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THE MOSTAR AND WIENER INDEX OF ALTERNATE LUCAS CUBES

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Abstract. The Wiener index and the Mostar index quantify two distance related properties of connected graphs: the Wiener index is the sum of the distances over all pairs of vertices and the Mostar index is a measure of how far the graph is from being distance-balanced. These two measures have been considered for a number of interesting families of graphs. In this paper, we determine the Wiener index and the Mostar index of alternate Lucas cubes. Alternate Lucas cubes form a family of interconnection networks whose recursive construction mimics the construction of the well-known Fibonacci cubes.

1. Introduction

The hypercube graph Q_n of dimension n (also called n-cube) is one of the basic models for interconnection networks. The vertex set of Q_n is represented by the set of all binary strings of length n and two vertices are made adjacent if and only if they differ in exactly one bit. Fibonacci cubes and (classical) Lucas cubes were introduced as new models of computation for interconnection networks [11, 15]. Both families are subgraphs of Q_n. These cubes decompose into two lower dimensional Fibonacci cubes and a perfect matching. They have interesting structural and enumerative properties which have been studied extensively [11, 13, 15]. *Alternate Lucas cubes* were introduced in [9] as an alternative for the Lucas cubes. They have a useful fundamental decomposition similar to that of the Fibonacci cubes; just as Fibonacci cubes are constructed from two lower dimensonal Fibonacci cubes and a perfect matching, alternate Lucas cubes are constructed from two lower dimensional alternate Lucas cubes and a perfect matching. They have many interesting structural and enumerative properties [9]. Here we remark that, the number of vertices and the number of edges of alternate Lucas cubes

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are equinumerous with those of the corresponding classical Lucas cubes, but they are not isomorphic graphs.

Distance related properties of graphs such as the Wiener index, irregularity and Mostar index have been studied for various families of graphs in the literature. Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The *Wiener index* $W(G)$ is defined as the sum of distances over all unordered pairs of vertices of G, that is,

$$W(G) = \sum_{\{u,v\} \in V(G)} d(u,v)$$

where the distance $d(u,v)$ is the length of the shortest path in G between u and v. It is determined for Fibonacci cubes and Lucas cubes in [14]. The irregularity of a graph measures how much the graph differs from a regular graph and *Albertson index* (irregularity) is defined as the sum of $|deg(u) - deg(v)|$ over all edges $uv \in E(G)$ [1]. The irregularity of Fibonacci cubes and Lucas cubes are studied in [3, 7] and the irregularity of alternate Lucas cubes are presented in [9]. For any $uv \in E(G)$ let $n_u(G)$ denote the number of vertices in $V(G)$ that are closer (w.r.t. the standard shortest path metric) to u than to v; and let $n_v(G)$ denote the number of vertices in $V(G)$ that are closer to v than to u. The *Mostar index* of G is defined in [6] as

$$\mathrm{Mo}(G) = \sum_{uv \in E(G)} |n_u(G) - n_v(G)| \ .$$

This index measures how far the given graph G is from being distance-balanced (see, [12]). For a survey on Mostar index of graphs see [2]. When G (and the edge $uv \in E(G)$) is clear from the context we will use $n_u = n_u(G)$ and $n_v = n_v(G)$. The relation between the Mostar index and the irregularity of graphs and their difference are investigated in [10]. Recently, the Mostar index of trees, product graphs, Fibonacci cubes and Lucas cubes have been investigated in [4, 8].

In this paper, we determine the Wiener index and the Mostar index of alternate Lucas cubes. In Section 2 we present the structure of alternate Lucas cubes and structural results we need for the calculation of these indices. In Section 3 we obtain an expression for the Mostar index of alternate Lucas cubes and we derive a closed formula for this expression in Section 4. Finally, we present the Wiener index of alternate Lucas cubes in Section 5.

2. Preliminaries

We use the notation $[n] = \{1, 2, \ldots, n\}$ for any $n \in \mathbb{Z}^+$. Let $B = \{0, 1\}$ and

$$B_n = \{b_1 b_2 \cdots b_n \mid b_i \in B, \ \forall i \in [n]\}$$

denote the set of all binary strings of length n. The n-dimensional hypercube Q_n is the simple graph with vertices represented by the 2^n binary strings in B_n. The edges of Q_n are between pairs of vertices whose binary representations differ in exactly one bit position. The n-dimensional Fibonacci cube Γ_n

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is the induced subgraph of Q_n, is obtained by removing all vertices containing consecutive 1's from Q_n. The vertices of Γ_n can be represented by the set FB_n of all *Fibonacci strings* of length n

$$FB_n = \{b_1 b_2 \cdots b_n \mid b_i \cdot b_{i+1} = 0,\ \forall i \in [n-1]\} \subseteq B_n \ .$$

The number of vertices of Γ_n is equal to $|FB_n| = f_{n+2}$, where $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ are the Fibonacci numbers. If we remove the vertices with $b_1 = b_n = 1$ from Γ_n, then one obtains the n-dimensional classical Lucas cube Λ_n, whose vertices can be represented by the set all Lucas strings of length n

$$\{b_1 b_2 \cdots b_n \mid b_i \cdot b_{i+1} = 0,\ \forall i \in [n-1] \text{ and } b_1 \cdot b_n = 0\} \subseteq B_n \ .$$

For $n \geq 1$, Λ_n has L_n vertices, where $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ are the Lucas numbers.

Every positive integer can be represented uniquely as the sum of distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This representation is called the *Zeckendorf representation*. By convention we assume that the integer 0 is represented by the n-bit string $(0 \cdots 0)$ when we are considering n-dimensional graphs with binary labels. A similar representation of integers using Lucas numbers is considered in [5] where it is shown that every positive integer m can be expressed uniquely as a sum of distinct Lucas numbers in the form

$$m = \sum_{i=0}^{n-1} b_{n-i} L_i \ ,$$

where $b_i \cdot b_{i+1} = 0$ for $1 \leq i \leq n-1$ and $b_n \cdot b_{n-2} = 0$. We call this representation the *Lucas representation* of integers. We will refer to the binary encoding of an integer via its coefficients b_i in this representation as its *binary alternate Lucas string*. By removing the vertices with $b_{n-2} = b_n = 1$ from Γ_n, one obtains the n-dimensional alternate Lucas cubes $\mathcal{L}_n$ [9], whose vertices can be represented by the set all binary alternate Lucas strings of length n

$$LB_n = \{b_1 b_2 \cdots b_n \mid b_i \cdot b_{i+1} = 0,\ \forall i \in [n-1] \text{ and } b_{n-2} \cdot b_n = 0\} \subseteq B_n \ .$$

For $n \geq 1$, $\mathcal{L}_n$ also has L_n vertices. It is clear from the above definitions that $\mathcal{L}_1 = \Lambda_1 = \Gamma_1 = K_1$ (complete graph with 1 vertex), $\mathcal{L}_2 = \Lambda_2 = P_3$ (path graph on 3 vertices) and $\mathcal{L}_3 = \Lambda_3$.

The following *fundamental decomposition* of Γ_n, Λ_n and $\mathcal{L}_n$ can be obtained easily from the definitions:

(1) For the Fibonacci cubes, the set FB_n splits into two subsets depending on $b_1 = 0$ or $b_1 = 1$. Therefore, Γ_n decomposes into a subgraph Γ_{n-1}, whose vertices are given by the strings that start with 0, and a subgraph Γ_{n-2} whose vertices are given by the strings that start with 10. This decomposition can be denoted as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$$

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where there is a perfect matching between $10\Gamma_{n-2}$ and $00\Gamma_{n-2} \subseteq 0\Gamma_{n-1}$, which are two copies of Γ_{n-2}.

(2) The Lucas cube Λ_n has a subgraph Γ_{n-1} whose vertices are represented by the Fibonacci strings starting with 0 and a subgraph Γ_{n-3} whose vertices are given by the Fibonacci strings that start with 10 and end with 0. Using these subgraphs Λ_n can be decomposed as

$$\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0$$

where there is a perfect matching between $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0 \subseteq 0\Gamma_{n-1}$. Here we note that the lower dimensional graphs that appear in the decomposition of Λ_n are not Lucas cubes, but Fibonacci cubes.

(3) For the alternate Lucas cube $\mathcal{L}_n$, the set LB_n is also splits into two subsets depending on $b_1 = 0$ or $b_1 = 1$. Therefore, $\mathcal{L}_n$ can be decomposed into two subgraphs induced by the vertices that start with 0 and 10 respectively. The vertices that start with 0 constitute a graph isomorphic to $\mathcal{L}_{n-1}$ and the vertices that start with 10 constitute a graph isomorphic to $\mathcal{L}_{n-2}$. For $n \geq 3$, the decomposition of $\mathcal{L}_n$ can be represented as

$$\mathcal{L}_n = 0\mathcal{L}_{n-1} + 10\mathcal{L}_{n-2} \tag{2.1}$$

where there is a perfect matching between $10\mathcal{L}_{n-2}$ and $00\mathcal{L}_{n-2} \subseteq 0\mathcal{L}_{n-1}$, analogous to the decomposition of Fibonacci cubes.

3. The Mostar index of alternate Lucas cubes

By direct inspection we observe that $\mathrm{Mo}(\mathcal{L}_1) = 0$, $\mathrm{Mo}(\mathcal{L}_2) = 2$ and $\mathrm{Mo}(\mathcal{L}_3) = 6$.

Lemma 3.1. *For $n \geq 3$, assume that $uv \in E(\mathcal{L}_n)$ with $u_k = 0$ and $v_k = 1$ for some $k \in [n]$. Then for $k \leq n-2$ we have $n_u(\mathcal{L}_n) = f_{k+1}L_{n-k}$ and $n_v(\mathcal{L}_n) = f_kL_{n-k-1}$; and for $k \in \{n-1, n\}$ we have $n_u(\mathcal{L}_n) = f_{n+1}$ and $n_v(\mathcal{L}_n) = f_{n-1}$.*

Proof. Assume that $1 < k < n-2$ and let $\alpha \in V(\mathcal{L}_n)$ have string representation $a_1a_2\cdots a_n$. Since $uv \in E(\mathcal{L}_n)$, u and v must be of the form $b_1\cdots b_{k-2}000b_{k+2}\cdots b_n$ and $b_1\cdots b_{k-2}010b_{k+2}\cdots b_n$, respectively. Then the difference between $d(\alpha,u)$ and $d(\alpha,v)$ depends on the value of a_k. If $a_k = 0$ we have $d(\alpha,u) = d(\alpha,v)-1$ and if $a_k = 1$ we have $d(\alpha,u) = d(\alpha,v)+1$. Therefore, $n_u(\mathcal{L}_n)$ and $n_v(\mathcal{L}_n)$ are equal to the number of vertices in $\mathcal{L}_n$ whose kth coordinate is 0 and 1, respectively. These vertices have string representation of the form $\beta_1 0\beta_2$ and $\beta_3 010\beta_4$ where β_1 and β_3 are Fibonacci strings of length $k-1$ and $k-2$; β_2 and β_4 are alternate Lucas strings of length $n-k$ and $n-k-1$. Consequently, $n_u(\mathcal{L}_n) = f_{k+1}L_{n-k}$ and $n_v(\mathcal{L}_n) = f_kL_{n-k-1}$.

For the case $k = 1$ we have $u \in V(0\mathcal{L}_{n-1})$ and $v \in V(10\mathcal{L}_{n-2})$. Then $n_u(\mathcal{L}_n) = |V(0\mathcal{L}_{n-1})| = L_{n-1} = f_2L_{n-1}$ and $n_v(\mathcal{L}_n) = |V(10\mathcal{L}_{n-2})| = L_{n-2} = f_1L_{n-2}$.

For $k = n-2$, $n_u(\mathcal{L}_n)$ counts the number of vertices in $\mathcal{L}_n$ having string representations $\beta000$, $\beta001$ and $\beta010$, which gives $n_u(\mathcal{L}_n) = 3f_{n-1} = f_{n-1}L_2$ since β can be any Fibonacci string of length

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$n-3$. Similarly, $n_v(\mathcal{L}_n) = f_{n-2} = f_{n-2}L_1$ since it counts the number of vertices in $\mathcal{L}_n$ having string representation of the form $\beta'0100$.

Similarly, for $k = n-1$, $n_u(\mathcal{L}_n)$ and $n_v(\mathcal{L}_n)$ count the number of vertices in $\mathcal{L}_n$ having string representations $\beta a_{n-2}0a_n$ and $\beta 010$ respectively. Since $a_{n-2}0a_n \in LB_3$, we have $\beta \in FB_{n-3}$ if $a_{n-2} = 0$ ($a_n \in B$) and $\beta \in FB_{n-4}$ if $a_{n-2} = 1$ ($a_n = 0$). Then we get $n_u(\mathcal{L}_n) = 2f_{n-1} + f_{n-2} = f_{n+1}$ and $n_v(\mathcal{L}_n) = f_{n-1}$.

Finally for $k = n$, $n_u(\mathcal{L}_n)$ and $n_v(\mathcal{L}_n)$ count the number of vertices in $\mathcal{L}_n$ having string representations $\beta 0$ and $\beta' 001$ respectively, which immediately gives $n_u(\mathcal{L}_n) = f_{n+1}$ and $n_v(\mathcal{L}_n) = f_{n-1}$. □

To find the Mostar index of the alternate Lucas cube $\mathcal{L}_n$, we need to find the number of edges $uv \in E(\mathcal{L}_n)$ for which $u_k = 0$ and $v_k = 1$ for each fixed $k \in [n]$ and add up these contributions over k.

Lemma 3.2. *For $n \geq 3$, assume that $uv \in E(\mathcal{L}_n)$ with $u_k = 0$ and $v_k = 1$ for some $k \in [n]$. Then the number of such edges in $\mathcal{L}_n$ is equal to $f_k L_{n-k-1}$ for $k \leq n-2$, and is equal to f_{n-1} for $k \in \{n-1, n\}$.*

Proof. As in the proof of Lemma 3.1 for $1 < k < n-1$, we know that u and v are of the form $a_1 \cdots a_{k-2}000a_{k+2} \cdots a_n$ and $a_1 \cdots a_{k-2}010a_{k+2} \cdots a_n$. Therefore the number edges uv in $\mathcal{L}_n$ satisfying $u_k = 0$ and $v_k = 1$ is equal to the number of vertices of the form $a_1 \cdots a_{k-2}000a_{k+2} \cdots a_n$, which gives the desired result.

For the boundary case $k = 1$ we need to find the number of vertices of the form $00a_3 \cdots a_n$ and for the cases $k = n-1$ and $k = n$ we need to find the number of vertices of the form $a_1 \cdots a_{n-3}000$. Clearly, these numbers are equal to $|LB_{n-2}| = L_{n-2} = f_1 L_{n-2}$ and $|FB_{n-3}| = f_{n-1}$. This completes the proof. □

Using Lemma 3.1 and Lemma 3.2 we obtain the following main result.

Theorem 3.3. *For $n \geq 1$, the Mostar index of alternate Lucas cube $\mathcal{L}_n$ is given by*

$$\mathrm{Mo}(\mathcal{L}_n) = 2f_{n-1}f_n + \sum_{k=1}^{n-2} f_k L_{n-k-1} \left(f_k L_{n-k-2} + f_{k-1} L_{n-k} \right) . \tag{3.1}$$

In the next section we derive a closed form formula for the expression (3.1).

4. A closed formula for $\mathrm{Mo}(\mathcal{L}_n)$

By the fundamental decomposition (2.1) of $\mathcal{L}_n$, the set of edges $E(\mathcal{L}_n)$ consists of three distinct types:

(1) The edges in $0\mathcal{L}_{n-1}$, which we denote by $E(0\mathcal{L}_{n-1})$.
(2) The link edges between $10\mathcal{L}_{n-2}$ and $00\mathcal{L}_{n-2} \subset 0\mathcal{L}_{n-1}$, which we denote by LE_n.
(3) The edges in $10\mathcal{L}_{n-2}$, which we denote by $E(10\mathcal{L}_{n-2})$.

In other words we have the partition

$$E(\mathcal{L}_n) = E(0\mathcal{L}_{n-1}) \cup LE_n \cup E(10\mathcal{L}_{n-2}) \ .$$

We keep track of the contribution of each part of this decomposition by setting for $n \geq 3$,

$$M_n(x,y,z) = \sum_{uv \in E(0\mathcal{L}_{n-1})} |n_u - n_v| x + \sum_{uv \in LE_n} |n_u - n_v| y + \sum_{uv \in E(10\mathcal{L}_{n-2})} |n_u - n_v| z \ . \tag{4.1}$$

Clearly, $\mathrm{Mo}(\mathcal{L}_n) = M_n(1,1,1)$. We define

$$\begin{aligned} M_0 &= M_1 = 0 \\ M_2 &= y + z \end{aligned} \tag{4.2}$$

and by direct inspection calculate

$$\begin{aligned} M_3 &= 4x + 2y \\ M_4 &= 11x + 3y + 6z \\ M_5 &= 42x + 12y + 17z \ . \end{aligned} \tag{4.3}$$

The initial conditions (4.2) and (4.3) in turn give

$$\begin{aligned} \mathrm{Mo}(\mathcal{L}_2) &= M_2(1,1,1) = 2 \\ \mathrm{Mo}(\mathcal{L}_3) &= M_3(1,1,1) = 6 \\ \mathrm{Mo}(\mathcal{L}_4) &= M_4(1,1,1) = 20 \\ \mathrm{Mo}(\mathcal{L}_5) &= M_5(1,1,1) = 71 \end{aligned}$$

consistent with the values that are calculated using Theorem 3.3.

By using the fundamental decomposition (2.1) of $\mathcal{L}_n$ and mimicking the proof of [8, Proposition 1] we obtain the following useful result.

Proposition 4.1. *For $n \geq 4$ the polynomial $M_n(x,y,z)$ satisfies*

$$M_n(x,y,z) = M_{n-1}(x+z, 0, x) + M_{n-2}(2x+z, x+z, x+z) + L_{n-3}(L_{n-2} + L_{n-4})x + L_{n-2}L_{n-3}y$$

where $M_0 = M_1 = 0$, $M_2(x,y,z) = y+z$ and $M_3(x,y,z) = 4x+2y$.

Proof. Considering the sums in definition (4.1), there are three cases to consider:

(1) $uv \in LE_n$ such that $u \in V(0\mathcal{L}_{n-1})$ and $v \in V(10\mathcal{L}_{n-2})$:
We know that $d(u,v) = 1$ and the string representations of u and v must be of the form $00b_3 \cdots b_n$ and $10b_3 \cdots b_n$, respectively. Then using Lemma 3.1 with $k = 1$ we have $|n_u - n_v| = L_{n-1} - L_{n-2} = L_{n-3}$ for each edge uv in LE_n. As $|LE_n| = L_{n-2}$ all of these edges contribute $L_{n-2}L_{n-3}y$ to $M_n(x,y,z)$.

(2) $uv \in E(10\mathcal{L}_{n-2})$:
Let the string representations of u and v be $10u_3 \cdots u_n$ and $10v_3 \cdots v_n$, respectively. Using the fundamental decomposition of $\mathcal{L}_n$ there exist vertices of the form $u' = 0u_3 \cdots u_n$ and $v' = 0v_3 \cdots v_n$ in $V(\mathcal{L}_{n-1})$; $u'' = u_3 \cdots u_n$ and $v'' = v_3 \cdots v_n$ in $V(\mathcal{L}_{n-2})$. Then n_u counts the number of vertices $0\alpha \in V(0\mathcal{L}_{n-1})$ and $10\beta \in V(10\mathcal{L}_{n-2})$ satisfying $d(0\alpha, u) < d(0\alpha, v)$

and $d(10\beta, u) < d(10\beta, v)$. For any $0\alpha \in V(0\mathcal{L}_{n-1})$ we know that $d(0\alpha, u) = d(\alpha, u') + 1$ and $d(0\alpha, v) = d(\alpha, 0v') + 1$. Therefore, for a fixed $0\alpha \in V(0\mathcal{L}_{n-1})$, $d(\alpha, u') < d(\alpha, v')$ if and only if $d(0\alpha, u) < d(0\alpha, v)$. Similarly, for any $10\beta \in V(10\mathcal{L}_{n-2})$ we have $d(10\beta, u) = d(\beta, u'')$ and $d(\beta, v) = d(\beta, v'')$. Then we can write

$$\begin{aligned}\sum_{uv \in E(10\mathcal{L}_{n-2})} \left|n_u(\mathcal{L}_n) - n_v(\mathcal{L}_n)\right| &= \sum_{u'v' \in E(\mathcal{L}_{n-1})} \left|n_{u'}(\mathcal{L}_{n-1}) - n_{v'}(\mathcal{L}_{n-1})\right| \\ &\quad + \sum_{u''v'' \in E(\mathcal{L}_{n-2})} \left|n_{u''}(\mathcal{L}_{n-2}) - n_{v''}(\mathcal{L}_{n-2})\right| .\end{aligned}$$

Note that $\mathcal{L}_{n-1} = 0\mathcal{L}_{n-2} + 10\mathcal{L}_{n-3}$ and the edge $u'v' \in E(\mathcal{L}_{n-1})$ is an edge in the set $E(0\mathcal{L}_{n-2})$. Furthermore $u''v'' \in E(\mathcal{L}_{n-2})$ is an arbitrary edge. Then by the definition (4.1) of M_n we have

$$\sum_{u'v' \in E(\mathcal{L}_{n-1})} \left|n_{u'}(\mathcal{L}_{n-1}) - n_{v'}(\mathcal{L}_{n-1})\right| = M_{n-1}(1, 0, 0)$$

and

$$\sum_{u''v'' \in E(\mathcal{L}_{n-2})} \left|n_{u''}(\mathcal{L}_{n-2}) - n_{v''}(\mathcal{L}_{n-2})\right| = M_{n-2}(1, 1, 1) .$$

Hence all of these edges $uv \in E(10\mathcal{L}_{n-2})$ contribute $\big(M_{n-1}(1, 0, 0)+M_{n-2}(1, 1, 1)\big)z$ to $M_n(x, y, z)$.

(3) $uv \in E(0\mathcal{L}_{n-1})$:

Since $0\mathcal{L}_{n-1} = 00\mathcal{L}_{n-2} + 010\mathcal{L}_{n-3}$ we have three subcases to consider here.

(a) $uv \in LE_{n-1}$ such that $u \in 00\mathcal{L}_{n-2}$ and $v \in 010\mathcal{L}_{n-3}$:

Then using Lemma 3.1 with $k = 2$ we have $|n_u - n_v| = f_3 L_{n-2} - f_2 L_{n-3} = 2L_{n-2} - L_{n-3} = L_{n-2} + L_{n-4}$ for each edge uv in LE_n. As $|LE_{n-1}| = L_{n-3}$ all of these edges contribute $L_{n-3}(L_{n-2} + L_{n-4})x$ to $M_n(x, y, z)$.

(b) $uv \in E(010\mathcal{L}_{n-3})$:

Let the string representations of u and v are of the form $010u_4 \cdots u_n$ and $010v_4 \cdots v_n$ respectively. Using the fundamental decomposition of $\mathcal{L}_n$ there exist vertices of the form $u' = 000u_4 \cdots u_n$ and $v' = 000v_4 \cdots v_n$ in $V(0\mathcal{L}_{n-1})$; $u'' = 0u_4 \cdots u_n$ and $v'' = 0v_4 \cdots v_n$ in $V(\mathcal{L}_{n-2})$. Then for any $10\alpha \in V(10\mathcal{L}_{n-2})$ we know that $d(10\alpha, u) = d(10\alpha, u') + 1 = d(\alpha, u'') + 2$ and we know that $d(10\alpha, v) = d(10\alpha, v') + 1 = d(\alpha, v'') + 2$. Therefore for all $10\alpha \in V(10\mathcal{L}_{n-2})$ we count their total contribution to M_n by $M_{n-2}(1, 0, 0)x$ in this case. Furthermore, as $uv \in E(010\mathcal{L}_{n-3})$ we have $uv \in E(0\mathcal{L}_{n-1})$, and for all $0\alpha \in V(0\mathcal{L}_{n-1})$ we count their total contribution to M_n by $M_{n-1}(0, 0, 1)x$ by using the definition of M_{n-1}. Hence, the edges $uv \in E(010\mathcal{L}_{n-3})$ contribute $\big(M_{n-1}(0, 0, 1) + M_{n-2}(1, 0, 0)\big)x$ to $M_n(x, y, z)$.

(c) $uv \in E(00\mathcal{L}_{n-2})$.

These edges are the ones of $E(0\mathcal{L}_{n-1})$ that are not in $E(010\mathcal{L}_{n-3})$ and LE_{n-1} (not created during the connection of $00\mathcal{L}_{n-2}$ and $010\mathcal{L}_{n-3}$). Then similar to the Case 2 and using the definition (4.1) of M_n these edges contribute $\big(M_{n-1}(1, 0, 0) + M_{n-2}(1, 1, 1)\big)x$ to $M_n(x, y, z)$.

Combining all of the above cases and noting $M_{n-1}(0,0,1)x = M_{n-1}(0,0,x)$, $M_{n-2}(1,0,0)x = M_{n-2}(x,0,0)$, $M_{n-2}(1,1,1)x = M_{n-2}(x,x,x)$ we complete the proof. □

If we write $M_n(x,y,z) = a_n x + b_n y + c_n z$, then from the recursion in Proposition 4.1, we obtain for $n \geq 4$ the system of equations

$$\begin{aligned} a_n &= a_{n-1} + c_{n-1} + 2a_{n-2} + b_{n-2} + c_{n-2} + L_{n-3}(L_{n-2} + L_{n-4}) \\ b_n &= L_{n-2}L_{n-3} \\ c_n &= a_{n-1} + a_{n-2} + b_{n-2} + c_{n-2}\ . \end{aligned} \tag{4.4}$$

The initial values for $0 \leq n \leq 3$ are given by (4.2) and (4.3). Let $A(t), B(t), C(t)$ be the generating functions of the sequences a_n, b_n, c_n, $(n \geq 0)$, respectively. We have the auxiliary result that ([16, A215602])

$$D(t) = \sum_{n \geq 4} L_{n-2}L_{n-3}t^n = \frac{t^4(3 + 6t - 2t^2)}{(1+t)(1-3t+t^2)}\ . \tag{4.5}$$

From (4.4) with (4.2) and (4.3) we obtain the system of equations

$$\begin{aligned} A(t) &= (t + 2t^2)A(t) + t^2 B(t) + (t + t^2)C(t) + (1+t)D(t) + 3t^3 + 2t^4 \\ B(t) &= D(t) + t^2 + 2t^3 \\ C(t) &= (t + t^2)A(t) + t^2 B(t) + t^2 C(t) + t^2\ . \end{aligned} \tag{4.6}$$

Solving the system (4.6) and using (4.5) we find

$$\begin{aligned} A(t) &= \frac{t^3(4 - 5t - 2t^2 + 2t^3)}{(1+t)^2(1-3t+t^2)^2} \\ B(t) &= \frac{t^2(1 - 3t^2 + 3t^3)}{(1+t)(1-3t+t^2)} \\ C(t) &= \frac{t^2(1 - 4t + 6t^2 + 3t^3 - 6t^4 + 9t^5 - 3t^6)}{(1+t)^2(1-3t+t^2)^2}\ . \end{aligned}$$

Since $\mathrm{Mo}(\Gamma_n) = M_n(1,1,1) = a_n + b_n + c_n$, adding the generating functions $A(t), B(t), C(t)$ gives

$$\sum_{n \geq 0} \mathrm{Mo}(\mathcal{L}_n)t^n = \frac{t^2(2 - 2t - 4t^2 + 11t^3 - 4t^4)}{(1+t)^2(1-3t+t^2)^2} = 2t^2 + 6t^3 + 20t^4 + 71t^5 + 220t^6 + \cdots \tag{4.7}$$

Using partial fractions decomposition in (4.7) and the expansions

$$\begin{aligned} \frac{1}{1-3t+t^2} &= \sum_{n \geq 0} f_{2n+2}t^n \\ \frac{1}{(1-3t+t^2)^2} &= \sum_{n \geq 0} \tfrac{1}{5}\big((4n+2)f_{2n+2} + (3n+3)f_{2n+1}\big)t^n \end{aligned} \tag{4.8}$$

we obtain after some simplification the following result.

Theorem 4.2. *For $n \geq 1$, the Mostar index of alternate Lucas cube $\mathcal{L}_n$ is given by*

$$\mathrm{Mo}(\mathcal{L}_n) = \frac{1}{25}\Big((16L_{2n} + (5n-28)L_{2n-1} - (15n-40)(-1)^n\Big) \ . \tag{4.9}$$

The expression in (4.9) for $\mathrm{Mo}(\mathcal{L}_n)$ is a closed form evaluation of the sum given in Theorem 3.3.

5. The Wiener index of Alternate Lucas cubes

By mimicking the proof of [14, Theorem 3.1] and using Lemma 3.1 we obtain the following result.

Theorem 5.1. *For $n \geq 1$, the Wiener index of alternate Lucas cube $\mathcal{L}_n$ is*

$$W(\mathcal{L}_n) = 2f_{n+1}f_{n-1} + \sum_{k=1}^{n-2} f_k f_{k+1} L_{n-k-1} L_{n-k} \ . \tag{5.1}$$

Next we obtain a closed form expression for the formula (5.1). Note that we have the generating functions

$$\begin{aligned} F(t) &= \sum_{k\geq 1} f_k f_{k+1} t^k = \frac{t}{(1+t)(1-3t+t^2)} \\ G(t) &= \sum_{k\geq 1} L_k L_{k+1} t^k = \frac{t(3+6t-2t^2)}{(1+t)(1-3t+t^2)} \\ H(t) &= \sum_{k\geq 1} f_{k-1} f_{k+1} t^k = \frac{t^2(2-t)}{(1+t)(1-3t+t^2)} \ . \end{aligned}$$

The sum that appears in (5.1) is a convolution. Therefore the generating function of $W(\mathcal{L}_n)$ is

$$\sum_{n\geq 1} W(\mathcal{L}_n) t^n = 2H(t) + tF(t)H(t) = \frac{t^2(4-7t+2t^2+6t^3-2t^4)}{(1+t)^2(1-3t+t^2)^2} \ . \tag{5.2}$$

From (5.2), first few terms of the sequence $W(\mathcal{L}_n)$ $(n \geq 1)$ are

$$0, 4, 9, 38, 118, 380, 1156, 3476, 10247, 29862, \ldots$$

in agreement with the values obtained from (5.1).

By using partial fractions decomposition of (5.2) and the expansions in (4.8) we obtain the following expression for the Wiener index of alternate Lucas cubes.

Theorem 5.2. *For $n \geq 1$, the Wiener index of the alternate Lucas cube $\mathcal{L}_n$ is given in closed form by*

$$W(\mathcal{L}_n) = \frac{1}{25}\Big((5n-6)L_{2n-1} + (5n+8)L_{2n-2} + (5n+20)(-1)^n\Big) \ . \tag{5.3}$$

We find from (5.3) that the asymptotic expression for the Wiener index of alternate Lucas cubes is

$$W(\mathcal{L}_n) \sim \tfrac{1}{5} n L_{2n} \ .$$

It is interesting that using the expression (4.9), we find that $\mathrm{Mo}(\mathcal{L}_n)$ is asymptotically given by

$$\mathrm{Mo}(\mathcal{L}_n) \sim \tfrac{1}{5} n L_{2n-1} \ .$$

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