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ON GROUPS IN WHICH SUBNORMAL SUBGROUPS OF INFINITE RANK ARE COMMENSURABLE WITH SOME NORMAL SUBGROUP

ULDERICO DARDANO* AND FAUSTO DE MARI

Abstract. We study soluble groups G in which each subnormal subgroup H with infinite rank is commensurable with a normal subgroup, i.e. there exists a normal subgroup N such that $H \cap N$ has finite index in both H and N. We show that if such a G is periodic, then all subnormal subgroups are commensurable with a normal subgroup, provided either the Hirsch-Plotkin radical of G has infinite rank or G is nilpotent-by-abelian (and has infinite rank).

1. Introduction and statement of results

A group G is said to be a T-group if normality in G is a transitive relation, i.e. if all subnormal subgroups are normal. The structure of soluble T-groups was well described in the 1960s by Gaschütz, Zacher and Robinson [14]. Then, taking these results as a model, several authors have studied soluble groups in which subnormal subgroups have some embedding property which "approximates" normality. In particular, Casolo [2] considered T^*-groups, that is groups in which any subnormal subgroup H has the property nn (nearly normal), i.e. the index $|H^G : H|$ is finite. Then Franciosi, de Giovanni and Newell [11] considered T_*-groups, that is groups in which any subnormal subgroup H has the property cf (core-finite, normal-by-finite), i.e. the index $|H : H_G|$ is finite. Here, as usual, H^G (resp. H_G) denotes the smallest (resp. largest) normal subgroup of G containing (resp. contained in) H

Recently in [4], in order to put those results in a common framework, we considered $T[*]$-groups, that is groups in which each subnormal subgroup H is cn, i.e. commensurable with a normal subgroup of G. Recall that two subgroups H and K are called commensurable if $H \cap K$ has finite index in both H and K, hence both nn and cf imply cn. Clearly all the above results rely on corresponding

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∗Corresponding author.
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previous results on groups in which *all* subgroups are nn, cf, cn resp. [13, 1, 3] resp. A similar approach was adopted in [7] where finitely generated groups in which subnormal subgroups are inert have been considered, where the term *inert* refers to a different generalization of both nn, cf (namely, an inert subgroup is a subgroup which is commensurable with each of its conjugates).

In the last decade, several authors have studied the influence on a soluble group of the behavior of its subgroups of *infinite rank* (see for instance [5, 10] or the bibliography in [9]). Recall that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property and *infinite rank* is there is no such r. For example, in [8] it was proved that *if G is a periodic soluble group of infinite rank in which every subnormal subgroup of infinite rank is normal, then G is a T-group indeed.* Then in [9], authors have considered groups of infinite rank with properties T_+ (T^+, resp.), that is groups in which the condition of being cf (resp. nn) is imposed only to subnormal subgroups *with infinite rank*. In fact, it has been shown that *a periodic soluble group of infinite rank G with property T_+ (T^+, resp.) has the full T_* (T^*, resp.) property, provided one of the following holds:*
(A) the Hirsch-Plotkin radical of G has infinite rank,
(B) the commutator subgroup G' is nilpotent.

In this paper we show that a similar statement is true also for the property cn. Moreover, by a corollary, we give some further information about the property cf as well. Let us call $T[+]$*-group* a group in which each subnormal subgroup of infinite rank is a cn-subgroup.

Theorem A. *Let G be a periodic soluble $T[+]$-group whose Hirsch-Plotkin radical has infinite rank. Then G is a $T[*]$-group.*

Corollary 1.1. *Let G be a periodic soluble $T[+]$-group (resp. T_+-group) of infinite rank such that $\pi(G')$ is finite. Then all subgroups of G are cn (resp. cf).*

Theorem B. *Let G be a periodic $T[+]$-group of infinite rank with nilpotent commutator subgroup. Then G is a $T[*]$-group.*

Note that if $G = A \rtimes B$ is the holomorph group of the additive group A of the rational numbers by the multiplicative group B of positive rationals (acting by usual multiplication), then, as noticed in [8], the only subnormal non-normal subgroups of G are those contained in A (which has rank 1) so that G is $T[+]$. However all proper non-trivial subgroups of A are not cn, since if they were cn then they were cf [6, Proposition 1], contradicting the fact that A is minimal normal in G.

Our notation and terminology is standard and can be found in [15, 16]

2. Proofs

By a standard argument one checks easily that *if H_1 and H_2 are cn- (resp. cf-) subgroups of G, then $H_1 \cap H_2$ is likewise cn (resp. cf). The same holds for H_1H_2, provided this set is a subgroup.*

Lemma 2.1. *Let G be a $T[+]$-group and let A be a subnormal subgroup of G. If A is the direct product of infinitely many non-trivial cyclic subgroups, then any subgroup of A is a cn-subgroup of G.*

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Proof. Let X be any subgroup of A, then X is a subnormal subgroup of G and X is likewise a direct product of cyclic groups [16, 4.3.16]. In order to prove that X is a *cn*-subgroup of G we may assume that X has finite rank. Then there exist subgroups A_1, A_2, A_3 of A with infinite rank such that $X \leq A_3$ and $A = A_1 \times A_2 \times A_3$. Thus XA_1 and XA_2 are subnormal subgroups of infinite rank, so that they are both *cn*-subgroups of G. Therefore $X = XA_1 \cap XA_2$ is likewise *cn* in G. □

Lemma 2.2. *Let G be a periodic $T[+]$-group. If G contains an abelian subnormal subgroup of infinite rank A, then G is a $T[*]$-group.*

Proof. By hypothesis there exists a normal subgroup N of G which is commensurable with A. Then $A \cap N$ has finite index in AN and hence N is an abelian-by-finite group of infinite rank. In particular, N contains a characteristic subgroup N_* of finite index which is an abelian group of infinite rank; hence replacing A by N_* it can be supposed that A is a normal subgroup. Since G is periodic and A has infinite rank, it follows that the socle S of A is a normal subgroup of G which is the direct product of infinitely many non-trivial cyclic subgroups. Application of Lemma 2.1 yields that all subgroups of S are *cn*-subgroups of G and hence by [3, Lemma 2.8], there exist G-invariant subgroups $S_0 \leq S_1$ of S such that S_0 and S/S_1 are finite and all subgroups of S lying between S_0 ad S_1 are normal in G

Let X be any subnormal subgroup of finite rank of G. Then $X \cap S_1$ has finite rank, hence $S_2 = S_0(X \cap S_1)$ is a normal subgroup of G of finite rank. Therefore, since S_2X is commensurable with X, we may assume $S_2 = \{1\}$. Clearly there exist subgroups S_3 and S_4 with infinite rank such that $S_1 = S_3 \times S_4$. Since both S_3 and S_4 are normal subgroups of G, we have that both XS_3 and XS_4 are subnormal subgroups of infinite rank of G and hence they are both *cn*. Thus $X = XS_3 \cap XS_4$ is likewise a *cn*-subgroup of G. □

Recall that any primary locally nilpotent group of finite rank is a Chernikov group [15, Part 2, p.38].

Lemma 2.3. *Let G a $T[+]$-group of infinite rank. If G is a Baer p-group, then G is a nilpotent $T[*]$-group.*

Proof. Let X be any subnormal subgroup of G with finite rank. Then X is a Chernikov group (see [15, Part 2, p.389]) and X contains an abelian divisible normal subgroup J of finite index. Hence J is subnormal in G, and so J^G is abelian and divisible [15, Part 1, Lemma 4.46]. If A is any abelian subnormal subgroup of G, the subgroup J^GA is nilpotent and $[J, A] = \{1\}$ [15, Part 1, Lemma 3.13]. Since G is generated by its subnormal abelian subgroups, it follows that $J \leq Z(G)$ and so X/X_G is finite. This proves that G is a $T[*]$-group, hence nilpotent [4, Proposition 20]. □

The following lemma is probably well-known but we are not able to find it in the literature, hence we write also the proof.

Lemma 2.4. *Let G be a periodic finite-by-abelian group of finite rank. Then $G/Z(G)$ is finite.*

Proof. Clearly $C = C_G(G')$ is a normal subgroup of finite index of G which is nilpotent and has finite rank; in particular, any primary component of C is a Chernikov group. Let $\pi = \pi(G')$ be the set of

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all primes p such that G' contains some element of order p. Then π is finite and so the subgroup C_π is a Chernikov group; hence $C_\pi Z(G)/Z(G)$ is finite [15, Part 1, Lemma 4.31]. On the other hand $C_{\pi'}$ is abelian, and so it follows that $C/Z(C)$ is finite. Thus G is both abelian-by-finite and finite-by-abelian and hence $G/Z(G)$ is finite. □

Proof of Theorem A. Assume, for a contradiction, that the statement is false and let X be a subnormal subgroup G which is not a *cn*-subgroup; in particular, X has finite rank. Among all counterexamples choose G in such a way that X has the smallest possible derived length. Then the derived subgroup $Y = X'$ of X is a *cn*-subgroup by the minimal choice on the derived length of X; on the other hand, Y has finite rank and so Y/Y_G is finite [6, Proposition 1]. Then X/Y_G is a finite-by-abelian group of finite rank, and hence its centre $Z/Y_G = Z(X/Y_G)$ has finite index in X/Y_G by Lemma 2.4. Thus Z is a subnormal subgroup of G which has finite index in X, so that the index $|Z : Z_G|$ is infinite and hence Z cannot be a *cn*-subgroup of G [6, Proposition 1]. Since Y_G has finite rank, the Hirsch-Plotkin radical of G/Y_G has infinite rank and so G/Y_G is also a counterexample; thus replacing G with G/Y_G and X with Z/Y_G it can be supposed that X is abelian. Hence X is contained in the Hirsh-Plotkin radical H of G.

Let P any primary component of H, and suppose that P has infinite rank. If F is the Fitting subgroup of P, then F is nilpotent by Lemma 2.3. Let A be a maximal abelian normal subgroup of F, then $A = C_F(A)$ (see [15, Part 1, Lemma 2.19.1]) and so A has infinite rank (see [15, Part 1, Theorem 3.29]). Hence G is a $T[*]$-group by Lemma 2.2. This contradiction proves that each primary component of H has finite rank. In particular, as H has infinite rank, there exist H_1 and H_2 subgroups of infinite rank such that $H = H_1 \times H_2$ and $\pi(H_1) \cap \pi(H_2) = \emptyset$. By the same reason, for $i \in \{1, 2\}$, two subgroups of infinite rank $H_{i,1}$ and $H_{i,2}$ can be found such that $H_i = H_{i,1} \times H_{i,2}$ and $\pi(H_{i,1}) \cap \pi(H_{i,2}) = \emptyset$. If $i, j \in \{1, 2\}$ and $i \neq j$, considered $\pi_i = \pi(H_i)$ and denoted by X_i the π_i-component of X, the subgroups $X_i H_{j,1}$ and $X_i H_{j,2}$ are subnormal subgroups of infinite rank of G, so that they are both *cn*-subgroups of G and hence $X_i = X_i H_{j,1} \cap X_i H_{j,2}$ is likewise a *cn*-subgroup of G. Therefore $X = X_1 X_2$ is a *cn*-subgroup of G and this final contradiction concludes the proof. □

Proof of Corollary. One may refine the derived series of G' to a series $G_1 = G' \geq \cdots \geq G_n = \{1\}$ whose factors $A_i = G_i/G_{i+1}$ are abelian p-groups (for possibly different primes). Let $C_i = C_G(A_i)$ for each i. If A_i has finite rank, then A_i is a Chernikov group and the same holds for G/C_i as a periodic group of automorphisms of a Chernikov group [15, Part 1, Theorem 3.29]. If A_i has infinite rank, then Lemma 2.2 yields that each subgroup of A_i is a *cn*-subgroup (resp *cf*-) of G. Hence, according to [4, Proposition 14], G/C_i is finite as a periodic group of power automorphisms of abelian p-groups [14, Lemma 4.1.2]. Thus if C is the intersection of all C_i's, then G/C is a Chernikov group and therefore has finite rank. It follows that C has infinite rank. On the other hand, C is nilpotent by a well-known fact (see [12]). Then by Theorem A, the group G has property $T[*]$ (resp. T_*). Further, by [4, Theorem 15], all subgroups of G are *cn* (resp. *cf*). □

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Proof of Theorem B. Assume that the statement is false. As in the first part of proof of Theorem A, there exists a counterexample G containing an abelian subnormal subgroup X that is not a *cn*-subgroup; in particular, X has finite rank and the index $|X : X_G|$ is infinite. Then $L = XG'$ is a nilpotent normal subgroup and hence has finite rank by Theorem A. Let $p \in \pi(X)$, then $L/L_{p'}$ is a nilpotent p-group of finite rank and hence it is a Chernikov group; thus $G/C_G(L/L_{p'})$ is finite (see [15, Part 1, Corollary p.85]) and hence $C_{G/L_{p'}}(L/L_{p'})$ is a nilpotent normal subgroup of infinite rank of $G/L_{p'}$. Thus Theorem A yields that $G/L_{p'}$ is a $T[*]$-group. Therefore $X_pL_{p'}$ is a *cn*-subgroup of G, and hence it is even cf because it has finite rank [6, Proposition 1]. The p-component of the core $(X_pL_{p'})_G$ of $X_pL_{p'}$in G is G-invariant, it coincides with the subgroup $X_p \cap (X_pL_{p'})_G$ and so has finite index in X_p, therefore X_p is cf. In particular, the set π of all primes p in $\pi(X)$ such that X_p is not normal in G is infinite. Replacing G by $G/L_{\pi'}$ it can be supposed that $\pi = \pi(L)$. Then there exists an infinite subset π_0 of π such that $G/L_{\pi_0'}$ contains a nilpotent normal subgroup of infinite rank (see [9, Corollary 11],); hence $G/L_{\pi_0'}$ is a $T[*]$-group by Theorem A. Therefore $X_{\pi_0}L_{\pi_0'}$ is a *cn*-subgroup of G and so even a cf-subgroup (see [6, Proposition 1]); hence X_{π_0} is cf and this is a contradiction because X_p is not normal in G for each $p \in \pi_0$.

References

[1] J. T. Buckley, J. C. Lennox, B. H. Neumann, H. Smith and J. Wiegold, Groups with all subgroups normal-by-finite, *J. Austral. Math. Soc. Ser. A*, **59** (1995) 384–398

[2] C. Casolo, Groups with finite conjugacy classes of subnormal subgroups, *Rend. Sem. Mat. Univ. Padova*, **81** (1989) 107–149

[3] C. Casolo, U. Dardano and S. Rinauro, Groups in which each subgroup is commensurable with a normal subgroup, *J. Algebra*, **496** (2018) 48–60

[4] U. Dardano and F. De Mari, Groups in which each subnormal subgroup is commensurable with some normal subgroup, *J. Group Theory*, https://doi.org/10.1515/jgth-2020-0076.

[5] U. Dardano and F. De Mari, On groups with all proper subgroups finite-by-abelian-by-finite, *Arch. Math. (Basel)*, https://doi.org/10.1007/s00013-021-01580-6.

[6] U. Dardano, F. De Mari and S. Rinauro, The weak minimal condition on subgroups which fail to be close to normal subgroups, *J. Algebra*, **560** (2020) 371–382

[7] U. Dardano, S. Rinauro, On soluble groups whose subnormal subgroups are inert, *Int. J. Group Theory*, **4** (2015) 17–24

[8] M. De Falco, F. de Giovanni, C. Musella, Y. P. Sysak, Groups of infinite rank in which normality is a transitive relation, *Glasg. Math. J.*, **56** (2014) 387–393

[9] M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi, Groups of infinite rank with finite conjugacy classes of subnormal subgroups, *J. Algebra* **431** (2015) 24–37

[10] F. De Mari, Groups with many modular or self-normalizing subgroups, to appear on *Comm. Algebra*, https://doi.org/10.1080/00927872.2020.1870999.

[11] S. Franciosi, F. de Giovanni, M. L. Newell, Groups whose subnormal subgroups are normal-by-finite, *Comm. Algebra*, **23** (1995) 5483–5497

[12] P. Hall, Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.*, **2** (1958) 787–801

[13] B. H. Neumann, Groups with finite classes of conjugate subgroups, *Math. Z.*, **63** (1955) 76–96

[14] D. J. S. Robinson, Groups in which normality is a transitive relation, *Proc. Cambridge Philos. Soc.*, **60** 1964 21–38

[15] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, New York-Berlin, 1972.

[16] D. J. S. Robinson, *A Course in the Theory of Groups*, Second edition. Graduate Texts in Mathematics, **80**, Springer-Verlag, New York, 1996.

Ulderico Dardano

Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II", Via Cintia - Monte S. Angelo, I-80126 Napoli, Italy

dardano@unina.it

Fausto De Mari

Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II", Via Cintia - Monte S. Angelo, I-80126 Napoli, Italy

fausto.demari@unina.it

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