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CONVOLUTION IDENTITIES INVOLVING THE CENTRAL BINOMIAL COEFFICIENTS AND CATALAN NUMBERS

NECDET BATIR, HAKAN KÜÇÜK AND SEZER SORGUN*

Abstract. We generalize some convolution identities due to Witula and Qi et al. involving the central binomial coefficients and Catalan numbers. Our formula allows us to establish many new identities involving these important quantities and recovers some known identities in the literature. Also, we give new proofs of Shapiro's Catalan convolution and a famous identity of Hajós.

1. Introduction

The *central binomial coefficient* B_n and *Catalan number* C_n are defined by

$$B_n = \binom{2n}{n} \quad \text{and} \quad C_n = \frac{1}{n+1}\binom{2n}{n},$$

respectively. These numbers appear in the series expansions of some elementary functions. For example, we have

$$\arcsin x = \sum_{n=0}^{\infty} \frac{B_n x^{2n+1}}{4^n(2n+1)}, \quad |x| < 1,$$

and

$$(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 B_n}, \quad |x| < 1;$$

see [7].

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*Corresponding author.
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In the literature, there exists many series representations for some important mathematical constants which involve B_n and C_n. As examples, we have

$$\sum_{n=0}^{\infty} \frac{(42n+5)B_n^3}{4^{6n}} = \frac{16}{\pi}.$$

This is due to Ramanujan; see [7, p. 188]. The representation

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 B_n},$$

which played a key role in the celebrated proof of irrationality of $\zeta(3)$ by Apéry [4]. The central binomial coefficients and Catalan numbers have been investigated by many authors in a various directions. Elezović [14] presents several asymptotic expansions for them. Qi et al. [23, Section 4.2], and Mansour and Sun [18] obtained the following elegant integral representations, respectively.

$$B_n = \frac{1}{\pi} \int_0^{\infty} \frac{dx}{(x^2+1/4)^{n+1}},$$

$$\text{and} \quad C_n = (-1)^{n+1} 2^{2n+1} \int_0^1 P_{2n+1}(x-1)dx,$$

where P_n are the classical *Legendre's polynomials*. The central binomial coefficients and Catalan numbers have important applications in combinatorial theory, graph theory, and statistics (see [1, 6, 26]). For basic properties, generalizations and modular properties of these numbers we refer to [8, 9, 11, 17, 22, 23, 24, 27]. The work in this paper is motivated by some recent works on convolution identities involving B_n and C_n. Witula et al. [31] proved the identity

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{2k+1} = \frac{16^n}{(2n+1)B_n}. \tag{1.1}$$

Alzer and Nagy [2] studied some identities related to (1.1) and they proved the following *convolution identity*:

$$\sum_{k=0}^{n-1} (-1)^k B_k (B_{n-k} - C_{n-k}) = (-1)^n \left(\frac{1}{2} B_{n+1} - 2^n B_{[(n+1)/2]} \right),$$

where $[.]$ is the greatest integer function. Qi et al. [21] provided two new proofs of identity (1.1) along with some possibly new convolution identities for B_n and C_n. The first aim of this paper is to generalize the identity (1.1) and to prove, for any $a \in \mathbb{R}$, which is not zero and a negative integer,

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+a} = \frac{4^n \Gamma(a) \Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)}.$$

The particular value $a = \frac{1}{2}$ leads to (1.1). Taking different values for a, we obtain [21, Eq. (4.1)] and some others given in [2]. Also, we can differentiate it with respect to a, and in this way, we may derive many new interesting identities.

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The following elegant identity has attracted the attention of many mathematicians.

$$\sum_{k=0}^{n} C_{2k}C_{2n-2k} = 4^n C_n. \tag{1.2}$$

In 2002, Shapiro [16, p. 23] observed that it can be easily proved by the generating function method. Andrews [3] formulated a q-analogue of (1.2), and he offered a combinatorial proof of the identity. Nagy [19] also gave a combinatorial proof of it. Hajnal and Nagy [19] have presented a bijective proof of this identity. Our second aim is to give a new and different proof of identity (1.2), based on the *Wilf–Zeilberger algorithm*.

In 1983, Sved [30] possed the following identity as a problem

$$\sum_{k=0}^{n} B_k B_{n-k} = 4^n \tag{1.3}$$

and requested a combinatorial solution. In the literature, many different types of proofs of this identity appeared [5, 10, 12, 15]. In [29] Sved recounts the story of this identity and its combinatorial proofs. In [13] the authors gave the first bijective proof of it. Our final aim in this paper is to provide a new proof of this identity by using the WZ-method. As it is well known the classical *gamma function* is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ $(x > 0)$. The *Legendre's duplication formula* for the gamma function states that

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi},\ n \in \mathbb{N} \cup \{0\}. \tag{1.4}$$

The *digamma function* ψ is defined by the logarithmic derivative of the gamma function, that is, $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. In the literature, the function ψ' is known to be *trigamma function*. For $m, n \in \mathbb{N}$ the *generalized harmonic number* $H_n^{(m)}$ of order m is defined by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}. \tag{1.5}$$

$H_0^{(m)} = 0$ and $H_n^{(1)} = H_n$ is the familiar harmonic number $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. The digamma and tetragamma functions, and the harmonic numbers are related with

$$\psi(n+1) = -\gamma + H_n, \tag{1.6}$$

and

$$\psi'(1) - \psi'(n+1) = H_n^{(2)} \quad (n \in \mathbb{N}), \tag{1.7}$$

where $\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \log n\right) = 0.57721\cdots$ is the *Euler-Mascheroni constant*. The following duplication formula is valid for the digamma function:

$$\psi\left(n + \frac{1}{2}\right) = 2\psi(2n) - \psi(n) - 2\log 2 = 2H_{2n} - H_n - 2\log 2 - \gamma; \tag{1.8}$$

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see [25]. We shall frequently use the following generalized binomial coefficient

$$\binom{s}{t} = \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)},$$

where t and s are real numbers which are not negative integers, and $t \leq s$.

2. The Wilf-Zeilberger Method

In this section, we recall briefly the *Wilf-Zeilberger method (WZ-method)*. A discrete function $A(n,k)$ is *hypergeometric* if both

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are rational functions in both n and k. A pair (F,G) of hypergeometric functions is said to be a $WZ-$ *pair (Wilf-Zeilberger pair)* if for $n = 0,1,2,\ldots$ and for all $k \in \mathbb{Z}$, they satisfy

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{2.1}$$

In this case, Wilf and Zeilberger [20, Chapter 7] and [30] proved that there exists a rational function $C(n,k)$ such that

$$G(n,k) = C(n,k)F(n,k).$$

They called $C(n,k)$ as certificate of the pair (F,G). Summing on $n \geq 0$ both sides of (2.1), one gets

$$\begin{aligned}\sum_{n=0}^{\infty}\{G(n,k+1) - G(n,k)\} &= \sum_{n=0}^{\infty}\{F(n+1,k) - F(n,k)\} \\ &= \lim_{n\to\infty} F(n,k) - F(0,k).\end{aligned} \tag{2.2}$$

In most applications, it is usually very easy to evaluate $F(0,k)$ and $\lim_{n\to\infty} F(n,k)$. So, taking particular values for k in (2.2), we can obtain many identities. We can also sum both sides of (2.1) over k's and in this case we get

$$\begin{aligned}\sum_{k=0}^{\infty}\{F(n+1,k) - F(,k)\} &= \sum_{k=0}^{\infty}\{G(n,k+1) - G(n,k)\} \\ &= \lim_{k\to\infty} G(n,k) - G(n,0).\end{aligned}$$

If $G(n,0) = 0$ and $\lim_{k\to\infty} G(n,k) = 0$, we get

$$\sum_{k=0}^{\infty}\{F(n+1,k) - F(n,k)\} = 0\,(n = 0,1,2,3,\ldots),$$

which implies that $\sum_{k=0}^{\infty} F(n,k)$ is a constant. Let us say $\sum_{k=0}^{\infty} F(n,k) = C$. Usually, it is very easy to evaluate this constant by choosing a particular value for k (usually $k = 0$), in other cases, we evaluate it by taking the limit as $k \to \infty$. It is worthy to note that for a given hypergeometric function $F(n,k)$ the package EKHAD based on MAPLE [20, 30] allows us to find a rational function $G(n,k)$ (if there exists) such that (F,G) is a $WZ-pair$. Now let's briefly give how to get WZ-pair in EKHAD. Suppose $F(n,k)$ is given summand for which one is interested in getting a recurrence for

$f(n) = \sum_k F(n,k)$. Then make a call for *ct(F(n,k), r, k, n, N)*, where r is the order of the recurrence. If the output of *ct* (creative telescoping algorithm) gives $\Big[N-1, C(n,k)\Big]$ for $r=1$, then we get $G(n,k)$ which is WZ-pair of $F(n,k)$ such that $G(n,k) = F(n,k)C(n,k)$. We also refer the interested readers to [20] and [30] for more information about the WZ- method.

Now we are ready to present our main results.

3. Main Results

Theorem 3.1. *Let a be any real number, which is not zero and a negative integer. Then we have*

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+a} = \frac{4^n \Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.1}$$

Proof. We prove by the WZ method. Let n and k be non-negative integers with $k \le n$. Consider the following discrete function.

$$F(n,k) = \frac{(n+2a)\Gamma(k+1/2)\Gamma(n-k+1/2)\Gamma(n+a+1)\Gamma(a+1/2)}{2\pi(k+a)(n-k+a)\Gamma(a)\Gamma(k+1)\Gamma(n-k+1)\Gamma(n+a+1/2)}. \tag{3.2}$$

The package EKHAD [20, 30] allows us to find the rational function G, where

$$G(n,k) = -\frac{(3n+2a-2k+3)\Gamma(n-k+3/2)\Gamma(n+a+1)\Gamma(k+1/2)\Gamma(a+1/2)}{2\pi(n-k+a+1)(n+1)\Gamma(k)\Gamma(a)\Gamma(n+a+3/2)\Gamma(n-k+2)} \tag{3.3}$$

($k \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$), such that (F,G) is a WZ-pair. That is,

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{3.4}$$

By replacing n by k, and k by j in (3.4), we obtain

$$F(k+1,j) - F(k,j) = G(k,j+1) - G(k,j).$$

Summing both sides of this equation from $j=0$ to $j=k+1$, we deduce

$$\sum_{j=0}^{k+1} F(k+1,j) - \sum_{j=0}^{k} F(k,j) - F(k,k+1)$$
$$= \sum_{j=0}^{k+1} (G(k,j+1) - G(k,j)) = G(k,k+2) - G(k,0).$$

Note that $G(k,k+2) = G(k,0) = 0$ and $F(k,k+1) = 0$ by identities in (3.2) and (3.3), thus, we get

$$\sum_{j=0}^{k+1} F(k+1,j) - \sum_{j=0}^{k} F(k,j) = 0.$$

Summing both sides from $k=0$ to $k=n-1$, we conclude that

$$\sum_{k=0}^{n-1} \left(\sum_{j=0}^{k+1} F(k+1,j) - \sum_{j=0}^{k} F(k,j) \right) = 0.$$

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This is a telescopic sum. We therefore have

$$\sum_{j=0}^{n} F(n,j) - F(0,0) = 0.$$

But since $F(0,0)=1$ this gives

$$\sum_{k=0}^{n} F(n,k) = 1$$

or

$$\sum_{k=0}^{n} \frac{\Gamma(k+1/2)\Gamma(n-k+1/2)}{(k+a)(n-k+a)\Gamma(k+1)\Gamma(n-k+1)} = \frac{2\pi\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}.$$

From this identity, by (1.4), we arrive at

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{(k+a)(n-k+a)} = \frac{2^{2n+1}\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.5}$$

Since

$$\frac{1}{(k+a)(n-k+a)} = \frac{1}{n+2a}\left(\frac{1}{k+a} + \frac{1}{n-k+a}\right),$$

we deduce from (3.5)

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{(k+a)(n-k+a)} = \frac{1}{n+2a}\sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+a} + \frac{1}{n+2a}\sum_{k=0}^{n} \frac{B_k B_{n-k}}{n-k+a}$$

$$= \frac{2}{n+2a}\sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+a}. \tag{3.6}$$

From (3.5) and (3.6) we arrive at

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+a} = \frac{4^n\Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)},$$

which is the desired result. □

Differentiating both sides of (3.1) with respect to a, we have the following corollary.

Corollary 3.2. *For $n = 0, 1, 2, \ldots$ we have*

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{(k+a)^2} = -\frac{4^n\Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)} \times \left(\psi(a) + \psi(n+a+1/2) - \psi(n+a+1) - \psi(a+1/2)\right), \tag{3.7}$$

where ψ is the digamma function.

We present now a new proof of the elegant convolution formula (given in (1.2)) of Shapiro.

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Theorem 3.3. *Let n be a non-negative integer. Then*

$$\sum_{k=0}^{n} C_{2k}C_{2n-2k} = 4^n C_n. \tag{3.8}$$

Proof. Let n and k be non-negative integers with $k \le n$, and

$$A(n,k) = \frac{\Gamma(2k+1/2)\Gamma(2n-2k+1/2)\Gamma(n+2)}{\sqrt{\pi}\Gamma(2k+2)\Gamma(2n-2k+2)\Gamma(n+1/2)}.$$

Then the Mapple package EKHAD [20, 30] allows to find its WZ-pair

$$B(n,k) = \frac{(2k-3n-4)\Gamma(n+1)\Gamma(2n-2k+5/2)\Gamma(2k+1/2)}{4\sqrt{\pi}\Gamma(2k)\Gamma(n+5/2)\Gamma(2n-2k+4)}$$

such that

$$A(n+1,k) - A(n,k) = B(n,k+1) - B(n,k), \quad (k = 0,1,2,\dots \quad (k \le n)). \tag{3.9}$$

By replacing n by k, and k by j in (3.9) we get

$$A(k+1,j) - A(k,j) = B(k,j+1) - B(k,j). \tag{3.10}$$

Summing both sides of (3.10) from $j = 0$ to $j = k+1$, we get

$$\sum_{j=0}^{k+1} A(k+1,j) - \sum_{j=0}^{k+1} A(k,j) = \sum_{j=0}^{k+1}(B(k,j+1) - B(k,j)).$$

The right-hand side is a telescoping sum, therefore this can be rewritten as follows:

$$\sum_{j=0}^{k+1} A(k+1,j) - \sum_{j=0}^{k} A(k,j) - A(k,k+1) = B(k,k+2) - B(k,0).$$

But since $A(k,k+1) = 0$, $B(k,k+2) = 0$, and $B(k,0) = 0$, it follows that

$$\sum_{j=0}^{k+1} A(k+1,j) - \sum_{j=0}^{k} A(k,j) = 0.$$

Summing both sides from $k = 0$ to $k = n-1$ yields

$$\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k+1} A(k+1,j) - \sum_{j=0}^{k} A(k,j)\right) = 0,$$

which is a telescoping sum again. We therefore have

$$\sum_{j=0}^{n} A(n,j) - A(0,0) = 0,$$

But since $A(0,0) = 1$, this leads to

$$\sum_{k=0}^{n} A(n,k) = 1.$$

If we use (1.4), after some simple computations, we see that this is equivalent to (3.8). □

Theorem 3.4. *Let n be a non-negative integer. Then*

$$\sum_{k=0}^{n} B_k B_{n-k} = 4^n \tag{3.11}$$

Proof. We define the discrete function

$$P(n,k) = \frac{1}{\pi}\frac{\Gamma(n-k+1/2)\Gamma(k+1/2)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

The package EKHAD [20, 30] allows us to find

$$Q(n,k) = \frac{1}{2\pi}\frac{(2n-2k+1)\Gamma(n-k+1/2)\Gamma(k+1/2)}{4^n(n+1)\Gamma(k+1)\Gamma(n-k+2)},$$

such that

$$(n+1)(P(n,k)-P(n+1,k)) = Q(n,k+1)-Q(n,k).$$

The replacement $n \to k$ and $k \to j$ leads to

$$P(k,j)-P(k+1,j)s = \frac{1}{k+1}(Q(k,j+1)-Q(k,j)).$$

We now sum both sides from $j=0$ to $j=k+1$ and we get

$$\begin{aligned}\sum_{j=0}^{k+1}(P(k,j)-P(k+1,j)) &= \frac{1}{k+1}\sum_{j=0}^{k+1}(Q(k,j+1)-Q(k,j))\\ &= \frac{Q(k,k+2)-Q(k,0)}{k+1}.\end{aligned}$$

Since $Q(k,k+2)=0$ and $Q(k,0)=0$ we have

$$\sum_{j=0}^{k+1} P(k,j) - \sum_{j=0}^{k+1} P(k+1,j) = 0$$

or

$$\sum_{j=0}^{k+1} P(k+1,j) - \sum_{j=0}^{k} P(k,j) - P(k,k+1) = 0.$$

But since $P(k,k+1)=0$, this becomes

$$\sum_{j=0}^{k+1} P(k+1,j) - \sum_{j=0}^{k} P(k,j) = 0.$$

Summing both sides of this equation from $k=0$ to $k=n-1$ one gets

$$\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k+1} P(k+1,j) - \sum_{j=0}^{k} P(k,j)\right) = 0,$$

which is a telescopic sum, and therefore we get

$$\sum_{k=0}^{n} P(n,k) = P(0,0) = 1. \tag{3.12}$$

Using (1.4), we can rewrite P as follows

$$P(n,k)=\frac{1}{4^n}\binom{2k}{k}\binom{2n-2k}{n-k},$$

which, by (3.12), leads to (3.11). □

In the last section we present some applications of our main results.

4. Applications

This section is devoted to the applications of our main results. Taking particular values for a in (3.1) we may obtain many convolution identities for B_n and C_n. Our first identity recovers (1.1).

Identity 4.1. *Let n be a non-negative integer. Then*

$$\sum_{k=0}^{n}\frac{B_kB_{n-k}}{2k+1}=\frac{16^n}{(2n+1)B_n}.$$

Proof. The proof follows from Theorem 3.1 with $a=1/2$ by (1.4). □

Identity 4.2. *Letting $a=1$ in Theorem 3.1 we get*

$$\sum_{k=0}^{n}C_kB_{n-k}=\frac{1}{2}B_{n+1}.$$

Identity 4.3. *For $n=0,1,2,\ldots$ we have*

$$\sum_{k=0}^{n}C_kC_{n-k}=C_{n+1}.$$

Proof. Since

$$\frac{1}{(k+1)(n-k+1)}=\frac{1}{n+2}\left(\frac{1}{k+1}+\frac{1}{n-k+1}\right),$$

we get

$$\begin{aligned}\sum_{k=0}^{n}C_kC_{n-k}&=\sum_{k=0}^{n}\frac{B_kB_{n-k}}{(k+1)(n-k+1)}\\&=\frac{1}{n+2}\left(\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+1}+\sum_{k=0}^{n}\frac{B_kB_{n-k}}{n-k+1}\right)\\&=\frac{2}{n+2}\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+1},\end{aligned}$$

from which the proof follows by (3.1) with $a=1$. □

Identity 4.4. *For $n=1,2,\ldots$ we have*

$$\sum_{k=1}^{n}\frac{B_kB_{n-k}}{k}=2(H_{2n}-H_n)B_n,$$

where H_n are the usual harmonic numbers.

Proof. Taking the first term of the left-hand side of (3.1) to the right and then letting $a \to 0$, the conclusion follows by L' Hospital rule, the duplication formula (1.4) and (1.6). □

Identity 4.5. *Let n be a non-negative integer. Then*

$$\sum_{k=1}^{n} \frac{B_k B_{n-k}}{k^2} = B_n \left(2H_{2n}^{(2)} - H_n^{(2)} - 2(H_n - H_{2n})^2\right),$$

where $H_n^{(2)}$ are the generalized harmonic numbers as defined by (1.5).

Proof. Taking the first term of the sum (3.7) to the right, and then simplifying the result, and finally letting $a \to 0$, we get

$$\sum_{k=1}^{n} \frac{B_k B_{n-k}}{k^2} = -\lim_{a\to 0} \frac{4^n\Gamma(a+1)\Gamma(n+a+1/2)G(a) - \Gamma(n+a+1)\Gamma(a+1/2)B_n}{\Gamma(n+a+1)\Gamma(a+1/2)a^2}.$$

Here

$$G(a) = a\psi(a+1) - 1 + a\psi(n+a+1/2) - a\psi(n+a+1) - a\psi(a+1/2).$$

Using L'Hospital rule, and taking into account the duplication formula (1.4), and the formulas (1.6), (1.7) and (1.8), we can easily evaluate this limit and we see that it is equal to the right-hand side of the statement of Identity4.5. But since it requires lengthy calculations we omit the details. □

Identity 4.6. *Let n be a non-negative integer. Then*

$$\sum_{k=1}^{n} \frac{C_k C_{n-k}}{k^2} = C_n \Bigg[2H_{2n}^{(2)} - H_n^{(2)} - 2(H_n - H_{2n})^2 - \frac{2n(H_{2n} - H_n)}{n+1} + \frac{n^3 + 2n^2 + 3n}{(n+2)(n+1)^2}\Bigg]$$

Proof. We have

$$\sum_{k=1}^{n} \frac{C_k C_{n-k}}{k^2} = \sum_{k=1}^{n} \frac{B_k B_{n-k}}{k^2(k+1)(n-k+1)}.$$

By partial fraction decomposition we have

$$\frac{1}{k^2(k+1)(n-k+1)} = \frac{1}{k^2(n+1)} - \frac{n}{k(n+1)^2} + \frac{1}{(k+1)(n+2)} + \frac{1}{(n+1)^2(n+2)(n-k+1)}.$$

Summing both sides of this equation from $k = 1$ to $k = n$, after some manipulations we get

$$\sum_{k=1}^{n} \frac{C_k C_{n-k}}{k^2} = \frac{1}{n+1}\sum_{k=1}^{n} \frac{B_k B_{n-k}}{k^2} + \frac{n^2+2n+2}{(n+1)^2(n+2)}\sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+1} - \frac{n}{(n+1)^2}\sum_{k=1}^{n} \frac{B_k B_{n-k}}{k} - \frac{(n^2+n+1)B_n}{(n+1)^3}.$$

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By Identity 4.1, Identity 4.2 and setting $a = 1$ in Theorem 3.1, after some simplifications, we get

$$\sum_{k=1}^{n} \frac{C_k C_{n-k}}{k^2} = C_n \Bigg[2H_{2n}^{(2)} - H_n^{(2)} - 2(H_n - H_{2n})^2 + \frac{n^3 + 2n^2 + 3n}{(n+2)(n+1)^2} - \frac{2n(H_{2n} - H_n)}{n+1} \Bigg].$$

□

Identity 4.7. *Let n be a non-negative integer. If we substitute $a = 1/2$ in Corollary 3.2 and use (1.4)* and (1.8), we get

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{(2k+1)^2} = \frac{16^n (H_{2n+1} - H_n)}{(2n+1) B_n}. \tag{4.1}$$

Identity 4.8. *For $m = 0, 1, 2, \ldots$ and any integer n with $n > m$ we have*

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{2k - 2m - 1} = 0.$$

Proof. If we substitute $a = -m - \frac{1}{2}$ $(m \in \mathbb{N})$ in (3.1) the proof follows. □

Identity 4.9. *Let n be a non-negative integer. If we substitute $a = m \in \mathbb{N}$ in Theorem 3.1, we get*

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{(k+m)^2} = \frac{2\binom{2m+2n}{m+n}}{m\binom{2m}{m}} \left(H_{m+n} + H_{2m} - H_m - H_{2m+2n} + \frac{1}{2m} \right). \tag{4.2}$$

Identity 4.10. *For $n = 0, 1, 2, \ldots$ we have*

$$\sum_{k=0}^{n} \frac{C_k C_{n-k}}{k+a} = \frac{(n+2a)(2n+1)C_n}{(n+2)(n+a+1)(a-1)} - \frac{4^n \Gamma(a)\Gamma(n+a+1/2)}{(a-1)\Gamma(n+a+2)\Gamma(a+1/2)}.$$

Proof. By partial fraction decomposition we get

$$\begin{aligned} \sum_{k=0}^{n} \frac{C_k C_{n-k}}{k+a} &= \sum_{k=0}^{n} \frac{B_k B_{n-k}}{(k+a)(k+1)(n-k+1)} \\ &= \frac{1}{n+2} \left(\frac{1}{a-1} \sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+1} - \frac{1}{a-1} \sum_{k=0}^{n} \frac{B_k B_{n-k}}{n-k+1} \right. \\ &\quad \left. + \frac{1}{n+a+1} \sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+a} + \frac{1}{n+a+1} \sum_{k=0}^{n} \frac{B_k B_{n-k}}{n-k+1} \right). \end{aligned} \tag{4.3}$$

Clearly we have

$$\sum_{k=0}^{n} \frac{B_k B_{n-k}}{n-k+1} = \sum_{k=0}^{n} \frac{B_k B_{n-k}}{k+1}.$$

Using this in (4.3), we get

$$\sum_{k=0}^{n}\frac{C_kC_{n-k}}{k+a}=\frac{1}{(n+2)(n+a+1)(a-1)}\times\left((n+2a)\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+1}-(n+2)\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+a}\right).$$

Now the conclusion follows from Theorem 3.1. □

Identity 4.11. *For $n=0,1,2,\dots$ we have*

$$\sum_{k=0}^{n}\frac{B_kC_{n-k}}{k+a}=\frac{1}{n+a+1}\left(\frac{4^n\Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)}+\frac{1}{2}B_{n+1}\right).$$

Proof. Note that

$$\frac{B_kC_{n-k}}{k+a}=\frac{B_kB_{n-k}}{(k+a)(n-k+1)}=\frac{1}{n+a+1}\left(\frac{B_kB_{n-k}}{k+a}+\frac{B_kB_{n-k}}{n-k+1}\right).$$

Hence, summing both sides one gets

$$\begin{aligned}\sum_{k=0}^{n}\frac{B_kC_{n-k}}{k+a}&=\frac{1}{n+a+1}\left(\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+a}+\sum_{k=0}^{n}\frac{B_kB_{n-k}}{n-k+1}\right)\\&=\frac{1}{n+a+1}\left(\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+a}+\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+1}\right).\end{aligned}$$

The proof now follows from Theorem 3.1. □

Identity 4.12. *Let n be a non-negative integer. Then we have*

$$\sum_{k=0}^{n}\frac{B_k(B_{n-k}-C_{n-k})}{n-k+2}=\frac{nB_{n+1}}{6(n+2)}.$$

Proof. The following rearrangement is valid

$$\sum_{k=0}^{n}\frac{B_k(B_{n-k}-C_{n-k})}{n-k+2}=\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+2}-\sum_{k=0}^{n}\frac{B_{n-k}C_k}{k+2}. \tag{4.4}$$

From the definitions and partial fraction decomposition we find

$$\sum_{k=0}^{n}\frac{B_{n-k}C_k}{k+2}=\sum_{k=0}^{n}\frac{B_kB_{n-k}}{(k+2)(k+1)}=\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+1}-\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+2}.$$

Thus, using (4.4), we obtain

$$\sum_{k=0}^{n}\frac{B_k(B_{n-k}-C_{n-k})}{n-k+2}=2\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+2}-\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+1}. \tag{4.5}$$

From Theorem 3.1 with $a=2$ we get

$$\sum_{k=0}^{n}\frac{B_kB_{n-k}}{k+2}=\frac{2n+3}{6(n+2)}B_{n+1}.$$

Substituting this in (4.5), and using Theorem 3.1 for $a=1$, the conclusion follows immediately. □

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Remark 4.13. *Identity* 4.12 recovers in [2, Theorem 4].

Identity 4.14. *Let n be a non-negative integer. Letting $a = 1$ in Identity* 4.11 we get

$$\sum_{k=0}^{n} \frac{B_k C_{n-k}}{k+1} = \frac{B_{n+1}}{n+2}.$$

Remark 4.15. *Using our results given here it is possible to evaluate all the following sums:*

$$\sum_{k=1}^{n} \frac{B_k B_{n-k}}{k^m}, \quad \sum_{k=0}^{n} \frac{B_k B_{n-k}}{(k+a)^m}, \quad \sum_{k=1}^{n} \frac{C_k C_{n-k}}{(k+a)^m}, \quad \textit{and} \quad \sum_{k=1}^{n} \frac{C_k C_{n-k}}{k^m}$$

but when m is large lengthy computations are required.

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Necdet Batır

Department of Mathematics, Nevşehir Hacı Bektaş Veli University, 50300, Nevşehir, Turkey

Email: nbatir@hotmail.com

Hakan Küçük

Department of Mathematics, Nevşehir Hacı Bektaş Veli University, 50300, Nevşehir, Turkey

Email: hakankucuk1979@gmail.com

Sezer Sorgun

Department of Mathematics, Nevşehir Hacı Bektaş Veli University, 50300, Nevşehir, Turkey

Email: srgnrzs@gmail.com