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TRILINEAR ALTERNATING FORMS AND RELATED CMLS AND GECS

NOUREDDINE MIDOUNE* AND MOHAMED ANOUAR RAKDI

Abstract. The classification of trivectors(trilinear alternating forms) depends essentially on the dimension n of the base space. This classification seems to be a difficult problem (unlike in the bilinear case). For $n \leq 8$ there exist finitely many trivector classes under the action of the general linear group $GL(n)$. The methods of Galois cohomology can be used to determine the classes of nondegenerate trivectors which split into multiple classes when going from $\bar{K}$(the algebraic closure of K) to K. In this paper, we are interested in the classification of trivectors of an eight dimensional vector space over a finite field of characteristic 3, $K = \mathbb{F}_{3^m}$. We obtain a 31 inequivalent trivectors, 20 of which are full rank. Having its motivation in the theory of the generalized elliptic curves and commutative moufang loop, this research studies the case of the forms over the 3 elements field. We use a transfer theorem providing a one-to-one correspondence between the classes of trilinear alternating forms of rank 8 over a finite field with 3 elements $\mathbb{F}_3$ and the rank 9 class 2 Hall generalized elliptic curves (GECs) of 3-order 9 and commutative moufang loop (CMLs). We derive a classification and explicit descriptions of the 31 Hall GECs whose rank and 3-order both equal 9 and the number of order 3^9-CMLs.

1. Introduction

Let $V = V(n, K)$ be an n-dimensional vector space over a field K, and let $\wedge^3 V$ denote the exterior power of degree 3 over V. The classification of trivectors is is concerned with the action of general linear group $GL(V)$ on the space $\wedge^3 V$ defined by $f.\omega = (\wedge^3 f)(\omega)$. The equivalence classes are the $GL(V)$-orbits under this action. As $\wedge^3 V^* \simeq (\wedge^3 V)^*$, there is no difference between trilinear alternating forms and trivectors. The support of the trivector ω is the least subspace F of V such that $\omega \in \wedge^3 F$. $r(\omega)$. If

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*Corresponding author.

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$r(\omega) = n$, the trivector ω are full rank.We designate as $Alt\,(n, K)$ the set of all the trilinear alternating forms over K.

This classification was done for the case $n \leq 7$ by [5] for a large family of fields including all finite fields. [7], D.Djokovic [6] and L. Noui [10] solved the case $n = 8$ for $K = C$, $K = R$ and K an algebraically closed field of arbitrary characteristic, respectively. The classification for finite fields, except for the characteristic 2 or 3, for $n = 8$ by [9] and the 8-dimensional case over $\mathbb{F}_2$ by [8].

The generalized elliptic curves (GECs) are pairs (G, T), where T is a family of triples (x, y, z) of "points" from the set G, which are characterized by equalities of the form $x.y = z$, where the law $x.y$ makes G into a totally symmetric quasigroup. Isotopic loops arise by setting $x * y = u.(x.y)$. When $(x.y).(a.b) = (x.a).(y.b)$, identically, (G, T) is an entropic GEC and $(G, *)$ is an abelian group. Similarly, a terentropic GEC may be characterized by $x^2.(a.b) = (x.a)(x.b)$, $(G, *)$ is then a commutative moufang loop (CMLs). Additionally, if $x^2 = x$, then Hall GECs and $(G, *)$ is an exponent 3 CML. In class 2 CMLs, the associator enjoys some pseudolinearity: $(x * x\prime, y, z) = (x, y, z) * (x\prime, y, z)$.

The GECs arose in combinatorics (in which they are called "Extended Triple Systems") and in quasigroup theory (in which they are identified with the related totally symmetric quasigroups $(G, .)$). The relationship between trivectors and CMLs, GEC can be seen in [1].

In this paper, we are interested in the classification of trivectors of an eight dimensional vector space over a finite field of characteristic 3, $K = \mathbb{F}_{3^m}$. We obtain 31 inequivalent trivectors, 20 of which are full rank. We use a transfer theorem providing a one-to-one correspondence between the classes of trilinear alternating forms of rank 8 over a finite field with 3 elements $\mathbb{F}_3$ and the rank 9 class 2 Hall GECs of 3-order 9 and CMLs. We derive a classification and explicit descriptions of the 31 Hall GECs whose rank and 3-order both equal 9 and the number of order 3^9-CMLs. Some undefined terms can be found in references [1], [3], [2] and [4].

2. Preliminaries

This section we recall specific result of trivectors of rank ≤ 7, definitions and and propertie which are needed in generalized elliptic curves (GECs) and commutative moufang loop (CMLs).

2.1. Trivectors.

We recall the known result of the trivectors of a rank ≤ 7 over a finite field K [5]; we focus on the special case $K = \mathbb{F}_3$.

Theorem 2.1. *Let V be a vector space of dimension 7 over a finite field with 3 elements $\mathbb{F}_3$,. Then any trivector of a rank ≤ 7 in $\wedge^3 V$ is equivalent to one of the trivectors in Table 1.*

2.2. GEC and CML.

Definition 2.2. *A quasigroup is a set G together with a binary operation "$\cdot$" where the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in G for all $a, b \in G$.*

Definition 2.3. *A loop L is a quasigroup and low "$\cdot$" which admits a neutral bilaterally 1 : $x \cdot 1 = 1 \cdot x = x$.*

Definition 2.4. *A loop L is called a commutative moufang loop (CMLs) if*

$$x \cdot y = y \cdot$$

TABLE 1. Trivectors of rank ≤ 7 over $\mathbb{F}_{3^m}$ (degenerate forms).

Name	*Trivector*
ω_3	$e_1e_2e_3$
ω_5	$e_1(e_2e_3 + e_4e_5)$
$\omega_{6,1}$	$e_1e_2e_3 + e_4e_5e_6$
$\omega_{6,1,-1}$	$e_1(e_3e_4 + e_5e_6) + e_2(e_3e_6 + e_4e_5)$
$\omega_{6,2}$	$e_1e_2e_4 + e_2e_3e_5 + e_1e_3e_6$
$\omega_{7,1}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7)$
$\omega_{7,2}$	$\omega_{7.1} + e_2e_4e_6$
$\omega_{7,3}$	$e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_7$
$\omega_{7,3,-1}$	$e_1(e_2e_5 + e_3e_7) + e_4(e_2e_3 - e_5e_7) + e_6e_5e_3$
$\omega_{7,4}$	$e_1(e_2e_3 + e_4e_5) + e_2e_4e_6 + e_3e_5e_7$
$\omega_{7,5}$	$\omega_{7.2} + e_3e_5e_7$

$$x^2.(y.z) = (x.y).(x.z)$$

are satisfied for every $x, y, z \in L$. We often write xy instead of $x.y$

A generalized elliptic curve (GEC for short) is a pair (G, T), where T is a given family of unordered triples from the set G such that any pair (x, y), not necessarily distinct points of G, is contained in exactly one triple $((x, y, z))$ from T.

Let us denote then $x.y = z$ and for any fixed element u from G we set $x \star_u y = u.(x.y)$. Both these binary laws are commutative. Besides, $u.(x.u) = x$, hence $(G, \star_u)$ is a commutative loop with an identity element u; we call it the related loop of origin u.

For any point x from G, the "tangential" is the unique point t such that $((xxt))$ belongs to T; in case $t = x$ we say that x is an inflexion point. The set $I(G)$ of all the inflexion points is the set of the idempotent elements in the totally symmetric quasigroup $(G, \cdot)$. The "rank" of a GEC (G, T) is the smallest cardinal number r such that G admits a generator subset whose cardinal is r. The entropic GECs are those in which $(x.y).(z.t) = (x.z).(y.t)$ identically. When this identity is only assumed to be fulfilled in any subsystem of rank ≤ 3, we say that (G, T) is a terentropic GEC (or TGEC). More particularly, the TGECs in which any point is an inflexion point are the Hall GECs (or HGECs for short).

Example 2.5. *(Zassenhaus commutative moufang loop) The first (and smallest) example of a non-associative exp3-CML was given in 1937 by Zassenhaus, by defining, on the set of all $81 = 3^4$, $(0, 1, 2)-$ sequences $x = (x_1, x_2, x_3, x_4)$.*

Let $\mathbb{F}_3$ denote the finite field with 3 elements. Let L be the set $\mathbb{F}_3^4$. We define a new multiplication on $\mathbb{F}_3^4$ for $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$, by

$$x.y = x + y + (0, 0, 0, (x_3 - y_3)(x_1y_2 - x_2y_1)) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 + (x_3 - y_3)(x_1y_2 - x_2y_1)).$$

Then $(L, .)$ is a commutative moufang loop that is not associative.

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Let (E,T) be a HGEC of order 3^{n+1} and of rank $n+1$. We designate by $e_0, e_1, e_2, \dots, e_n$ a generator subset of E and by $(E,+)$ the exponent 3 CML related to the origin e_0(e_0 is the identity element).

If (E,T) is HGEC of class 2, of rank $n+1$ and 3-order $n+1$, then, there exist a unique alternate trilinear form ω from E^3 to $\mathbb{F}_3$, $\omega \in Alt(n, \mathbb{F}_3)$ is said to be the "factorized associator"of (E,T) and $(E,+)$.

Let us now reverse the process, from a given alternate trilinear from $Alt(n, \mathbb{F}_3)$. We want to recover the corresponding HGECs and exponent 3 CMLs of class 2. Consider an arbitrary codimension 1 subspace R in $\Lambda^3 V$, where $V = V(n, \mathbb{F}_3)$, vector space with a fixed basis $e_i, i = 1, 2, \dots, n$. The quotient $W = \Lambda^3 V/R$ is trivially generated by the $\binom{n}{3}$ cosets of the trivectors $\overline{e_i} \wedge \overline{e_j} \wedge \overline{e_k} = (e_i \wedge e_j \wedge e_k + R)$ where $1 \le i < j < k \le n$.

Every vector x from the direct sum $E = V \oplus W$ may be written as a linear combination; we have

$$x = \sum_{i=1,2,\dots,n} x_i e_i + \sum_{1 \le i<j<k \le n} x_{ijk} \overline{e_i} \wedge \overline{e_j} \wedge \overline{e_k}$$

where, x_i and x_{ijk} belong to $\mathbb{F}_3$. We may define an exponent 3 CML binary law by setting

$$x * y = x + y + \sum_{1 \le i<j<k \le 8} (x_i y_j - y_i x_j)(x_k - y_k) \overline{e_i} \wedge \overline{e_j} \wedge \overline{e_k}$$

The so-determined sum $x * y$ is well-defined, though the x_{ijk}'s are not uniquely determined from x. A straightforward verification proves that $(E, *)$ has class 2. The family of unordered triples $((x, y, z))$ characterized by $x * y * z = 0$ endows the set E with a structure of HGEC of class 2 whose factorized associator (trilinear alternatig forms) is

$$\omega(x, y, z) = \sum_{1 \le i<j<k \le 8} e^*_{ijk}(x, y, z) \overline{e_i} \wedge \overline{e_j} \wedge \overline{e_k}.$$

Theorem 2.6. *(correspodence) for $n \ge 3$ we have one-to-one correspodence between the classes of trilinear alternating forms of rank n over $\mathbb{F}_3$ and the isomorphy classes of CCHGs (respective of CMLs exponent 3) of rank $n+1$ (resp. n), and of 3 -order $n+1$, and of class 2.*

Theorem 2.7. *(The number of isomorphy classes). For $n \ge 3$, the number of trivectors classes coincides with the maximum number of pairwise non-isomorphic rank $n+1$ HGECs of 3- order $n+1$ and of class 2.*

Theorem 2.8. *(The number of isomorphy classes) For $n = 7$, the number of trivectors classes coincides with the maximum number of pairwise non-isomorphic rank 8 HGECs of 3- order 8 and of class 2. There are exactly 5 such HGECs that are centrally irreducible.*

3. Main Results

The classification of trilinear alternate forms.

3.1. Trivectors of rank 8 over a finite field $K = \mathbb{F}_{3^m}$ of characteristic 3.

Theorem 3.1. *Let V be an eight dimensional vector space over a finite field $K = \mathbb{F}_{3^m}$ of characteristic 3. There are 32 inequivalent trivectors in $\wedge^3 V$, 20 of which are full rank, then any trivector of rank 8 in $\wedge^3 V$ is exactly equivalent to one of the trivectors $\omega_{8,i}$ given in Table 2.*

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TABLE 2. Trivectors of rank 8 over $\mathbb{F}_{3^m}$.

name	*Trivector*
$\omega_{8,1}$	$e_1(e_2e_3+e_4e_5)+e_6e_7e_8$
$\omega_{8,2}$	$e_1(e_2e_3+e_4e_5+e_6e_7)+e_5e_6e_8$
$\omega_{8,3}$	$e_1(e_3e_4+e_5e_6)+e_2(e_3e_5+e_7e_8)$
$\omega_{8,4}$	$e_1(e_2e_3+e_4e_5)+e_6(e_2e_7+e_4e_8)$
$\omega_{8,4,-1}$	$e_5(e_1e_2+e_3e_4)+e_6(e_1e_3-e_2e_4)+e_7(e_1e_4)+e_8(e_2e_3)$
$\omega_{8,5}$	$e_1(e_2e_3+e_4e_5)+e_6(e_2e_3+e_7e_8)$
$\omega_{8,5,-1}$	$e_1(e_5e_4+e_7e_8+e_6e_3)+e_2(e_5e_3+e_4e_6)$
$\omega_{8,5,a}$	$e_7(e_1e_2+e_3e_4+e_5e_6)+e_8\left[e_1(e_4+ae_5)+e_2e_6+a^2e_3e_5\right]$
$\omega_{8,6}$	$e_1(e_2e_3+e_4e_5+e_6e_7)+e_8(e_4e_3+e_5e_6)$
$\omega_{8,7}$	$e_1(e_2e_3+e_4e_5+e_6e_7)+e_2(e_5e_6+e_7e_8)$
$\omega_{8,8}$	$e_1(e_2e_8+e_3e_6+e_4e_7)+e_6e_7e_8+e_3e_4e_5$
$\omega_{8,9}$	$e_1\left[e_2(e_3+e_4)+e_5e_6\right]+e_3e_5e_7+e_4e_6e_8$
$\omega_{8,9,-1}$	$e_3(e_5e_1-e_6e_2)+e_4(e_5e_2+e_6e_1)+e_7(e_1e_2+e_8e_6)$
$\omega_{8,9,a}$	$e_2\left[(e_3+e_4)e_1+a^2e_7e_6\right]+e_8\left[a(e_6+e_7)e_1+e_3e_4\right]+e_5(e_3e_7+e_6e_4)$
$\omega_{8,10}$	$e_1(e_2e_8+e_6e_7)+e_2e_3e_5+e_3e_4e_6+e_4e_5e_7$
$\omega_{8,10,-1}$	$e_5(e_1e_2)+e_6\left[e_2e_3+(e_1-e_8)e_4\right]+e_7\left[e_2e_4-(e_1+e_8)e_3\right]$
$\omega_{8,11}$	$e_1(e_3e_7+e_5e_4+e_8e_2)+e_8(e_4e_3+e_6e_7)+e_2e_4e_6$
$\omega_{8,12}$	$e_1\left[(e_4-e_7)(e_3-e_8)+e_5e_7\right]+e_2(e_3e_4+e_5e_6)+e_6e_7e_8$
$\omega_{8,13}$	$e_1\left[e_5(e_3+e_7)+e_8e_4\right]+e_2(e_3e_4+e_5e_6)+e_6e_7e_8$
$\omega_{8,13,-1}$	$e_8\left[(e_2-e_1)e_7-e_3e_6-e_4e_5\right]+e_3(e_4e_1+e_5e_7)+e_6(e_2e_5-e_7e_4)$

Here, $a\in(\mathbb{F}^*_{3^m})^3$

If ω is a trivector defined over the field K, a K-form of ω is another trivector of the same type as that of ω, defined over K which is isomorphic to ω over $\overline{K}$, the algebraic closure of K.

The classification for finite fields, except for characteristic 2 or 3, for $n=8$ has been done in [9]; some of the previous results are still true in characteristic 3, hence, it is sufficient to study the case of orbits of type $\omega_{8,i}$, for $i=5,9$.

Proposition 3.2. *In characteristic* 3*, the trivectors of type* $\omega_{8,i}$ *for* $i=5$ *and* 9 *are written as follows:*
$\omega_{8,5,a}=e_7(e_1e_2+e_3e_4+e_5e_6)+e_8\left[e_1(e_4+ae_5)+e_2e_6+a^2e_3e_5\right]$
$\omega_{8,9,a}=e_2\left[(e_3+e_4)e_1+a^2e_7e_6\right]+e_8\left[a(e_6+e_7)e_1+e_3e_4\right]+e_5(e_3e_7+e_6e_4)$

Proof. If L is a cubic extension of K, there exists a trivector $\omega_L\in\wedge^3V$ such that $\omega_L\not\simeq\omega_{8,5}$ and $\omega_L\otimes L\in\wedge^3(V\otimes_K L)$ is L-isomorphic to $\omega_{8,5}$. We construct ω_L as follows. Let F be a 3-dimensional $L-$space. We choose a basis$\{e_1,e_2,e_3\}$ of F and let the determinant form $\varphi:\wedge^3F\to L$ be such that $\varphi(e_1,e_2,e_3)=1$ and $Tr:L\to K$ be the trace form, hence $\omega_L=Tr_L\circ\varphi:F\times F\times F\to K$ is an alternating trilinear form of rank 9 on $V=F$ viewed as vector space over K.

If $cara\,(L) = 3$, we take $L = K\,(t)$ with $t^3 - t = a$, an explicit calculation shows that by choosing a suitable K-basis of V, we find that ω_L can be expressed as: $\omega_L = e_7\,(e_1 e_2 + e_3 e_4 + e_5 e_6) + e_8\left[e_1\,(e_4 + a e_5) + e_2 e_6 + a^2 e_3 e_5\right] = \omega_{8,5,a}$

We can also prove that $\omega_{8,5}$ is not equivalent to $\omega_{8,5,a}$ by using the arithmetical invariant $d_1\,(\omega)$ [10] Similar arguments apply to the case for $\omega_{8,9}$.

The cardinality $\mid Aut\,(\omega_i)|$ of the automorphisms groups $Aut\,(\omega_i)$ for trivectors ω_i in $\wedge^3 \mathbf{F}_q^8$, is given in Tables 3 and 4.

Table 3. The cardinality of the automorphisms groups $Aut\,(\omega_i)$ for trivectors ω_i of rank ≤ 7 over $\mathbf{F}_{3^m}$.

Trivector	$\mid Aut\,(\omega_i)\mid$
ω_3	$q^{28}\left(q^5-1\right)\left(q^4-1\right)\left(q^3-1\right)^2\left(q^2-1\right)^2(q-1)$
ω_5	$q^{26}\left(q^4-1\right)\left(q^3-1\right)\left(q^2-1\right)^2(q-1)^2$
$\omega_{6,1}$	$2q^{19}\left(q^3-1\right)^2\left(q^2-1\right)^3(q-1)$
$\omega_{6,1,-1}$	$2q^{19}\left(q^6-1\right)\left(q^4-1\right)\left(q^2-1\right)(q-1)$
$\omega_{6,2}$	$q^{24}\left(q^3-1\right)\left(q^2-1\right)^2(q-1)^2$
$\omega_{7,1}$	$q^{22}\left(q^6-1\right)\left(q^4-1\right)\left(q^2-1\right)(q-1)^2$
$\omega_{7,2}$	$q^{22}\left(q^3-1\right)\left(q^2-1\right)(q-1)^2$
$\omega_{7,3}$	$2q^{19}\left(q^2-1\right)^2(q-1)^3$
$\omega_{7,3,-1}$	$2q^{19}\left(q^4-1\right)\left(q^2-1\right)(q-1)$
$\omega_{7,4}$	$q^{17}\left(q^2-1\right)^2(q-1)^2$
$\omega_{7,5}$	$\epsilon q^{13}\left(q^6-1\right)\left(q^2-1\right)(q-1)$

In order to prove that all orbits are determined, using Table 3 and Table 4, we compute the number $\mid O\,(\omega_i)| = \frac{|GL_8(\mathbf{F}_q)|}{|Aut(\omega_i)|}$ of orbits of trivectors ω_i in $\wedge^3 \mathbf{F}_q^8$ over $\mathbf{F}_{3^m}$. An easy computation shows that

$$\sum_{r(\omega_i)\quad\leq 7} \mid O\,(\omega_i)| + \sum_{r(\omega_i)\quad =8} \mid O\,(\omega_i)| = q^{56} = \left|\wedge^3 \mathbf{F}_q^8\,\right|. \qquad \square$$

Using classifications of trilinear alternating forms over the field of 3 elements $\mathbb{F}_3$, we enumerate non-associative exp3-CML. The number of isomorphy classes of 3^s-order rank n exp3-CMLs (or 3^n-order rank $n+1$ HGECs) may be summarized in Table 5.

3.2. **Transfer theorem.**

Proposition 3.3. *(The number of isomorphy classes) For* $n = 8, K = \mathbb{F}_3$, *the number of inequivalent classes of trivectors coincides with the maximum number of pairwise non-isomorphic rank* 9 *HGECs of* 3*-order* 9 *and of classe* 2*. There are exactly* 20 *inequivalent trivectors of full rank such HGEC that are centrally irreducible.*

Proof. We use theorem 3, for $n = 8$, and we obtain this result. $\square$

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TABLE 4. The cardinality of the automorphisms groups $Aut(\omega_i)$ for trivectors ω_i of rank 8 over $\mathbf{F}_{3^m}$.

Trivector	$\|Aut(\omega_{8,i})\|$
$\omega_{8,1}$	$q^{11}(q^4-1)(q^3-1)(q^2-1)^2(q-1)$
$\omega_{8,2}$	$q^{18}(q^2-1)^2(q-1)^2$
$\omega_{8,3}$	$q^{11}(q^2-1)^2(q-1)^2$
$\omega_{8,4}$	$2q^{14}(q^2-1)^2(q-1)^2$
$\omega_{8,4,-1}$	$2q^{14}(q^4-1)(q^2-1)$
$\omega_{8,5}$	$6q^9(q^2-1)^3(q-1)$
$\omega_{8,5,-1}$	$2q^9(q^4-1)(q^2-1)(q-1)$
$\omega_{8,5,a}$	$3q^9(q^6-1)(q-1)$
$\omega_{8,6}$	$q^{14}(q^2-1)(q-1)^2$
$\omega_{8,7}$	$q^{17}(q^2-1)(q-1)^2$
$\omega_{8,8}$	$q^{10}(q^2-1)(q-1)^2$
$\omega_{8,9}$	$6q^9(q-1)^3$
$\omega_{8,9,-1}$	$2q^9(q^2-1)(q-1)$
$\omega_{8,9,a}$	$3q^9(q^3-1)$
$\omega_{8,10}$	$2q^7(q^2-1)(q-1)^2$
$\omega_{8,10,-1}$	$2q^7(q^2-1)^2$
$\omega_{8,11}$	$\epsilon.q^{13}(q^2-1)(q-1)$
$\omega_{8,12}$	$q^6(q^2-1)(q-1)$
$\omega_{8,13}$	$2\epsilon.q^3(q^3+1)(q^2-1)$.
$\omega_{8,13,-1}$	$2\epsilon.q^3(q^3+1)(q^2-1)$

TABLE 5. The number of isomorphy classes.

Rank n	3	4	5	6	7	8
exp CMLs and HGECs \ order	3^4	3^5	3^6	3^7	3^8	3^9
rank n exponent 3 CMLs rank $n+1$ HGECs	1	1	2	5	11	31
factorized associator	ω_3	ω_3	ω_3, ω_5	ω_3, ω_5, $\omega_{6,1}$, $\omega_{6,2}$, $\omega_{6,1,-1}$	see Table 1	see Table 2

Proposition 3.4. *There are exactly* 31 *rank* 9 *HGECs whose* 3*-order is* 9*, and among them only* 20 *are centrally irreducible and admit as factorized associators one of the previously defined alternate trilinear forms,* $\omega_{8,i}$, $i = 1, \ldots, 20$,

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TABLE 6. The factorized associator of trivectors (non-degenerate) over $\mathbb{F}_{3^m}$.

factorized associator $\omega_{8,i}$	$\phi_{\omega_{8,i}}(x,y)$
$\omega_{8,1} = e_1(e_{23} + e_{45}) + e_{678}$	$(x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4)(x_1 - y_1) + (x_7y_8 - x_8y_7)(x_6 - y_6)$
$\omega_{8,2} = e_1(e_{23} + e_{45} + e_{67}) + e_{568}$	$(x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4 + x_6y_7 - x_7y_6)(x_1 - y_1)$ $+(x_6y_8 - x_8y_6)(x_5 - y_5)$
$\omega_{8,3} = e_1(e_{34} + e_{56}) + e_2(e_{35} + e_{78})$	$(x_3y_4 - x_4y_3 + x_5y_6 - x_6y_5)(x_1 - y_1)$ $+(x_3y_5 - x_5y_3 + x_7y_8 - x_8y_7)(x_2 - y_2)$
$\omega_{8,4} = e_1(e_{23} + e_{45}) + e_6(e_{27} + e_{48})$	$(x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4)(x_1 - y_1) + (x_6y_7 - x_7y_6)(x_2 - y_2)$ $+(x_6y_8 - x_8y_6)(x_4 - y_4)$
$\omega_{8,4,-1} = e_5(e_{12} + e_{34}) + e_6(e_{13} - e_{24})$ $+e_7(e_{14}) + e_8(e_{23})$	$(x_2y_5 - x_5y_2 + x_3y_6 - x_6y_3 + x_4y_7 - x_7y_4)(x_1 - y_1)$ $+(x_4y_6 - x_6y_4 + x_3y_8 - x_8y_3)(x_2 - y_2) + (x_4y_5 - x_5y_4)(x_3 - y_3)$
$\omega_{8,5} = e_1(e_{23} + e_{45}) + e_6(e_{23} + e_{78})$	$(x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4)(x_1 - y_1) + (x_3y_6 - x_6y_3)(x_2 - y_2)$ $+(x_7y_8 - x_8y_7)(x_6 - y_6)$
$\omega_{8,5,-1} = e_1(e_{54} + e_{78} + e_{63}) + e_2(e_{53} + e_{46})$	$(x_4y_5 - x_5y_4 + x_7y_8 - x_8y_7 + x_3y_6 - x_6y_3)(x_1 - y_1)$ $+(x_3y_5 - x_5y_3 + x_4y_6 - x_6y_4)(x_2 - y_2)$
$\omega_{8,5,1} = e_7\,(e_1e_2 + e_3e_4 + e_5e_6)$ $+e_8\,[e_1\,(e_4 + e_5) + e_2e_6 + e_3e_5]$	$(x_2y_7 - x_7y_2 + x_4y_8 - x_8y_4 + x_5y_8 - x_8y_5)(x_1 - y_1)$ $+(x_6y_8 - x_8y_6)(x_2 - y_2) + (x_4y_7 - x_7y_4 + x_5y_8 - x_8y_5)(x_3 - y_3)$ $+(x_6y_7 - x_7y_6)(x_5 - y_5)$
$\omega_{8,6} = e_1(e_{23} + e_{45} + e_{67}) + e_8(e_{43} + e_{56})$	$(x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4 + x_6y_7 - x_7y_6)(x_1 - y_1)$ $+(x_4y_8 - x_8y_4)(x_3 - y_3) + (x_6y_8 - x_8y_6)(x_5 - y_5)$
$\omega_{8,7} = e_1(e_{23} + e_{45} + e_{67}) + e_2(e_{56} + e_{78})$	$(x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4 + x_6y_7 - x_7y_6)(x_1 - y_1)$ $+(x_5y_6 - x_6y_5 + x_7y_8 - x_8y_7)(x_2 - y_2)$
$\omega_{8,8} = e_1(e_{28} + e_{36} + e_{47}) + e_{678} + e_{345}$	$(x_2y_8 - x_8y_2 + x_3y_6 - x_6y_3 + x_4y_7 - x_7y_4)(x_1 - y_1)$ $+(x_4y_5 - x_5y_4)(x_3 - y_3) + (x_7y_8 - x_8y_7)(x_6 - y_6)$
$\omega_{8,9} = e_1[e_2(e_3 + e_4) + e_{56}] + e_{357} + e_{468}$	$(x_2y_3 - x_3y_2 + x_2y_4 - x_4y_2 + x_5y_6 - x_6y_5)(x_1 - y_1)$ $+(x_5y_7 - x_7y_5)(x_3 - y_3) + (x_6y_8 - x_8y_6)(x_4 - y_4)$
$\omega_{8,9,-1} = e_3(e_{51} - e_{62}) + e_4(e_{52} + e_{61})$ $+e_7(e_{12} + e_{86})$	$(x_3y_5 - x_5y_3 + x_4y_6 - x_6y_4 + x_2y_7 - x_7y_2)(x_1 - y_1)$ $+(x_3y_6 - x_6y_3 + x_4y_5 - x_5y_4)(x_2 - y_2) + (x_7y_8 - x_8y_7)(x_6 - y_6)$
$\omega_{8,9,1} = e_2\,[(e_3 + e_4)\,e_1 + e_7e_6]$ $+e_8\,[(e_6 + e_7)\,e_1 + e_3e_4] + e_5\,(e_3e_7 + e_6e_4)$	$[x_2y_3 - x_3y_2 + x_2y_4 - x_4y_2 - (x_6y_8 - x_8y_6 + x_7y_8 - x_8y_7)]\,(x_1 - y_1)$ $-(x_6y_7 - x_7y_6)(x_2 - y_2) + [x_4y_8 - x_8y_4 - (x_5y_7 - x_7y_5)]\,(x_3 - y_3)$ $+(x_5y_6 - x_6y_5)(x_4 - y_4)$
$\omega_{8,10} = e_1(e_{28} + e_{67}) + e_{235} + e_{346} + e_{457}$	$(x_2y_8 - x_8y_2 + x_6y_7 - x_7y_6)(x_1 - y_1) + (x_3y_5 - x_5y_3)(x_2 - y_2)$ $+(x_4y_6 - x_6y_4)(x_3 - y_3) + (x_5y_7 - x_7y_5)(x_4 - y_4)$
$\omega_{8,10,-1} = e_5(e_{12}) + e_6[e_{23} + (e_1 - e_8)e_4]$ $+e_7[e_{24} - (e_1 + e_8)e_3]$	$(x_2y_5 - x_5y_2 + x_4y_6 - x_6y_4 + x_3y_7 - x_7y_3)(x_1 - y_1)$ $+(x_3y_6 - x_6y_3 + x_4y_7 - x_7y_4)(x_2 - y_2) + (x_7y_8 - x_8y_7)(x_3 - y_3)$ $+(x_6y_8 - x_8y_6)(x_4 - y_4)$
$\omega_{8,11} = e_1(e_{37} + e_{54} + e_{82}) + e_8(e_{43} + e_{67})$ $+e_{246}$	$(x_3y_7 - x_7y_3 + x_4y_5 - x_5y_4 + x_2y_8 - x_8y_2)(x_1 - y_1)$ $+(x_4y_6 - x_6y_4)(x_2 - y_2) + (x_4y_8 - x_8y_4)(x_3 - y_3)$ $+(x_7y_8 - x_8y_7)(x_6 - y_6)$
$\omega_{8,12} = e_1[(e_4 - e_7)(e_3 - e_8) + e_{57}]$ $+e_2(e_{34} + e_{56}) + e_{678}$	$(x_3y_4 - x_4y_3 + x_4y_8 - x_8y_4 + x_3y_7 - x_7y_3 + x_5y_7 - x_7y_5)(x_1 - y_1)$ $+(x_3y_4 - x_4y_3 + x_5y_6 - x_6y_5)(x_2 - y_2) + (x_7y_8 - x_8y_7)(x_6 - y_6)$
$\omega_{8,13} = e_1[e_5(e_3 + e_7) + e_{84}] + e_2(e_{34} + e_{56})$ $+e_{678}$	$(x_3y_5 - x_5y_3 + x_5y_7 - x_7y_5 + x_4y_8 - x_8y_4)(x_1 - y_1)$ $+(x_3y_4 - x_4y_3 + x_5y_6 - x_6y_5)(x_2 - y_2) + (x_7y_8 - x_8y_7)(x_6 - y_6)$
$\omega_{8,13,-1} = e_8[(e_2 - e_1)e_7 - e_{36} - e_{45}]$ $+e_3(e_{41} + e_{57})\ +e_6(e_{25} - e_{74})$	$(x_7y_8 - x_8y_7 + x_3y_4 - x_4y_3)(x_1 - y_1)$ $+(x_5y_6 - x_6y_5 + x_7y_8 - x_8y_7)(x_2 - y_2) + (x_6y_8 - x_8y_6)(x_3 - y_3)$ $+(x_5y_8 - x_8y_5 + x_6y_7 - x_7y_6)(x_4 - y_4)$

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Proof. Let the trivector $\omega_{8,1}$ and R be the hyperplane of $\wedge^3 V$ generated by : $e_{123}-e_{145}$, $e_{123}-e_{678}$ and all e_{ijk}, for $i<j<k$, and for (i,j,k) different from : $(1,2,3)$, $(1,4,5)$, $(6,7,8)$. The quotient $E(\omega_{8,1}) = V \oplus \wedge^3 V/R$ is generated by $e_i, i=1,\ldots,8$ and by $u=\overline{e_{ijk}}$, and the compound in the sense of the loop is

$$x * y = x + y + \phi_{\omega_{8,i}}(x,y) \cdot u$$

with,

$$\phi_{\omega_{8,1}}(x,y) = (x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4)(x_1-y_1) + (x_7y_8 - x_8y_7)(x_6-y_6).$$

Similar arguments apply to the case for $\omega_{8,i}$, $i=2,\ldots,13$. □

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Noureddine Midoune

Department of Mathematics, University of MSILA, P.O.Box 166, Msila, Algeria

Email: noureddine.midoune@univ-msila.dz

Mohamed Anouar Rakdi

Department of Mathematics, University of MSILA, P.O.Box 166 Msila, Algeria

Email: mohamedanouar.rakdi@univ-msila.dz

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