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A SHORT NOTE ON THE TOPOLOGICAL DECOMPOSITION OF THE CENTRAL PRODUCT OF GROUPS

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Abstract. It has been recently observed that a topological decomposition of the Pauli group, as central product of the quaternion group of order eight and the cyclic group of order four, influences some significant dynamical systems in mathematical physics. The connection between groups of symmetries and dynamical systems is in fact well known, but looking specifically at the algebraic and topological decompositions of the Pauli group, we find conditions for the existence of a Riemannian 3-manifold whose fundamental group is epimorphically mapped onto a central product.

1. Introduction

Some classical results of mathematical physics illustrate that the knowledge of the group of symmetries in a Hamiltonian dynamical system is very important for the description of the entire dynamical system. In particular, in the paper [15], the authors show that the groups of symmetries of mathematical models that admit Riemannian manifolds have a simpler structure of the Hamiltonian. Standard results can be found in [2] on differential geometry and dynamical systems. Of course, Noether's Theorem [2, p. 88] motivates a series of important connections between mathematical physics and group theory since long time, but more recently Riemannian manifolds and groups have been used in quantum mechanics, in order to explain certain processes of quantization and the corresponding variational principles (see for instance [6]).

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An arbitrary group G is an *(internal) central product* of its subgroups H and K, denoted $G = H \circ K$, if $G = HK$ and the commutator subgroup $[H, K] = \langle [h, k] : h \in H, k \in K \rangle$ is trivial, where $[h, k] = h^{-1}k^{-1}hk$. It turns out that both H and K are normal in G in this situation; moreover $H \cap K \leqslant Z(H) \cap Z(K)$, where $Z(G)$ is the usual symbol to denote the center of a group G. It is useful to compare with [10, Theorem 5.3]. Along with the notion of central product, we will also refer to the usual *connected sum*, #, between two manifolds (i.e.: two compact and connected locally Euclidean topological spaces), as indicated in classical references of algebraic topology such as [12, p. 257] and [13, p. 79]. As is standard, we will denote the fundamental group of a (path connected) space X as $\pi(X)$. The last notion that we need to mention is that of *action* of a group on a topological space, and again this is a well known fact which can be found in [13, Chapter 5] and motivates the notion of orbits and space of orbits under a given action.

It is now appropriate to note from [3, 16] that a topological decomposition of the Pauli group, that is, of the finite group of order 16

$$P = \langle i, j, y \mid i^4 = 1, i^2 = j^2, ji = -ij, y^4 = 1, [i, y] = [j, y] = 1, i^2 = y^2 \rangle, \tag{1.1}$$

is possible via a quotient of the fundamental group of the usual 3-sphere $\mathbb{S}^3$. Note that $\mathbb{S}^3$ is the prototype of a low dimensional Riemannian manifold (see [5, Chapter 5] for basic facts of Riemannian geometry), the cyclic group of order four is

$$\mathbb{Z}(4) = \langle y \mid y^4 = 1 \rangle; \tag{1.2}$$

and the quaternion group of order eight is

$$Q_8 = \langle i, j \mid i^4 = 1, i^2 = j^2, ji = -ij \rangle = \{\pm 1, \pm i, \pm j, \pm k\}; \tag{1.3}$$

where $k = ij$ and of course $i^2 = j^2 = k^2 = -1$. More exactly:

Theorem 1.1. [3, Theorem 1.1] *There exist two compact path connected orbit spaces $U = \mathbb{S}^3/Q_8$ and $V = \mathbb{S}^3/\mathbb{Z}(4)$ such that the following conditions hold:*

(i) *$U \cup V$ is a compact path connected space with $U \cap V \neq \emptyset$, $\pi(U \cap V)$ cyclic of order 2 and $P \cong \pi(U \cup V)/N$ for some normal subgroup N of $\pi(U \cup V)$;*

(ii) *$U \# V$ is a Riemannian manifold of $\dim(U \# V) = 3$ and $P \cong \pi(U \# V)/L$ for some normal subgroup L of $\pi(U \# V)$.*

Both in case (i) and (ii), P is central product of $\pi(U)$ and $\pi(V)$.

Theorem 1.1 has an interesting effect on the Hamiltonian of a dynamical system with group of symmetries that is given by the Pauli group (see [3, Theorem 1.2]). In particular, it was conjectured something of more general.

Conjecture 1.2. [3, Conjecture 6.1] *Groups of the form $A = Q_8 \circ B$, where B is an abelian group containing at most one element of order 2, may have a construction of the Hamiltonian (in the sense of [3, Theorem 1.2]), which is similar to the case of B cyclic of order four.*

Without entering the details of the mathematical physics of [3, Theorem 1.2], it is interesting to work on the above conjecture because it provides new examples of groups, which may have significant applications. Note that the groups in Conjecture 1.2 may be infinite in principle, so the real issue is to know how the structure of central product in a group is related with the configurations of equilibrium of the motion in a dynamical system. This is still unknown, but we present a partial step in the direction of Conjecture 1.2. Of course here, $A \times B$ denotes the direct product of the groups A and B.

Theorem 1.3. *If G is a group which may be written as central product of its finite subgroups H and K and if H and K satisfy the following conditions:*

(i) *contain no subgroups isomorphic to $\mathbb{Z}(p) \times \mathbb{Z}(p)$ with p prime,*
(ii) *contain at most one element of order 2,*

then there exists a Riemannian manifold X of dimension 3 and a surjective homomorphism of groups from the fundamental group $\pi(X)$ of X onto G.

The proof of Theorem 1.3 uses essentially the sames ideas of [3, Proof of Theorem 1.1], but it shows that some technical steps may be reformulated in wider assumptions. Section 2 is in fact devoted to this proof and final applications are shortly mentioned from the point of view of the combinatorial group theory. The notation is standard and follows [2, 5, 7, 11, 12, 13, 14].

2. Proof and some observations

It is useful to recall here that a group G acts *freely* on a topological space X, if $g \cdot x \neq x$ for all $x \in X$, $g \in G$, $g \neq 1$; and we say that the action of G on X is *properly discontinuous* if, for each $x \in X$, there is an open neighbourhood V of x such that $g \cdot V \cap g' \cdot V = \emptyset$ for all $g, g' \in G$ with $g \neq g'$.

Proof 2.1. *Note that since H and K are finite and satisfy the conditions of* [7, Theorem 19.1] *and* [12, Page 75]*, they act freely on $\mathbb{S}^3$. And since $\mathbb{S}^3$ is a Hausdorff 3-manifold, they act properly discontinuously on $\mathbb{S}^3$ by* [13, Theorem 17.2]*. By* [13, Exercise 11.2 (d)] *both $U = \mathbb{S}^3/H$ and $V = \mathbb{S}^3/K$ are 3-manifolds and so* [13, Exercise 11.5 (b)] *implies that $U \# V$ is a 3-manifold. So now we need to show that $U \# V$ can be endowed with a Riemannian structure. In short the Riemannian metric (see* [5, Chapter 5.1.3]*) on $U \# V$ is inherited from the usual metric on the sphere $\mathbb{S}^3$. We start by observing from* [5] *that the action of H on the Riemannian sphere $\mathbb{S}^3$ with the round metric $d_{\mathbb{S}^3}$ that is derived from the Riemannian metric*

$$ds^2 = \frac{4\|dx\|^2}{(1 + \|x\|)^2},$$

(where $\|dx\|^2$ is the usual Riemannian metric on $\mathbb{R}^2$) produces the Riemannian space of orbits $U = \mathbb{S}^3/H$, with the canonical quotient map $p_U : \mathbb{S}^3 \to U$ with the induced distance function on $u_1, u_2 \in U$,

$$d_U(u_1, u_2) = \inf_{x \in p_U^{-1}(u_1), y \in p_U^{-1}(u_2)} d_{\mathbb{S}^3}(x, y)$$

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and analogously for V with d_V, because the actions of H and K are free and properly discontinuous (see [13, Theorem 17.2]*) on $\mathbb{S}^3$ (see* [4, Remark 41]*).*

A pair (M, Γ) where M is a Riemannian manifold and Γ is a properly discontinuous group acting (see [4, Definition 22 and Proposition 20]*) on M is called "good Riemannian orbifold". The underlying space of the "orbifold" is M/Γ. In the case of a good Riemannian orbifold (M, Γ) it follows that for $x, y \in M/\Gamma$,*

$$d(x,y) = d_M(\pi^{-1}(x), \pi^{-1}(y)) := \inf_{\tilde{x}\in\pi^{-1}(x), \tilde{y}\in\pi^{-1}(y)} d_M(\tilde{x}, \tilde{y})$$

and this is what we have, where in our case $M = \mathbb{S}^3$ and Γ can be H or K. This means that we can define a metric on the disjoint union of the subspaces U' and V' (which are U and V without the open balls of a connected sum construction) by:

$$d'(x,y) = \begin{cases} d_{U'}(x,y), & if\ x, y \in U' \\ d_{V'}(x,y), & if\ x, y \in V' \\ \infty, & otherwise. \end{cases}$$

Using this metric we find the quotient semi-metric on $U \cup V$ in the sense of [5, Definition 3.1.12]*, with respect to the relation #*

$$d_{\#}(x,y) = \inf\{\sum_{i=1}^{k} d'(p_i, q_i) \mid p_1 = x, q_k = y, k \in \mathbb{N}, p_i \in U \cup V, q_i \in U \cup V\}.$$

Since $U \cup V$ is compact, then $d_{\#}(x, y)$ a Riemannian metric on $U\#V$, (see [5, Exercise 3.1.14]*). Furthermore $\mathbb{S}^3$ is simply connected so that $\pi(U) \cong H$ and $\pi(V) \cong K$ by* [13, Corollary 19.4]*. Then $\pi(X) = \pi(U\#V) \cong \pi(U) * \pi(V) = H * K$, since we know from* [12] *that the connected sum at the level of topological spaces corresponds to the free product at the level of groups. By* [14, Problems for Section 4.1 (13)] *we have a quotient map $p : \pi(X) \to H \times K$, and* [7, Proposition 19.5] *shows that there exists an epimorphism $\varepsilon : H \times K \to G$, as claimed. Finally, $\varepsilon p : \pi(X) \to G$ is surjective as a composition of two epimorphisms and the result follows.*

Looking at Theorem 1.3, we choose $H = Q_8$ and so both (i) and (ii) of Theorem 1.3 are satisfied by H; then we choose $K = \mathbb{Z}(4)$, again this group satisfies both (i) and (ii) of Theorem 1.3. Therefore Theorem 1.1 becomes a corollary of Theorem 1.3. This is a first immediate application.

On the other hand, we may say more in connection with graph theory. Following [11, Chapter IV.A, Definition 2, p.76], a finite group G with generating set S may be always obtained as homomorphic image via the natural epimorphism $\alpha : F(S) \to G$ from the free group on S to G and the *word length* $l_S(x)$ of an element $x \in G$ is the smallest integer n for which there exists a sequence of elements $(s_1, s_2, \ldots, s_n)$ of $S \cup S^{-1}$ such that $x = \alpha(s_1 s_2 \cdots s_n)$. The *word metric* on G is defined by $d_S(x, y) = l_S(x^{-1}y)$ and in this way G is a metric space and G acts on itself from the left by isometries. In this situation, the *Cayley graph* $\Gamma(G, S)$ of G is the simple graph with vertex set G and in which two vertices x, y are the two ends of an edge if and only if $d_S(x, y) = 1$. In order to avoid loops in $\Gamma(G, S)$ one has to forbid the presence of the unit of G in S. Note that $\Gamma(G, S)$ is

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connected since S generates G. Classical properties of Cayley graphs are described in [11, Chapters IV and V].

We want to give a rough idea of construction of the Cayley graph of the Pauli group $P = Q_8 \circ \mathbb{Z}(4)$ via a method which relies on the ideas of construction of central product of groups.

Consider the Cayley graph of $\mathbb{Z}(4)$ for (1.2)

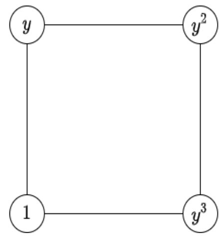

FIGURE 1. The Cayley graph of $\mathbb{Z}(4)$.

successively that of Q_8 for (1.3).

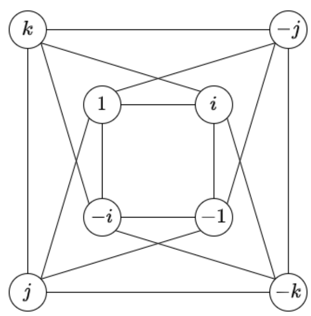

FIGURE 2. The Cayley graph of Q_8.

There are results in literature, showing that it is possible to construct a graph from two prescribed ones mimicking the ideas of the semidirect products of groups or of the wreath products of groups (see [1, 8, 9, 17, 18]). One of the main problems is to understand how the topological properties of the original graphs are preserved, or lost, along the construction.

Remark 2.2. *From the algebraic perspective, it is possible to construct a presentation for P, following prescribed transformations and starting from the presentations of Q_8 and $\mathbb{Z}(4)$. Roughly speaking one obtains (1.1) by*

1. *taking generators and relations from (1.2) and (1.3),*
2. *recognizing the centers $Z(Q_8) = \{1, -1\}$ and $Z(\mathbb{Z}(4)) = \mathbb{Z}(4)$,*
3. *identifying the cyclic subgroup of order two in $Z(Q_8)$ and $Z(\mathbb{Z}(4))$.*

This process is described in [3], and inspired to more general techniques in [14].

At the level of $\Gamma(P, \{i, j, y\})$, we may argue in a similar way, beginning from $\Gamma(\mathbb{Z}(4), \{y\})$ and $\Gamma(Q_8, \{i, j\})$, but it is more delicate to understand the combinatorial properties which are possessed

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by $\Gamma(\mathbb{Z}(4), \{y\})$ and $\Gamma(Q_8, \{i, j\})$ and then lost, or preserved along the construction. In other words, it is interesting to:

Problem 2.3. *Understand the structure of central products for Cayley graphs.*

The above problem is relatively clear for wreath products, direct products and other forms of products in [8, 17, 18], but apparently there is not too much literature for the case of central products of two graphs. Why should we be interested in Cayley graphs in the present context?

Remark 2.4. *The operations 1, 2 and 3 of Remark 2.2 may be formalized at the level of geometric group theory for Cayley graphs of groups via the notion of Tietze transformation (see [11]). In fact properties of the Cayley graphs correspond to properties of the word metric on the group, and viceversa. Therefore the notion of central product for Cayley graphs could offer an alternative (perhaps more general) geometric point of view for the proof of Theorem 1.3, involving combinatorics of words and formal languages.*

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