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A CHARACTERIZATION OF GVZ GROUPS IN TERMS OF FULLY RAMIFIED CHARACTERS

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Abstract. In this paper, we obtain a characterization of GVZ-groups in terms of commutators and monolithic quotients. This characterization is based on counting formulas due to Gallagher.

Throughout this paper, all groups are finite. For a group G, we write $\mathrm{Irr}(G)$ for the set of irreducible characters of G. In this paper, we present a new characterization of GVZ-groups. A group G is a *GVZ-group* if every irreducible character $\chi \in \mathrm{Irr}(G)$ satisfies that χ vanishes on $G \setminus Z(\chi)$.

The term GVZ-group was introduced by Nenciu in [12]. Nenciu continued the study of GVZ-groups in [13] and the second author further continued these studies in [10]. In our paper [2], we showed that GVZ-groups can be characterized in terms of another class of groups that have appeared in the literature.

An element $g \in G$ is called *flat* if the conjugacy class of G is $g[g, G]$. In [14], they defined a group G to be *flat* if every element in G is flat. In fact, groups satisfying this condition had been studied even earlier. Predating each of these references, Murai [11] referred to such groups as *groups of Ono type*. In [14], they proved that if G is nilpotent and flat, then G is a GVZ-group. Improving this result, we prove in [2] that a group G is a GVZ-group if and only if it is flat.

In this paper, we characterize GVZ-groups using fully ramified characters. For a normal subgroup N of G, we say that the character $\chi \in \mathrm{Irr}(G)$ is *fully ramified* over N if χ_N is homogeneous and $\chi(g) = 0$ for every element $g \in G \setminus N$.

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Following the literature, a group G is called *central type* if there is an irreducible character of G that is fully ramified over the center $Z(G)$. Results about central type groups are in [4], [5], [6], and [7].

With this as motivation, we define an irreducible character χ of G to be *central type* if χ, considered as a character of $G/\ker(\chi)$, is fully ramified over $Z(G/\ker(\chi))$. (I.e., $G/\ker\chi$ is a group of central type with faithful character χ.) It is not difficult to see that G is a GVZ-group if and only if every character $\chi \in \mathrm{Irr}(G)$ is of central type.

Recall from the literature that a group is called *monolithic* if it has a unique minimal normal subgroup. It is easy to see that if N is a normal subgroup of G and G/N is monolithic, then N appears as the kernel of some irreducible character of G. Also an irreducible character χ is called *monolithic* if the quotient group $G/\ker(\chi)$ is monolithic. Thus, monolithic quotients correspond to monolithic characters.

The purpose of this paper is to give a new characterization of central type characters based on ideas of Gallagher that are encapsulated in [9, Theorem 1.19 and Lemma 1.20], thereby obtaining a new characterizations of GVZ-groups. In particular, we prove the following theorem.

Theorem 1. *Let G be a nonabelian group. Then the following are equivalent:*

(1) *G is a GVZ-group.*
(2) *For every monolithic character $\chi \in \mathrm{Irr}(G)$ and for every element $g \in G \setminus Z(\chi)$, there exists an element $x \in G$ so that $[g,x] \in Z(\chi) \setminus \ker(\chi)$.*
(3) *G is nilpotent, and for every normal subgroup N of G for which G/N is monolithic and for every element $g \in G$ satisfying $[g,G] \not\leq N$, there exists an element $x \in G$ such that $[g,x] \notin N$ and $[[g,x],G] \leq N$.*

Our proof relies on the following lemma, which we will see is an immediate consequence of some arguments of Gallagher that can be found in [9, Theorem 1.19 and Lemma 1.20]. For an element $g \in G$, we set $D_G(g) = \{x \in G \mid [x,g] \in Z(G)\}$. Observe that $D_G(g)/Z(G) = C_{G/Z(G)}(gZ(G))$, so $D_G(g)$ is always a subgroup of G.

Lemma 2. *Let G be a group. If the character $\vartheta \in \mathrm{Irr}(Z(G))$ is faithful, then ϑ is fully ramified with respect to $G/Z(G)$ if and only if $[g, D_G(g)] \neq 1$ for every element $g \in G \setminus Z(G)$.*

Proof. By [9, Theorem 1.19 and Lemma 1.20], the number of irreducible constituents of ϑ^G equals the number of conjugacy classes of cosets $gZ(G) \in G/Z(G)$ that satisfy $[g, D_G(g)] = 1$. Observe that if $g \in Z(G)$, then $[g, D_G(g)] = 1$. Hence, the only way that there can be only one conjugacy class of elements of in $G/Z(G)$ satisfying this condition is if $[g, D_G(g)] \neq 1$ for all elements $g \in G \setminus Z(G)$. Since ϑ is fully ramified with respect to $G/Z(G)$ if and only if ϑ^G has a unique irreducible constituent, it follows that ϑ is fully ramified with respect to $G/Z(G)$ if and only if there is only one conjugacy class satisfying the condition. This gives the desired result. □

We get a slightly stronger statement without much difficulty.

Lemma 3. *Let G be a group. If $\lambda \in \mathrm{Irr}(Z(G))$ is a character, then λ is fully ramified with respect to $G/Z(G)$ if and only if $[g, D_G(g)] \not\leq \ker(\lambda)$ for every element $g \in G \setminus Z(G)$.*

Proof. Let $Z = Z(G)$ and let $K = \ker(\lambda)$. Suppose first that λ is fully ramified with respect to G/Z. Since λ is fully ramified with respect to G/Z, it follows that $Z/K = Z(G/K)$. Applying Lemma 2, we have that $[gK, D_{G/K}(gK)] \neq 1$ for all cosets $gK \in G/K \setminus Z/K$. It is not difficult to see that this implies that $[g, D_G(g)] \nleq K$ for all elements $g \in G \setminus Z$. Conversely, suppose that $[g, D_G(g)] \nleq K$ for all $g \in G \setminus Z$. Hence, we have $[gK, D_{G/K}(gK)] \neq 1$ for all $gK \in G/K \setminus Z/K$. This implies that $[gK, G/K] \neq 1$ for all cosets $gK \in G/K \setminus Z/K$, and so $Z(G/K) \leq Z/K$. Since $Z/K \leq Z(G/K)$ obviously holds, we have $Z(G/K) = Z/K$. Notice that λ is a faithful character of Z/K, so we may apply Lemma 2 to see that λ is fully ramified with respect to $\dfrac{G/K}{Z/K} \cong \dfrac{G}{Z}$. □

Let G be a group, fix a character $\chi \in \mathrm{Irr}(G)$, and write $\chi_{Z(G)} = \chi(1)\lambda$ for some character $\lambda \in \mathrm{Irr}(Z(G))$. Note that $\ker(\lambda) = \ker(\chi) \cap Z(G)$. Consider an element $g \in G$. Since $[g, D_G(g)] \leq Z(G)$, we have $[g, D_G(g)] \nleq \ker(\lambda)$ if and only if $[g, D_G(g)] \nleq \ker(\chi)$. Furthermore, $[g, D_G(g)] \nleq \ker(\chi)$ if and only if there exists an element $x \in G$ so that $[g, x] \in Z(G) \setminus \ker(\chi)$. Hence, Lemma 3 can be equivalently stated as follows.

Lemma 4. *Let G be a group. A character $\chi \in \mathrm{Irr}(G)$ is fully ramified over $Z(G)$ if and only if for every element $g \in G \setminus Z(G)$, there exists an element $x \in G$ for which $[g, x] \in Z(G) \setminus \ker(\chi)$.*

This yields the desired characterization of central type characters.

Theorem 5. *The character $\chi \in \mathrm{Irr}(G)$ has central type if and only if for every element $g \in G \setminus Z(\chi)$, there exists an element $x \in G$ for which $[g, x] \in Z(\chi) \setminus \ker(\chi)$.*

Proof. Note that χ is a faithful irreducible character of $G/\ker(\chi)$ and $Z(G/\ker(\chi)) = Z(\chi)/\ker(\chi)$. Thus we see from Lemma 4 that χ, regarded as a characer of $G/\ker(\chi)$, has central type if and only if for every element $g \in G \setminus Z(\chi)$, there exists an element $x \in G$ for which $1 \neq [g, x]\ker(\chi) \in Z(G/\ker(\chi))$. It is easy to see that this is equivalent to the statement that was to be proved. □

Remark 6. *Observe that Theorem 5 implies the well-known result that χ has central type if $G/Z(\chi)$ is abelian (see* [8, Theorem 2.31]*, for example).*

Before proceeding, we discuss monolithic groups and characters. We need one more result to prove Theorem 1. This result is proved in our paper [1].

Theorem 7. *The group G is nilpotent if and only if $Z(\chi) > \ker(\chi)$ for each nonprincipal, monolithic character $\chi \in \mathrm{Irr}(G)$.*

We now prove Theorem 1.

Proof of Theorem 1. First note the the statement (1) implies (2) follows immediately from Theorem 5.

Next we show that (2) implies (3). Let $\chi \in \mathrm{Irr}(G)$ be monolithic. By Theorem 5, χ has central type. In particular $\chi(1)^2 = |G : Z(\chi)|$, from which we deduce that $Z(\chi) > \ker(\chi)$ if χ is nonprincipal. Thus G is nilpotent by Theorem 7. Now, let N be a normal subgroup of G for which G/N is monolithic.

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Then G/N has a faithful irreducible character, and thus $N = \ker(\chi)$ for some character $\chi \in \mathrm{Irr}(G)$. Let $g \in G$ such that $[g, G] \nleq N$. Then $gN \notin Z(G/N) = Z(\chi)/N$ and so $g \notin Z(\chi)$. By (2), there exists $x \in G$ such that $[g, x] \in Z(\chi) \setminus N$. Since $Z(\chi)/N = Z(G/N)$, we see that $[[g, x], G] \leq N$.

To complete the proof, we show that (3) implies (1). Fix a prime p that divides $|G|$, a Sylow subgroup $P \in \mathrm{Syl}_p(G)$, and a character $\psi \in \mathrm{Irr}(P)$. Consider the character $\xi = \psi \times 1_H \in \mathrm{Irr}(G)$, where H is a normal p-complement of G. Then $G/\ker(\xi) \cong P/\ker(\psi)$ is monolithic, by [8, Theorem 2.32]. So ξ is fully ramified over $Z(\xi) = Z(\psi) \times H$ by Theorem 5, and this implies that ψ is fully ramified over $Z(\psi)$. Now, consider a character $\chi \in \mathrm{Irr}(G)$. To show that G is a GVZ-group, it suffices to show that χ is fully ramified over $Z(\chi)$. Suppose that $G = P_1 \times \cdots \times P_r$ is a factorization of G into a direct product of its Sylow subgroups. Then there exist characters $\nu_i \in \mathrm{Irr}(P_i)$ so that $\chi = \nu_1 \times \cdots \times \nu_r$. Observe that $Z(\chi) = Z(\nu_1) \times \cdots \times Z(\nu_r)$. We have already shown that each ν_i is fully ramified over $Z(\nu_i)$ and so it follows that χ is fully ramified over $Z(\chi)$, as desired. This proves (1). □

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