \triangleright (a \triangleright y) = L_a(x) \triangleright L_a(y)$ for all $x, y \in Q$ and since $L_a(Q)$ is a permutation, then, $L_a : Q \to Q$ is an automorphism.

(2) The proof of this is similar to 1. above.

(3) **(i):** $L_x(R_x(y)) = x \triangleright (y \triangleright x) = y = (x \triangleright y) \triangleright x = R_x(L_x(y))$ since Q is a CIPQ.
  (ii): Since L_x and R_x are automorphisms (by 1. and 2. above) then their composition is again an autormorphism.

□

Remark 3.4. *Lemma 3.3 above shows that LIPQs, RIPQs and CIPQs are classically different latin quandles.*

Lemma 3.5. *Let Q be a finite IPQ, then for all $x \in Q$ the following properties hold and can easily be verified.*

(1) $R_x = L_x(R_x^{-1} = L_x^{-1})$
(2) $R_xR_x = L_xL_x$
(3) $L_xL_x^{-1} = L_x^{-1}L_x$
(4) $R_x^{-1}R_x = L_x^{-1}L_x$
(5) $R_xR_x^{-1} = L_xL_x^{-1}$
(6) $L_xL_x^{-1} = R_xL_x^{-1}$

The proofs of the above properties hold by symmetry, and the properties show that IPQs are symmetric. An IPQ is a reflection of itself. It is completely self-involutory. It is the only quandle that is an involutory quandle in the true sense of it where the left and right translations on Q are involutions and coincide. These properties are rare when compared with other latin quandles. Behold an IPQ comprises of the properties of LIPQ, RIPQ and CIPQ, which suggests that a CIPQ possesses another kind of involution that is worthy of investigation. Hence, this paper is investigating this class of involution. The CIPQs are special involutory quandles where the left translation is the inverse of the right translation and vice versa. This operation induces a cross inverse property action on the elements.

Lemma 3.6. *A latin quandle $(Q, \cdot)$ is symmetric if and only if it is an IPQ.*

Proof. Q is symmetric means that, it is $ax = xa$ and $x(xa) = a$ for all $a, x \in Q \Rightarrow Q$ is a commutative LIPQ. Then, $x(xa) = (xa)x = (ax)x$ (a RIPQ). Thus, Q is an IPQ.

Conversely, Q an IPQ implies that $x(xa) = (ax)x$. Thus, Q is commutative by left and right cancelation laws. □

Theorem 3.7. *Let $Q(\cdot)$ be a cyclic group of odd order n such that*

$$x + y = (y^{-1}x)x, \forall x, y \in Q$$

Then, $(Q, +)$ is a left involutory property latin quandle (LIPQ) of odd order n.

Proof. We need to verify that Definition 2.7 is satisfied. Obviously Q is a quasigroup. Next,

(1) Let $x \in Q$, then $x + x = (x^{-1}x)x = x$ (idempotent law)
(2) Let $x, y, z \in Q$, then $(x + y) + z = (y^{-1}x)z^{-1}x(y^{-1}x)x$
  and $(x + z) + (y + z) = (y^{-1}x)z^{-1}x(y^{-1}x)x$
  Therefore, $(x + y) + z = (x + z) + (y + z)$ (right distributive law)
  Similarly, the left distributive law is shown the same way.
(3) Next, $x + (x + y) = x + (y^{-1}x)x = [[(y^{-1}x)x]^{-1}x]x = y$ (Left involutory law)

Thus, $(Q, +)$ is a LIPQ of odd order n. □

Example 3.1. *A LIPQ of order 5 constructed from a cyclic group of order 5 using Theorem 3.7.*

TABLE 1. LIPQ of order 5

$\cdot$	1	2	3	4	5
1	1	5	4	3	2
2	3	2	1	5	4
3	5	4	3	2	1
4	2	1	5	4	3
5	4	3	2	1	5

http://dx.doi.org/10.22108/IJGT.2022.132799.1780

Theorem 3.8. *Let $Q(\cdot)$ be a cyclic group of odd order n such that*

$$x + y = y(yx^{-1}), \forall x, y \in Q$$

Then, $(Q, +)$ is a right involutory property latin quandle (RIPQ) of odd order n.

Proof. We also need to show that Q obeys the necessary properties in Definition 2.7.

(1) Let $x \in Q$, then $x + x = x(xx^{-1}) = x$ (idempotent law)
(2) Let $x, y, z \in Q$, then $(x + y) + z = z(zy^{-1})xy^{-1}$
and $(x + z) + (y + z) = z(zy^{-1})xy^{-1}$
Therefore, $(x + y) + z = (x + z) + (y + z)$ (right distributive law)
Similarly, the left distributive law is shown the same way.
(3) Next, $(x + y) + y = y[y[y(yx^{-1})]^{-1}] = x$ (right involutory)

Thus, $(Q, +)$ is an RIPQ of odd order n. □

Example 3.2. *An RIPQ constructed from a cyclic group of order 5 using Theorem 3.8.*

TABLE 2. RIPQ of order 5

$\cdot$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

Remark 3.9. *Latin quandles in Table 1 and Table 2 are non-isomorphic.*

Theorem 3.10. *Let $Q(\cdot)$ be a commutative group (not cyclic) of order 3^n such that*

$$x + y = x(y^{-1}x), \forall x, y \in Q$$

Then, a commutative $(Q, +)$ is an involutory property latin quandle (IPQ) of order $3^n, n \geq 1$.

Proof. The proof is similar to that of Theorem 3.7 and Theorem 3.8

(1) Let $x \in Q$, then $x + x = x(x^{-1}x) = x$ (idempotent law)

(2) Let $x, y, z \in Q$, then $(x+y)+z = x(y^{-1}x)z^{-1}x(y^{-1}x)$
and $(x+z)+(y+z) = x(y^{-1}x)z^{-1}x(y^{-1}x)$
Therefore, $(x+y)+z = (x+z)+(y+z)$ (right distributive law)
Similarly, the left distributive law is shown the same way.

(3) Next, $x+(x+y) = x[[x(y^{-1}x)]^{-1}x] = y$ (LIPQ)

(4) commutativity justifies $(x+y)+y = y+(x+y) = y+(y+x) = y[[y(x^{-1}y)]^{-1}y] = x$ (RIPQ)

Thus, $(Q,+)$ is an IPQ of order $3^n, n \geq 1$. □

Example 3.3. *An example of an involutory latin quandle of order 9 by Theorem 3.10.*

Table 3. IPQ of order 9

·	1	2	3	4	5	6	7	8	9
1	1	3	2	7	9	8	4	6	5
2	3	2	1	9	8	7	6	5	4
3	2	1	3	8	7	9	5	4	6
4	7	9	8	4	6	5	1	3	2
5	9	8	7	6	5	4	3	2	1
6	8	7	9	5	4	6	2	1	3
7	4	6	5	1	3	2	7	9	8
8	6	5	4	3	2	1	9	8	7
9	5	4	6	2	1	3	8	7	9

Corollary 3.11. *A commutative LIPQ (RIPQ) is an IPQ.*

Proof. The proof follows from Theorem 3.10. □

Conjecture 3.1. *There is no IPQ of order* $4, 5, 7$ *and* 8

Example 3.4. *Consider an example of a latin quandle of order 7.*

http://dx.doi.org/10.22108/IJGT.2022.132799.1780

TABLE 4. A latin quandle of order 7 that is not a CIPQ nor LIPQ(RIPQ)

·	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

Example 3.5. *A CIPQ of order 7.*

TABLE 5. CIPQ of order 7

·	1	2	3	4	5	6	7
1	1	6	4	2	7	5	3
2	4	2	7	5	3	1	6
3	7	5	3	1	6	4	2
4	3	1	6	4	2	7	5
5	6	4	2	7	5	3	1
6	2	7	5	3	1	6	4
7	5	3	1	6	4	2	7

Remark 3.12. *The above latin quandle (in Table 5) is not a LIPQ nor a RIPQ. Consequently, it is not an IPQ but a CIPQ. The smallest CIPQ that is not an IPQ is the only latin quandle of order* 4.

Proposition 3.13. *The necessary and sufficient conditions for a latin quandle that is not a CIPQ to be a CIPQ is that it is a commutative LIPQ (or a commutative RIPQ).*

Proof. The necessary part: Let Q be a commutative LIPQ or a commutative RIPQ. Then, for all $x, y \in Q$ we have $x(xy) = y = (yx)x$. Implies that $(xy)x = y = x(yx)$-a CIPQ.
Also implies that $((xx)y)x = (xx)(yx)$-a Jordan identity.
The sufficient part: Suppose Q is a latin quandle, then it obeys Jordan identity $((xx)y)x = (xx)(yx) \Leftrightarrow (xy)x = x(yx)$
Since, Q is a commutative LIPQ (RIPQ), then $(xy)x = y = x(yx)$-a CIPQ. □

Corollary 3.14. *A commutative CIPQ is an IPQ.*

Proof. Consider $x(yx) = y = (yx)x = (xy)x = x(xy)$ (LIPQ and RIPQ), which gives an IPQ. □

Lemma 3.15. *Alexander quandles that are left and right distributive must be latin.*

Proof. All Alexander quandles are left distributive(see [38]) and idempotent with a unique right division law holding.
We only need to prove that a unique left division and right distributivity hold.
Consider $a \star x = b$ This implies that $x = \frac{t(b-a)+(1-t)b}{1-t}$ for $t \neq 1$. Next, we show uniqueness. Suppose there exists x_1 and x_2 such that $ta + (1-t)x_1 = b$ and $ta + (1-t)x_2 = b$ hold. This implies that $x_1 = x_2$. Thus, left division law holds and it is unique provided $t \neq 1$.
Next, we show right distributive law: $(a \star b) \star a = (a \star a) \star (b \star a)$
Consider $(a \star b) \star a = t((ta + (1-t)b) + (1-t)a = t^2 a + (1-t)(tb + a) = (a \star a) \star (b \star a)$ (Definition 2.7 holds). Therefore, Alexander quandles that are left and right distributive must be latin Alexander quandles. □

Theorem 3.16. *A latin Alexander quandle $Q(\star)$ is a CIPQ if and only if $b = t^2 a + (1-t)(tb + a)$*

Proof. A CIPQ obeys $(a \star b) \star a = b$ and

$$(a \star b) \star a = t^2 a + (1-t)(tb + a) = b$$

Then, the converse follows by Lemma 3.15. Q is a latin Alexander quandle. □

Remark 3.17. *A latin Alexander quandle of order* 4n *can be constructed from the direct product of a CIPQ of order* 4 *and a latin quandle of order n. Behold, this latin Alexander quandle of order* 4n *is also a CIPQ, and CIPQ of order* 4 *is the only latin quandle of order* 4.

Example 3.6. *An example of a latin Alexander quandle of order* 12.

Table 6. Latin Alexander quandle of order 12 that is a CIPQ

$\star$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	3	4	2	9	11	12	10	5	7	8	6
2	4	2	1	3	12	10	9	11	8	6	5	7
3	2	4	3	1	10	12	11	9	6	8	7	5
4	3	1	2	4	11	9	10	12	7	5	6	8
5	9	11	12	10	5	7	8	6	1	3	4	2
6	12	10	9	11	8	6	5	7	4	2	1	3
7	10	12	11	9	6	8	7	5	2	4	3	1
8	11	9	10	12	7	5	6	8	3	1	2	4
9	5	7	8	6	1	3	4	2	9	11	12	10
10	8	6	5	7	4	2	1	3	12	10	9	11
11	6	8	7	5	2	4	3	1	10	12	11	9
12	7	5	6	8	3	1	2	4	11	9	10	12

Theorem 3.18. *Let Q be a latin quandle. Then, the following are equivalent.*

(1) *Q is a IPQ*
(2) *Q is a SLQ*
(3) *Q is a commutative LIPQ(RIPQ)*
(4) *Q is a CIPQ*

Proof. (1)⇒(2). By definition $IPQ = SLQ$
(2)⇒(3). Q is an SLQ implies that Q is a commutative LIPQ (RIPQ).
(3)⇒(4). Since Q is commutative LIPQ (RIPQ), then Q is a CIPQ.
(4)⇒(1). A commutative CIPQ is a IPQ □

Remark 3.19. *Theorem 3.18 is simply the characterization of IPQ.*

4. Conclusion

The discussion shows that LIPQs, RIPQs, CIPQs and IPQs are strictly non-isomorphic involutory latin quandles. The left translations are involutions in LIPQ, the right translations are involutions

http://dx.doi.org/10.22108/IJGT.2022.132799.1780

in RIPQ. Either of them is an involution in CIPQ, and both of them are involutions and coincide in IPQ. Therefore, an IPQ generalizes the other three. Examples of IPQs are sparse. A quick example is the latin quandle of order 3, the next example is of order 9 as conjectured in this paper. In this work, however IPQs of order $3^n, n \geq 1$ were constructed in addition to LIPQs and RIPQs. The illustrative examples in this work were all verified by Maple software ([46]).

Acknowledgments

The authors wish to thank the anonymous referee for taking his time to go through the work.

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