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CLASSIFICATION OF THE PENTAVALENT SYMMETRIC GRAPHS OF ORDER $8pq$

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Abstract. A graph X is symmetric if its automorphism group is transitive on the arc set of the graph. Let p and q be two prime integers. In this paper, a complete classification is determined of connected pentavalent symmetric graphs of order $8pq$.

1. Introduction

In this paper all graphs are assumed to be finite, simple, connected and undirected. For a graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group of X, respectively. Let G be a permutation group on a set Ω and let $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if G is transitive and semiregular.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$, and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$. A 1-arc just called an *arc*. A graph X is said to be (G,s)-*arc-transitive* or (G,s)-*regular* if G acts transitively or regularly on the set of s-arcs of X, and (G,s)-*transitive* if G acts transitively on the s-arcs but not on the $(s+1)$-arcs of X, where G is a subgroup of $\mathrm{Aut}(X)$. A graph X is said to be *s-arc-transitive*, *s-regular* and *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular and $(\mathrm{Aut}(X), s)$-transitive, respectively. In particular, 0-arc-transitive means *vertex-transitive* and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph X is *edge-transitive* if $\mathrm{Aut}(X)$ is transitive on the edge set $E(X)$.

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Characterizing symmetric graphs have been received considerable attention in the literature, beginning with a classification of symmetric graphs of prime order p [6]. Later on, lots of work was done on the classification of symmetric graphs of order pq, where p and q are two prime numbers in [7, 26, 29]. Following these, by using some results about the structure of the vertex stabilizer G_v of $v \in V(X)$ [15] and by note that a vertex- and edge-transitive graph of odd valency is symmetric [28], symmetric and semi-symmetric graphs of valency three and five with restricted order were classified. For the cubic case, see [2–4, 10–13] for example. For the pentavalent case, see [1, 19, 22, 25, 30, 31] for example. In this paper, we shall classify pentavalent symmetric graphs of order $8pq$ for prime integers q and p. Thus the main result of this paper is the following theorem.

Theorem 1.1. *Let X be a connected pentavalent symmetric graph of order $8pq$ where $q \leq p$ be two prime integers. Then X is contained in Table 1.*

TABLE 1. Pentavalent symmetric graphs of order $8pq$

X	$\mathrm{Aut}(X)$	(q, p)
$\mathcal{G}_{48}$	$\mathrm{SL}(2,5) \rtimes \mathrm{D}_8$	$(2,3)$
Q_5	$\mathbb{Z}_2^5 \rtimes \mathrm{S}_5$	$(2,2)$
$\mathcal{G}_{32}$	$(\mathrm{D}_8 \circ Q_8) \rtimes \mathrm{A}_5$	$(2,2)$
$\mathcal{G}_{72}^1$	$\mathrm{PGL}(2,9)$	$(3,3)$
$\mathcal{G}_{72}^2$	$\mathrm{Aut}(A_6) \times \mathbb{Z}_2$	$(3,3)$
$\mathcal{G}_{120}^1$	$\mathrm{A}_5 \times \mathrm{D}_{10} \times \mathbb{Z}_2$	$(3,5)$
$\mathcal{G}_{120}^2$	$\mathrm{S}_5 \times \mathrm{D}_{10}$	$(3,5)$
$\mathcal{G}_{264}^i$	$\mathrm{PGL}(2,11) \times \mathbb{Z}_2$	$(3,11),\ i=1,2,4$
$\mathcal{G}_{264}^3$	$\mathrm{PSL}(2,11) \rtimes \mathrm{D}_8$	$(3,11)$
$\mathcal{G}_{408}^1$	$\mathrm{PSO}^-(4,4)$	$(3,17)$
$\mathcal{G}_{408}^{(2)}$	$\mathrm{PSL}(2,16)$	$(3,17)$
$\mathcal{G}_{248}^2$	$\mathrm{PSL}(2,31) \times \mathbb{Z}_2$	$(2,31)$
$\mathcal{G}_{4108}^{(2)}$	$\mathrm{PSL}(2,79) \times \mathbb{Z}_2$	$(13,79)$
$\mathcal{G}_{915848}$	$\mathrm{PGL}(2,479)$	$(239,479)$

2. Preliminaries

In this section, we introduce some necessary preliminary results which will be used later. Throughout this paper, we will denote by $\mathbb{Z}_n$, F_n, D_{2n}, A_n and S_n the cyclic group of order n, the Frobenius group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively.

Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. The *quotient graph* X_N of X relative to the orbits of N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [23, Theorem 9], we have the following.

Proposition 2.1. *Let X be a connected pentavalent (G, s)-arc-transitive graph for some $s \geq 1$, and let N be a normal subgroup of G with more than two orbits on $V(X)$. Then X_N is also a pentavalent symmetric graph and N is the kernel of the action of G on $V(X_N)$. Moreover, N is semiregular on $V(X)$ and G/N is an s-arc-transitive subgroup of $\mathrm{Aut}(X_N)$.*

Let X be an arc-transitive graph. The *standard double cover* $X^{(2)}$ of X is also an arc-transitive graph, which is defined as the graph with vertex set $V(X) \times \{1, 2\}$ (Cartesian product) such that vertices (u, i) and (v, j) are adjacent if and only if $i \neq j$ and u is adjacent to v in X. It is easy to see that $X^{(2)}$ and X have the same valency and $X^{(2)}$ is connected s-transitive if and only if X is connected s-transitive and X is not a bipartite graph. By [17, Proposition 2.6], we have the following lemma.

Lemma 2.2. *Let X be a G-arc-transitive graph with $G \leqslant \mathrm{Aut}(X)$. Suppose that $N \lhd G$ acts semiregularly on $V(X)$. Then X is the standard double cover of the normal quotient graph X_N if and only if $N \cong \mathbb{Z}_2$, and there is $H \lhd \mathrm{Aut}(X)$ such that $G = N \times H$ and H has exactly two orbits on $V(X)$.*

In order to construct pentavalent symmetric graphs, we need to introduce the so called coset graph. Let G be a finite group and $H \leq G$. Suppose D is a union of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G, and the edge set $\{\{Hg, Hdg\} \big| g \in G, d \in D\}$. The graph $\mathrm{Cos}(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G [24, 27]. Clearly, $\mathrm{Cos}(G, H, D) \cong \mathrm{Cos}(G, H^\alpha, D^\alpha)$ for each $\alpha \in \mathrm{Aut}(G)$. Let S be a generator subset of G with $1 \notin S$ and $S = S^{-1}$. Clearly, the coset graph $\mathrm{Cos}(G, 1, S)$ is a connected undirected simple graph, which is called a Cayley graph on G with respect to S and denoted by $\mathrm{Cay}(G, S)$.

Proposition 2.3. *[19, Proposition 2.9] Let X be a graph and let G be a vertex-transitive subgroup of $\mathrm{Aut}(X)$. Then X is isomorphic to a coset graph $\mathrm{Cos}(G, H, D)$, where $H = G_v$ is the stabilizer of $v \in V(X)$ in G and D consists of all elements of G which map V to one of its neighbors. Further,*

(1) *X is connected if and only if D generates the group G;*
(2) *X is G-arc-transitive if and only if D is a single double coset. In particular, if $g \in G$ interchanges V and one of its neighbors, then $g^2 \in H$ and $D = HgH$;*
(3) *The valency of X equal to $|D|/|H| = |H : H \cap H^g|$.*

In the next proposition, we apply [15] to describe vertex stabilizers of connected pentavalent symmetric graphs.

Proposition 2.4. *Let X be a connected pentavalent (G, s)-transitive graph for some $G \leqslant \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. If G_v is solvable, then $|G_v| \big| 80$ and $s \leq 3$. If G_v is insolvable, then $|G_v| \big| 2^9 \cdot 3^2 \cdot 5$ and $2 \leq s \leq 5$. Furthermore, one of the following holds:*

(1) *For $s = 1$, $G_v \cong \mathbb{Z}_5$, D_{10} or D_{20};*
(2) *For $s = 2$, $G_v \cong \mathrm{F}_{20}$, $\mathrm{F}_{20} \times \mathbb{Z}_2$, A_5 or S_5;*
(3) *For $s = 3$, $G_v \cong \mathrm{F}_{20} \times \mathbb{Z}_4$, $\mathrm{A}_4 \times \mathrm{A}_5$, $\mathrm{S}_4 \times \mathrm{S}_5$ or $(\mathrm{A}_4 \times \mathrm{A}_5) \rtimes \mathbb{Z}_2$ with $\mathrm{A}_4 \rtimes \mathbb{Z}_2 = \mathrm{S}_4$ and $\mathrm{A}_5 \rtimes \mathbb{Z}_2 = \mathrm{S}_5$;*

(4) *For* $s = 4$, $G_v \cong \mathrm{ASL}(2,4)$, $\mathrm{AGL}(2,4)$, $\mathrm{A\Sigma L}(2,4)$ *or* $\mathrm{A\Gamma L}(2,4)$*;*

(5) *For* $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2,4)$.

We are now ready to present some results associated with finite groups. By [14, pp. 134-136] and [20], we have the following lemma.

Lemma 2.5. *Let* $q < p$ *be two prime integers, and let* G *be a non-abelian simple group such that* $|G| \big| 2^{12} \cdot 3^2 \cdot 5 \cdot p \cdot q$. *Then* G *is isomorphic to one of the groups listed in Table 2.*

Table 2. Some non-abelian simple $\{2, 3, 5, p, q\}-$groups

G	Order	G	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2,19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(2,31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$\mathrm{PSL}(3,4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\mathrm{PSL}(2,25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{Sz}(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$
$\mathrm{PSL}(2,2^3)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSp}(4,4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$
$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$\mathrm{PSL}(3,3)$	$2^4 \cdot 3^3 \cdot 13$	$\mathrm{PSL}(5,2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
$\mathrm{PSU}(3,3)$	$2^5 \cdot 3^3 \cdot 7$	$\mathrm{PSL}(2,2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
$\mathrm{PSU}(3,4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$\mathrm{PSL}(2,2^8)$	$2^8 \cdot 3^2 \cdot 5 \cdot 17 \cdot 257$
$\mathrm{PSL}(2,11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	$\mathrm{PSL}(2,p)$	p is an odd prime
$\mathrm{PSL}(2,2^4)$	$2^4 \cdot 3 \cdot 5 \cdot 17$		

From [21, Section 239], we have the following lemma.

Lemma 2.6. *Let* p *be a prime, and* $q = p^n \geq 5$. *Then a maximal subgroup of* $\mathrm{PSL}(2,q)$ *is isomorphic to one of the following groups:*

(1) $\mathrm{D}_{\frac{2(q-1)}{d}}$, *where* $d = (2, q-1)$ *and* $q \neq 5, 7, 9, 11$*;*

(2) $\mathrm{D}_{\frac{2(q+1)}{d}}$, *where* $d = (2, q-1)$ *and* $q \neq 7, 9$*;*

(3) $\mathbb{Z}_q \rtimes \mathbb{Z}_{\frac{(q-1)}{d}}$*;*

(4) A_4, *when* $q = p = 5$, *or* $q = p \equiv 3, 13, 27, 37 \pmod{40}$*;*

(5) S_4, *when* $q = p \equiv \pm 1 \pmod 8$*;*

(6) A_5, *when* $q = p \equiv \pm 1 \pmod 5$, *or* $q = p^2 \equiv -1 \pmod 5$, *with* p *an odd prime;*

(7) $\mathrm{PSL}(2,r)$, *when* $q = r^m$ *with* m *an odd prime;*

(8) $\mathrm{PGL}(2,r)$, *when* $q = r^2$.

For reduction, we need some information about the connected pentavalent symmetric graphs of order $4pq$, $2pq$, $16p$, and $24p$, stated in the following lemmas. Following the notation in [16, 18, 19, 22, 25], $\mathcal{G}_n$ and $\mathcal{G}_n^i$ $i \in \{1, \ldots, 4\}$, in the Tables 1, 3, 4, 5, 6 are used for the corresponding graphs of order n.

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The Icosahedron graph I_{12} is a platonic solid bounded by 20 equilateral triangles and has 12 vertices and 30 edges. The n-cube graph is the Cartesian product of n factors K_2, and denoted by Q_n. The graph CL_{16} denotes the graph obtained by connecting all long diagonal of 4-cube Q_4.

By [18, 25], we can describe pentavalent symmetric graphs of order $4pq$ in the following lemma.

Lemma 2.7. *Let X be an arc transitive connected pentavalent graph of order $4pq$, where $q < p$ are primes. Then X is isomorphic to one of the graphs in Table 3.*

Table 3. Pentavalent symmetric graphs of order $4pq$

X	$\mathrm{Aut}(X)$	(q, p)
CL_{16}	$\mathbb{Z}_2^4 \rtimes \mathrm{S}_5$	$(2, 2)$
$I_{12}^{(2)}$	$(\mathrm{A}_5 \times \mathbb{Z}_2^2) \rtimes \mathbb{Z}_2$	$(2, 3)$
$\mathcal{G}_{248}$	$\mathrm{PSL}(2, 31)$	$(2, 31)$
$\mathcal{G}_{574}^2$	$\mathrm{PSL}(2, 41) \times \mathbb{Z}_2$	$(7, 41)$
$\mathcal{G}_{4108}$	$\mathrm{PSL}(2, 79)$	$(13, 79)$

Let m and l be integers such that $m \geq 31$ and $l^4 + l^3 + l^2 + l + 1 \equiv 0 \pmod m$. Suppose $\mathcal{CD}_{2m}^l$ denotes the Cayley graph $\mathrm{Cay}(G, \{b, ab, a^{l+1}b, a^{l^2+l+1}b, a^{l^3+l^2+l+1}b\})$, where G is the dihedral group $D_{2m} = \langle a, b | a^m = b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

From [19], we give some information about pentavalent symmetric graphs of order $2pq$, in the following lemma.

Lemma 2.8. *Let X be a connected pentavalent symmetric graph of order $2pq$, where $5 \leq q < p$ are primes. Then X is isomorphic to one of the graphs in Table 4.*

Table 4. Pentavalent symmetric graphs of order $2pq$

X	$\mathrm{Aut}(X)$	(q, p)
$\mathcal{G}_{66}$	$\mathrm{PGL}(2, 11)$	$(3, 11)$
$\mathcal{G}_{42}$	$\mathrm{Aut}(\mathrm{PSL}(3, 4))$	$(3, 7)$
$\mathcal{G}_{114}$	$\mathrm{PGL}(2, 19)$	$(3, 19)$
$\mathcal{G}_{406}$	$\mathrm{PGL}(2, 29)$	$(7, 29)$
$\mathcal{G}_{3422}$	$\mathrm{PGL}(2, 59)$	$(29, 59)$
$\mathcal{G}_{3782}$	$\mathrm{PGL}(2, 61)$	$(31, 61)$
$\mathcal{G}_{574}$	$\mathrm{PSL}(2, 41)$	$(7, 41)$
$\mathcal{G}_{170}$	$\mathrm{PSp}(4, 4).\mathbb{Z}_4$	$(5, 17)$
$\mathcal{CD}_{2pq}^l$	$\mathrm{D}_{2pq} \rtimes \mathbb{Z}_5$	*for some l satisfying* $l^4 + l^3 + l^2 + l \equiv 0 \pmod{pq}$

From [16], pentavalent symmetric graphs of order $16p$, are determined in the following lemma.

Lemma 2.9. *Let X be a connected pentavalent symmetric graph of order $16p$ for a prime p. Then X is isomorphic to one of the graphs in Table 5.*

TABLE 5. Pentavalent symmetric graphs of order $16p$

X	$\mathrm{Aut}(X)$	*Remark*
Q_5	$\mathbb{Z}_2^5 \rtimes S_5$	$p = 2$
$\mathcal{G}_{32}$	$(\mathrm{D}_8 \circ \mathrm{Q}_8) \rtimes \mathrm{A}_5$	$p = 2$
$\mathcal{G}_{248}^2$	$\mathrm{PSL}(2, 31) \times \mathbb{Z}_2$	$p = 31$

From [22], pentavalent symmetric graphs of order $24p$ are determined in the following lemma.

Lemma 2.10. *Let X be a connected pentavalent symmetric graph of order $24p$ for a prime p. Then X is isomorphic to one of the graphs in Table 6.*

TABLE 6. Pentavalent symmetric graphs of order $24p$

X	$\mathrm{Aut}(X)$	*Remark*
$\mathcal{G}_{48}$	$\mathrm{SL}(2,5) \rtimes \mathrm{D}_8$	$p = 2$
$\mathcal{G}_{72}^1$	$\mathrm{PGL}(2, 9)$	$p = 3$
$\mathcal{G}_{72}^2$	$\mathrm{Aut}(\mathrm{A}_6) \times \mathbb{Z}_2$	$p = 3$
$\mathcal{G}_{120}^1$	$\mathrm{A}_5 \times \mathrm{D}_{10} \times \mathbb{Z}_2$	$p = 5$
$\mathcal{G}_{120}^2$	$\mathrm{S}_5 \times \mathrm{D}_{10}$	$p = 5$
$\mathcal{G}_{264}^i$	$\mathrm{PGL}(2, 11) \times \mathbb{Z}_2$	$p = 11, i = 1, 2, 4$
$\mathcal{G}_{264}^3$	$\mathrm{PSL}(2, 11) \rtimes \mathrm{D}_8$	$p = 11$
$\mathcal{G}_{408}^1$	$\mathrm{PSO}_{-}(4, 4)$	$p = 17$
$\mathcal{G}_{408}^2$	$\mathrm{PSL}(2, 16)$	$p = 17$

3. The Proof of Theorem 1.1

In this section, we will prove the main result of the paper. We start by restating Theorem 1.1 and giving its proof.

Theorem 1.1. Let X be a connected pentavalent symmetric graph of order $8pq$ where $q \le p$ be two prime integers. Then X is contained in Table 1.

Proof. Let X is a pentavalent symmetric graph of order $8pq$, where p and q be two prime integers such that $q \le p$. Set $A := \mathrm{Aut}(X)$. Take $v \in V(X)$. By Proposition 2.4, $|A_v| \big| 2^9 \cdot 3^2 \cdot 5$. Hence $|A| = 2^i \cdot 3^j \cdot 5 \cdot p \cdot q$ with $3 \le i \le 12$ and $0 \le j \le 2$.

If $q = 2$, then X has order $16p$ and by Lemma 2.9, we have $p \in \{2, 31\}$ and X is isomorphic to Q_5, $\mathcal{G}_{32}$ or $\mathcal{G}_{248}^{(2)}$. If $q = 3$, then X is a pentavalent symmetric graph of order $24p$. Therefore by [22], $p \in \{2, 3, 5, 11, 17\}$ and X is contained in Table 6. Now assume that $3 < q = p$. Then $|V(X)| = 2^3p^2$.

However, by [30] there does not exist a connected pentavalent symmetric graph of order 2^3p^2 for any prime $p > 3$. For the remainder of this paper, we let $5 \leq q < p$ and let N be a minimal normal subgroup of A. We consider two cases on whether A has a solvable minimal normal subgroup N or not.

First, we suppose that A contains no solvable minimal normal subgroup. Let N be an insolvable minimal normal subgroup of A. Then $N = T^r$ for some positive integer r and a non-abelian simple group T. As N is insolvable, by Proposition 2.1, N has at most two orbits on $V(X)$. Thus $4pq\big||N|$ and N has at least three distinct prime divisors. As $p\big||A|$ and $p^2 \nmid |A|$, $p^2 \nmid |N|$. It shows that $r = 1$. Hence N is a non-abelian simple group of order divisible at most five primes. Let $\pi(N)$ be the set of primes which divides the order of N. So $3 \leq |\pi(N)| \leq 5$. If $|\pi(N)| = 3$, then $\pi(N) = \{2, p, q|5 \leq q < p\}$. However, such a group does not exist (see Table 2).

Note that $3^3 \nmid |A|$ and $p^2 \nmid |A|$, therefore $3^3 \cdot p^2 \nmid |N|$. Suppose that $|\pi(N)| = 4$. If $q = 5$, then $5^2\big||N|$, so from Table 2, we have $N \cong \mathrm{PSL}(2, 25)$, $\mathrm{PSU}(3, 4)$ or $\mathrm{PSp}(4, 4)$. Now assume that, $q > 5$, then by Table 2, $N \cong \mathrm{Sz}(8)$.
Let $N \cong \mathrm{PSL}(2, 25)$, then $|N_v| = 15$ or 30. But by Magma [5], $\mathrm{PSL}(2, 25)$ has no subgroups of order 15 or 30.
Suppose that $N \cong \mathrm{PSU}(3, 4)$, then $|N_v| = 120$ or 240. However, by Magma [5], $\mathrm{PSU}(3, 4)$ has no subgroups of order 120 or 240, a contradiction. Similarly, $N \ncong \mathrm{PSp}(4, 4)$ or $\mathrm{Sz}(8)$.
Now we consider the case when $|\pi(N)| = 5$. Thus $5 < q < p$. Set $C := \mathrm{C}_A(N)$, the centralizer of N in A. By simplification of N, we have $C \cap N = 1$. Since $5^2, p^2, q^2 \nmid |A|$, and $5 \cdot p \cdot q\big||N|$, one has $5, p, q \nmid |C|$. It implies that C is a $\{2, 3\}-$group, and so C is solvable. Which is impossible because A has no solvable minimal normal subgroup. Therefore $C = 1$ and hence $N < A \lesssim \mathrm{Aut}(N)$, that is, A is an almost simple group. Since $N \lhd A$ and A_v acts primitively on the set of neighbors of v in X, $5\big||N_v|$ (if $N_v = 1$, then $|N|\big|8pq$ and $|N|$ has at most three distinct prime divisors, which is a contradiction to assumption $|\pi(N)| = 5$). Thus N is a non-abelian simple $\{2, 3, 5, p, q\}-$group. By Table 2, N is isomorphic to one of the following groups:

$$\mathrm{M}_{22},\ \mathrm{PSL}(5, 2),\ \mathrm{PSL}(2, 2^6),\ \mathrm{PSL}(2, 2^8) \text{ or } \mathrm{PSL}(2, p) \text{ (for an odd prime } p).$$

Suppose that $N \cong \mathrm{M}_{22}$. Then $|N_v| = 720$ or 1440. Since M_{22} has no subgroup of order 1440, N is transitive on $V(X)$, and by Proposition 2.4, $N_v \cong \mathrm{A}_4 \times \mathrm{A}_5$. However, by Atlas [8], M_{22} has no subgroup isomorphic to $\mathrm{A}_4 \times \mathrm{A}_5$, a contradiction.

Suppose that $N \cong \mathrm{PSL}(2, 2^6)$. Then $|N_v| = 720$ or 360. By Magma [5], $\mathrm{PSL}(2, 2^6)$ has no subgroups of order 720 or 360, a contradiction.

Suppose that $N \cong \mathrm{PSL}(5, 2)$. If N has two orbits on $V(X)$, then X is bipartite and $|N_v| = 2^8 \cdot 3^2 \cdot 5$. Recall that A is almost simple. So $N < A \lesssim \mathrm{Aut}(N)$. Since $\mathrm{Aut}(N) \cong \mathrm{PSL}(5, 2) \rtimes \mathbb{Z}_2$, we have $A \cong \mathrm{PSL}(5, 2) \rtimes \mathbb{Z}_2$ and $|A_v| = 2^6 \cdot 3^2 \cdot 5$, which is impossible by Proposition 2.4. Thus N is transitive on $V(X)$ and $N_v \cong \mathrm{A\Gamma L}(2, 4)$. From Proposition 2.3, $X = \mathrm{Cos}(N, N_v, N_v g N_v)$ where g is a $2-$element in N such that $g^2 \in N_v$ and $< N_v, g > = N$. By Magma [5], there is no such $g \in N$, a contradiction.

Now assume that, $N \cong \mathrm{PSL}(2,2^8)$. If N is transitive on $V(X)$, then $|N_v| = 2^5 \cdot 3 \cdot 5$, contradicts by Proposition 2.4. Hence N has two orbits on $V(X)$ and $|N_v| = 2^6 \cdot 3 \cdot 5$. However, by Magma [5], $\mathrm{PSL}(2,2^8)$ has no subgroups of order $2^6 \cdot 3 \cdot 5$.

Finally, assume that $N = \mathrm{PSL}(2,p)$, for an odd prime p. Then N is a $\{2,3,5,p,q\}-$group. Since $|N : N_v| = 4pq$ or $8pq$, we have $3\big||N_v|$, It shows that $|A_v| \nmid 80$, so A_v is insolvable by Proposition 2.4. Since $\mathrm{Out}(\mathrm{PSL}(2,p)) = \mathbb{Z}_2$, $A_v \leqslant N_v.\mathbb{Z}_2$, implying N_v is non-solvable. In other words N_v is an insolvable subgroup of $\mathrm{PSL}(2,p)$. As $\mathrm{Aut}(\mathrm{PSL}(2,p)) = \mathrm{PGL}(2,p)$ has no subgroup isomorphic to $\mathrm{A}_5 \rtimes \mathbb{Z}_2$, $N_v = A_v = \mathrm{A}_5$ (see [19, Proposition 2.4]). So $|N| = 2^4 \cdot 3 \cdot 5 \cdot p \cdot q$ or $2^5 \cdot 3 \cdot 5 \cdot p \cdot q$. It implies that $8\big||N| = |\mathrm{PSL}(2,p)| = \dfrac{p(p-1)(p+1)}{2}$. Since p and 16 are coprime, $16\big|(p^2-1)$, that is, $p \equiv \pm 1 \pmod 8$. Observe that $(\dfrac{p-1}{2}, \dfrac{p+1}{2}) = 1$. If $q\big|p-1$, then $p+1 = 2^i 3^j 5^k$, where $1 \le i \le 5$, $0 \le j,\ k \le 1$. Therefore we can conclude that $p \in \{7, 23, 47, 79, 239\}$. Similarly, if $q\big|p+1$, then $p \in \{7, 17, 31, 41, 241, 401, 479\}$. Since N has exactly five prime divisors, we conclude that $p = 79, 41$ and 479.

If $N \cong \mathrm{PSL}(2,79)$, then $|N_v| = 30$ or 60. Since $\mathrm{PSL}(2,79)$ has no subgroups of order 30, N has two orbits on $V(X)$. Hence $A = \mathrm{PGL}(2,79)$ and $A_v \cong \mathrm{A}_5$, by Proposition 2.4. Thus by Proposition 2.3, $X \cong \mathrm{Cos}(A, A_v, A_v g A_v)$, where g is a $2-$element in A such that $g^2 \in A_v$ and $< A_v, g > = A$. However by Magma [5], there is no such $g \in A$, a contradiction.
If $N \cong \mathrm{PSL}(2,41)$, then $|N_v| = 15$ or 30, which is not possible as $N = \mathrm{PSL}(2,41)$ has no subgroups of oder 15 and 30.

Finally, if $N \cong \mathrm{PSL}(2,479)$, then $|N_v| = 60$ or 120. Suppose that $|N_v| = 120$, then N has two orbits on $V(X)$. As A is almost simple, $A = \mathrm{PGL}(2,479)$. So $|A_v| = 120$. Proposition 2.4, implies that $A_v \cong \mathrm{S}_5$. Which is impossible, as $\mathrm{PGL}(2,479)$ has no subgroups isomorphic to S_5. Therefore N is transitive on $V(X)$, and by Proposition 2.4, N_v is isomorphic to A_5. Set $N_v := H$. By Proposition 2.3, $X \cong \mathrm{Cos}(N, H, HgH)$, for some $g \in N \setminus H$ such that $g^2 \in H$ and $< H, g >= N$. It implies that $g \in \mathrm{N}_N(H)$. By Magma [5], $\mathrm{N}_N(H) \cong \mathrm{S}_4$ and there are six involutions $g \in \mathrm{S}_4$, such that $g^2 \in H$ and $< H, g >= N$. Since these involutions are conjugate in N, the coset graphs $\mathrm{Cos}(N, H, HgH)$ corresponding to the six involutions are isomorphic to each other. It follows that $X = \mathrm{Cos}(N, H, HgH)$. We denote by $\mathcal{G}_{915848}$ the coset graph $\mathrm{Cos}(N, H, HgH)$.

We now turn to the case where A has a solvable minimal normal subgroup N. Then N is an elementary abelian group. Since $|V(X)| = 8pq$, we have $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_p$ or $\mathbb{Z}_q$. It is easy to see that N has more than two orbits on $V(X)$, (otherwise $4pq\big||N|$, a contradiction). Therefore by Proposition 2.1, N is semiregular, and the quotient graph X_N of X relative to N is a pentavalent symmetric graph with A/N as an arc-transitive group of automorphism of X_N.

If $N \cong \mathbb{Z}_2^3$, then the quotient graph X_N of X relative to N is a pentavalent symmetric graph of order pq, which is a contradiction. Because pq is an odd integer.

Suppose that $N \cong \mathbb{Z}_2$, then X_N is a pentavalent symmetric graph of order $4pq$. By Lemma 2.7, $X_N \cong \mathcal{G}_{574}^{(2)}$ or $\mathcal{G}_{4108}$.
If $X_N \cong \mathcal{G}_{574}^{(2)}$, then $q = 7$ and $p = 41$, and $A/N \lesssim \mathrm{Aut}(\mathcal{G}_{574}^{(2)}) \cong \mathrm{PSL}(2,41) \times \mathbb{Z}_2$. Furthermore, A/N

is arc-transitive on X_N. Therefore $2^2 \cdot 5 \cdot 7 \cdot 41 \big| |A/N| \big| 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 41$. By Lemma 2.6 and note on maximal subgroups of $\mathrm{PSL}(2,41)$, we have $A/N \cong \mathrm{PSL}(2,41) \times \mathbb{Z}_2$. It implies that A/N has a unique normal subgroup $H/N \cong \mathrm{PSL}(2,41)$. Since $\mathrm{Mult}(\mathrm{PSL}(2,41)) = \mathbb{Z}_2$, we have $H \cong \mathrm{PSL}(2,41) \times \mathbb{Z}_2$ or $\mathrm{SL}(2,41)$. For the former case, A has a normal subgroup $M \cong \mathrm{PSL}(2,41)$. Because $\mathrm{PSL}(2,41) \trianglelefteq H/N$ and H/N is characteristic in A/N. By Proposition 2.1, M has at most two orbits on $V(X)$. By Atlas [8], $\mathrm{PSL}(2,41)$, has no subgroups of orders 15 and 30. Hence $H \cong \mathrm{SL}(2,41)$. So A has a normal subgroup M isomorphic to $\mathrm{SL}(2,41)$. Obviously, M has at most two orbits on $V(X)$. Let M be transitive on $V(X)$. Then $|M_V| = 30$, which is impossible by Proposition 2.4. Let M has two orbits on $V(X)$. Then Proposition 2.4, implying that $M_v \cong \mathrm{A}_5 \leqslant \mathrm{SL}(2,41)$. However, by [9, Lemma 2.7], the group $\mathrm{GL}(2,a)$ for a prime $a \geq 5$ contains no non-abelian simple subgroup, which is a contradiction.
If $X_N \cong \mathcal{G}_{4108}$, then $q = 13$ and $p = 79$. By Proposition 2.1, $2^2 \cdot 5 \cdot 13 \cdot 79 \big| |A/N|$ and $A/N \leqslant \mathrm{PSL}(2,79)$. Lemma 2.6 implies that $A/N \cong \mathrm{PSL}(2,79)$. So A is isomorphic to either $\mathrm{SL}(2,79)$ or $\mathrm{PSL}(2,79) \times \mathbb{Z}_2$. Suppose that $A \cong \mathrm{SL}(2,79)$. Then $|A_v| = 60$ and by Proposition 2.4, following that $A_v \cong \mathrm{A}_5$. Arguing similarly as above, $\mathrm{SL}(2,79)$ contains no non-abelian simple group. Hence $A \ncong \mathrm{SL}(2,79)$.
If $A \cong \mathrm{PSL}(2,79) \times \mathbb{Z}_2$. Then A always has a normal subgroup M isomorphic to $\mathrm{PSL}(2,79)$. By Proposition 2.1, M has at most two orbits on the vertex set of X. Thus $|M_v| = 30$ or 60. Since $\mathrm{PSL}(2,79)$ has no subgroup of order 30, M has exactly two orbits on $V(X)$. Furthermore, since $N \cong \mathbb{Z}_2$, by Lemma 2.2, implying that X is the standard double cover of the normal quotient graph $\mathcal{G}_{4108}$, that is, $X \cong \mathcal{G}_{4108}^{(2)}$.

Suppose that $N \cong \mathbb{Z}_q$. Then X_N is a pentavalent symmetric graph of order $8p$. Since $5 \leq q < p$, from Lemma 2.7, we have $p = 31$ and $X_N \cong \mathcal{G}_{248}$. Arc transitivity of A/N on X_N implies that $2^3 \cdot 5 \cdot 31 \big| |A/N|$ and $A/N \leqslant \mathrm{Aut}(\mathcal{G}_{248})$. So we can conclude that $A = \mathbb{Z}_q \times \mathrm{PSL}(2,31)$. It follows that A has a normal subgroup $M \cong \mathrm{PSL}(2,31)$. It is easy to verify that M has at most two orbits on $V(X)$. Thus $4 \cdot q \cdot 31 \big| |M|$. It shows that $q = 5$. If M is transitive on $V(X)$, then $|M_v| = 12$, which is impossible by Proposition 2.4. If M has two orbits on $V(X)$, then $X = \mathrm{Cos}(A, H, HgH)$, where $H := A_v$ and g is a $2-$element in A such that $\frac{|H|}{|H \cap H^g|} = 5$. However, by Magma [5], there is no such $g \in A$. So $N \ncong \mathbb{Z}_q$. Similarly, we obtain that $N \ncong \mathbb{Z}_p$.

Now suppose that, $N \cong \mathbb{Z}_2^2$. Then X_N is a pentavalent symmetric graph of order $2pq$ $(5 \leqslant q < p)$. So by Lemma 2.8, $X_N \cong \mathcal{G}_{406}, \mathcal{G}_{3422}, \mathcal{G}_{3782}, \mathcal{G}_{574}, \mathcal{G}_{170}$ or $\mathcal{CD}_{2pq}^{l}$ for some l satisfying $l^4 + l^3 + l^2 + l \equiv 0 (\mathrm{mod}\ pq)$.

Let $X_N \cong \mathcal{G}_{406}$. Then $A/N \leqslant \mathrm{Aut}(\mathcal{G}_{406}) = \mathrm{PGL}(2,29)$. From the arc transitivity of A/N on X_N, we have $5 \cdot 2 \cdot 7 \cdot 29 \big| |A/N|$. Since $\mathrm{PSL}(2,29)$ has exactly two orbits on $V(X_N)$ and $\mathrm{Out}(\mathrm{PSL}(2,29)) = \mathbb{Z}_2$, we have $A = (\mathrm{PSL}(2,29) \times \mathbb{Z}_2).\mathbb{Z}_2^2$ or $\mathrm{SL}(2,29).\mathbb{Z}_2^2$. It implies that $|A_v| = 60$, and so by Proposition 2.4, $A_v \cong \mathrm{A}_5$. For the former case, A has a normal subgroup $M \cong \mathrm{PSL}(2,29)$ which has exactly two orbits on $V(X)$ and hence $|M_v| = 15$, a contradiction. As $M_v \leqslant A_v$ and A_5 contains no subgroup of order 15. For the latter, A has a normal subgroup M isomorphic to $\mathrm{SL}(2,29)$. By Proposition 2.1, M is not transitive on $V(X)$. Hence M has exactly two orbits on $V(X)$. It follows that $|M_v| = 30$, which is a contradiction, as $M_v \leqslant A_v = \mathrm{A}_5$ and A_5 has no subgroup of order 30.

Let $X_N \cong \mathcal{G}_{574}$. Then $A/N \leqslant \mathrm{Aut}(\mathcal{G}_{574}) = \mathrm{PSL}(2,41)$. From the arc transitivity of A/N on X_N, we have $5 \cdot 2 \cdot 7 \cdot 41 \big| |A/N|$. Therefore $A/N \cong \mathrm{PSL}(2,41)$ and it implies that $A \cong \mathbb{Z}_2^2 \times \mathrm{PSL}(2,41)$ or $\mathbb{Z}_2 \times \mathrm{SL}(2,41)$. For the former case, A has a normal subgroup $M \cong \mathrm{PSL}(2,41)$. By Proposition 2.1, M has at most two orbits on $V(X)$. Therefore $|M_v| = 15$ or 30. However by Lemma 2.6, $\mathrm{PSL}(2,41)$ has no subgroups of order 15 and 30. Thus $A \ncong \mathbb{Z}_2^2 \times \mathrm{PSL}(2,41)$. For the latter, A has a normal subgroup $M \cong \mathrm{SL}(2,41)$. Since $|\mathrm{SL}(2,41)| \nmid 8 \times 7 \times 41$, M has at most two orbits on $V(X)$. If M is transitive on $V(X)$, then $|M_v| = 30$, which is impossible by Proposition 2.4. If M has exactly two orbits on $V(X)$, then $|M_v| = 60$ and $A_v \cong \mathrm{S}_5$. As $M_v \leqslant \mathrm{A}_v$, $M_v \cong \mathrm{A}_5$. Which is not possible as $\mathrm{GL}(2,41)$ contains no non-abelian simple group.

Let $X_N \cong \mathcal{G}_{3422}$. Then $q = 29$, $p = 59$ and $A/N \leqslant \mathrm{Aut}(\mathcal{G}_{3422}) = \mathrm{PGL}(2,59)$. By Magma [5], the minimal arc transitive subgroup of $\mathrm{Aut}(X_N)$ is isomorphic to $\mathrm{PGL}(2,59)$. Thus $A/N \cong \mathrm{PGL}(2,59)$. Hence $A \cong \mathbb{Z}_2^2 \times \mathrm{PGL}(2,59)$ or $\mathbb{Z}_2 \times (\mathrm{SL}(2,59) \rtimes \mathbb{Z}_2)$. For the former case, A has a normal subgroup $M \cong \mathrm{PGL}(2,59)$. If M is transitive on $V(X)$, then $|M_v| = 30$, which is a contradiction by Proposition 2.4. Thus $|A_v| = 60$. So $H := A_v \cong \mathrm{A}_5$, and Proposition 2.3, imply that $X = \mathrm{Cos}(A, H, HgH)$, where g is a $2-$element in A such that $g^2 \in H$ and $|H : H \bigcap H^g| = 5$. By Magma [5], there is no such $g \in A$, a contradiction. For the latter case, A has a normal subgroup isomorphic to $\mathrm{SL}(2,59) \rtimes \mathbb{Z}_2$, say M. Since M has no subgroup of order 30, M is transitive on $V(X)$, implying that $|M_v| = 15$, which is impossible by Proposition 2.4.

Let $X_N \cong \mathcal{G}_{3782}$. Then $q = 31$, $p = 61$ and $A/N \leqslant \mathrm{Aut}(\mathcal{G}_{3782}) = \mathrm{PGL}(2,61)$. By Magma [5], the minimal arc transitive subgroup of $\mathrm{Aut}(X_N)$ is isomorphic to $\mathrm{PGL}(2,61)$. Thus $A/N \cong \mathrm{PGL}(2,61)$ implying that $A \cong \mathbb{Z}_2^2 \times \mathrm{PGL}(2,61)$ or $\mathbb{Z}_2 \times (\mathrm{SL}(2,61) \rtimes \mathbb{Z}_2)$. With similar discussion as above, we have $X_N \ncong \mathcal{G}_{3782}$.

Now assume that $X_N \cong \mathcal{G}_{170}$. Then $q = 5$, $p = 17$ and $A/N \leqslant \mathrm{Aut}(\mathcal{G}_{170}) = \mathrm{PSp}(4,4).\mathbb{Z}_4$. As $\mathrm{PSp}(4,4)$ is not transitive on $V(X_N)$, $\mathrm{PSp}(4,4).\mathbb{Z}_2 \leqslant A/N \leqslant \mathrm{PSp}(4,4).\mathbb{Z}_4$. Since the Schur Multiplier of $\mathrm{PSp}(4,4)$ is trivial, $A = (\mathbb{Z}_2^2 \times \mathrm{PSp}(4,4)).\mathbb{Z}_2$ or $A = (\mathbb{Z}_2^2 \times \mathrm{PSp}(4,4)).\mathbb{Z}_4$. Therefore A always has a normal subgroup M isomorphic to $\mathrm{PSp}(4,4)$. It is easy to verify that M has at most two orbits on $V(X)$. Thus $|M_v| = 1440$ or 2880. However by Magma [5], $\mathrm{PSp}(4,4)$, has no subgroup of order 1440 or 2880.

Finally, assume that $X_N \cong \mathcal{CD}^l_{2pq}$ for some l satisfying $l^4 + l^3 + l^2 + l \equiv 0 \pmod{pq}$. As $2 \cdot 5 \cdot q \cdot p \big| |A/N|$ and $A/N \leqslant \mathrm{Aut}(X_N) = D_{2pq} \rtimes \mathbb{Z}_5$, implying that $|A| = 2^3 \cdot 5 \cdot p \cdot q$ and $|A_v| = 5$. Therefore by Proposition 2.4, $A_v \cong \mathbb{Z}_5$. Since $p \big| |A|$ and $p^2 \nmid |A|$, there is a unique $g \in A$ of order p. Set $H :=< g >$. Hence H is a normal minimal subgroup of A such that has more than two orbits on $V(X)$. So by Proposition 2.1, H is semiregular on $V(X)$ and X_H is a pentavalent symmetric graph of order $8q$. Hence by Lemma 2.7, $q = 31$. It implies that $A = \mathbb{Z}_p \times \mathrm{PSL}(2,31)$, a contradiction. This completes the proof of Theorem 1.1. □

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