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A NOTE ON THE AUTOMORPHISM GROUP OF THE HAMMING GRAPH

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Abstract. Let $m > 1$ be an integer and Ω be an m-set. The Hamming graph $H(n, m)$ has Ω^n as its vertex-set, with two vertices are adjacent if and only if they differ in exactly one coordinate. In this paper, we provide a new proof on the automorphism group of the Hamming graph $H(n, m)$. Although our result is not new (CE Praeger, C Schneider, Permutation groups and Cartesian decompositions, Cambridge University Press, 2018), we believe that our proof is shorter and more elementary than the known proofs for determining the automorphism group of Hamming graph.

1. Introduction

Let $m > 1$ be an integer and Ω be an m-set. The *Hamming graph* $H(n, m)$ has Ω^n as its vertex-set, with two vertices are adjacent if and only if they differ in exactly one coordinate. The latter graph is very famous and much is known about it. For instance this graph is actually the Cartesian product of n complete graphs K_m, that is, $K_m\square\cdots\square K_m$. Without lose of generality, we can assume that $\Omega = \{1, 2, \dots, m\}$. In general, the connection between Hamming graphs and coding theory is of major importance. If $m = 2$, then $H(n, m) = Q_n$, where Q_n is the hypercube of dimension n. Since the automorphism group of the hypercube Q_n has been already determined [12], in the sequel, we assume that $m \geq 3$. Figure 1, displays $H(2, 3)$ in the plane. Note that in Figure 1, we denote the vertex (x, y) by xy.

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It follows from the definition of the Hamming graph $H(n,m)$ that if $\theta \in Sym([n])$, then

$$f_\theta : V(H(n,m)) \longrightarrow V(H(n,m)),\ f_\theta(x_1,\dots,x_n) = (x_{\theta(1)},\dots,x_{\theta(n)}),$$

is an automorphism of the Hamming graph $H(n,m)$, and the mapping $\psi : Sym([n]) \longrightarrow Aut(H(n,m))$ defined by the rule $\psi(\theta) = f_\theta$ is an injection. Therefore, the set $H = \{f_\theta \mid \theta \in Sym([n])\}$ is a subgroup of $Aut(H(n,m))$ which is isomorphic with $Sym([n])$. Hence, $Sym([n]) \le Aut(H(n,m))$.

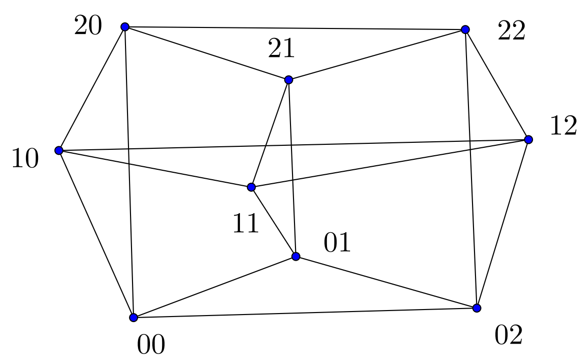

Figure 1. The Hamming graph H(2,3)

Let A, B be non-empty sets. Let $Fun(A,B)$ be the set of functions from A to B. If B is a group, then we can turn $Fun(A,B)$ into a group by defining a product $(fg)(a) = f(a)g(a),\ f,g \in Fun(A,B),\ a \in A$, where the product on the right of the equation is in B. Let $\Gamma = H(n,m)$. If $f \in Fun([n], \mathrm{Sym}([m]))$, then we define the mapping,

$$A_f : V(\Gamma) \to V(\Gamma), \quad \text{by the rule } A_f(x_1,\dots,x_n) = (f(1)(x_1),\dots,f(n)(x_n)).$$

It is easy to check that the mapping A_f is an automorphism of the graph Γ, and hence the group $F = \{A_f \mid f \in Fun([n], \mathrm{Sym}([m]))\}$ is a subgroup of the Hamming graph $H(n,m)$. Therefore, the subgroup which is generated by H and F in the group $Aut(\Gamma)$, namely, $W = \langle F, H\rangle$ is a subgroup of $Aut(\Gamma)$. In this paper, we wish to show that,

$$Aut(H(n,m)) = W = \langle F, H\rangle = F \rtimes H \cong Sym([m])wr_I Sym([n])$$

where $I = [n] = \{1,\dots,n\}$. Although our result is not new (see [20]), we believe that our proof is shorter and more elementary than the known proofs for determining the automorphism group of Hamming graph. Moreover, our proof is self contained [2].

There are various important families of graphs Γ in which we know that for a particular group G we have $G \le Aut(\Gamma)$, but showing that in fact we have $G = Aut(\Gamma)$ is a difficult task. For example note the following cases.

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(1) The *Boolean lattice* $BL_n, n \geq 1$ is the graph whose vertex-set is the set of all subsets of $[n] = \{1, 2, \ldots, n\}$, where two subsets x and y are adjacent if and only if their symmetric difference has precisely one element. The *hypercube* Q_n is the graph whose vertex set is $\{0, 1\}^n$, where two n-tuples are adjacent if they differ in precisely one coordinates. It is easy to show that $Q_n \cong BL_n$, and $Q_n \cong Cay(\mathbb{Z}_2^n; S)$, where $\mathbb{Z}_2$ is the cyclic group of order 2, and $S = \{e_i \mid 1 \leq i \leq n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 at the ith position. It is easy to show that the set $H = \{f_\theta \mid \theta \in Sym([n])\}$, $f_\theta(\{x_1, \ldots, x_n\}) = \{\theta(x_1), \ldots, \theta(x_n)\}$ is a subgroup of $Aut(BL_n)$, and hence H is a subgroup of the group $Aut(Q_n)$. We know that in every Cayley graph $\Gamma = Cay(G; S)$ the group $Aut(\Gamma)$ contains a subgroup isomorphic with the group G. Therefore, $\mathbb{Z}_2^n$ is a subgroup of $Aut(Q_n)$. Now, showing that $Aut(Q_n) = \langle \mathbb{Z}_2^n, Sym([n]) \rangle (\cong \mathbb{Z}_2^n \rtimes Sym([n]))$ is not an easy task [12].

(2) Let $n, k \in \mathbb{N}$ with $k < \frac{n}{2}$, and Let $[n] = \{1, \ldots, n\}$. The *Kneser graph* $K(n, k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices v,w are adjacent if and only if $|v \cap w|$=0. The Kneser graph $K(n, k)$ is a vertex-transitive graph [7]. It is easy to see that the group $H = \{f_\theta \mid \theta \in \mathrm{Sym}([n])\}$, $f_\theta(\{x_1, \ldots, x_k\}) = \{\theta(x_1), \ldots, \theta(x_k)\} (\cong Sym([n]))$, is a subgroup of $Aut(K(n, k))$ [7]. But, showing that $H = Aut(K(n, k))$ is rather a difficult work [7, Chapter 7, 16].

(3) Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, \ldots, n\}$. The *Johnson graph* $J(n, k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices v,w are adjacent if and only if $|v \cap w| = k - 1$. The Johnson graph $J(n, k)$ is a vertex-transitive graph [7]. It is easy to see that the set $H = \{f_\theta \mid \theta \in \mathrm{Sym}([n])\}$, $f_\theta(\{x_1, \ldots, x_k\}) = \{\theta(x_1), \ldots, \theta(x_k)\}$ is a subgroup of $Aut(J(n, k))$ [7]. It has been shown that $Aut(J(n, k)) \cong Sym([n])$ if $n \neq 2k$, and $Aut(J(n, k)) \cong Sym([n]) \times \mathbb{Z}_2$ if $n = 2k$ [4, 8, 13, 15].

2. Preliminaries

In this paper, a graph $\Gamma = \Gamma(V, E)$ is considered as a simple undirected graph with vertex-set $V(\Gamma) = V$, and edge-set $E(\Gamma) = E$. For all the terminology and notation not defined here, we follow [1, 3, 6, 7].

The group of all permutations of a set V is denoted by $Sym(V)$ or just $\mathrm{Sym}(n)$ when $|V| = n$. A *permutation group* G on V is a subgroup of $Sym(V)$. In this case we say that G act on V. If Γ is a graph with vertex set V, then we can view each automorphism as a permutation of V and so $Aut(\Gamma)$ is a permutation group. If G acts on V, we say that G is *transitive* (or G acts *transitively* on V) if there is just one orbit. This means that given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$.

Let Γ, Λ be arbitrary graphs with vertex set V_1, V_2 respectively. An isomorphism from Γ to Λ is a bijection $\psi : V_1 \longrightarrow V_2$ such that $\{x, y\}$ is an edge in Γ if and only if $\{\psi(x), \psi(y)\}$ is an edge in Λ. An isomorphism from a graph Γ to itself is called an automorphism of the graph Γ. The set of automorphisms of graph Γ with the operation of composition of functions is a group called the automorphism group of Γ and denoted by $Aut(\Gamma)$. Although, in most situations, it is difficult to determine the automorphism group of a graph and how it acts on its vertex-set, but there are various results in the literature and some of the recent works appear in the references [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21].

The graph Γ is called *vertex-transitive* if $Aut(\Gamma)$ acts transitively on $V(\Gamma)$. In other words, given any vertices u, v of Γ, there is an $f \in Aut(\Gamma)$ such that $f(u) = v$. For $v \in V(\Gamma)$ and $G = Aut(\Gamma)$ the *stabilizer subgroup* G_v is the subgroup of G containing of all automorphisms fixing v. In the vertex-transitive case all stabilizer subgroups G_v are conjugate in G, and consequently isomorphic. In this case, the index of G_v in G is given by the equation, $|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|$. If each stabilizer G_v is the identity group, then every element of G, except the identity, does not fix any vertex and we say that G act semiregularly on V. We say that G act regularly on V if and only if G acts transitively and semiregularly on V, and in this case we have $|V| = |G|$.

Let G be any abstract finite group with identity 1, and suppose Ω is a subset of G with the properties: (i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega$, (ii) $1 \notin \Omega$.
The *Cayley graph* $\Gamma = \Gamma(G; \Omega)$ is the (simple) graph whose vertex-set and edge-set are defined as follows: $V(\Gamma) = G$, $E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}$.

Let H, K be groups, with H acting on K in such way that the group structure of K is preserved (for example H is a subgroup of automorphisms of the group K). So for each $u \in K$ and $x \in H$ the mapping $u \longrightarrow u^x$ is an automorphism of K (Note that the action of H on K is not specified directly). The semi-direct product of K by H denoted by $K \rtimes H$ is the set

$$K \rtimes H = \{(u, x) \mid u \in K, x \in H\},$$

with binary operation $(u, x)(v, y) = (uv^{x^{-1}}, xy)$. Note that the identity element of the group $K \rtimes H$ is $(1_K, 1_H)$, and the inverse of the element (k, h) is the element $((k^{-1})^h, h^{-1})$.

3. Main Results

Let Γ be a connected graph with diameter D. Then we can partition the vertex-set $V(\Gamma)$ with respect to the distances of vertices from a fixed vertex. Let v be a fixed vertex of graph Γ. We denote the set of vertices at distance i from v, by $\Gamma_i(v)$. Thus, it is obvious that $\{v\} = \Gamma_0(v)$ and $N(v) = \Gamma_1(v)$, the set of adjacent vertices to vertex v, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(x), \ldots, \Gamma_D(x)$. If $\Gamma = H(n, m)$, then it is clear that two vertices are at distance k if and only if they differ in exactly k coordinates. Then the maximum distance occurs when the two vertices (regarded as ordered n-tuples) differ in all n coordinates. Thus the diameter of $H(n, m)$ is equal to n.

Lemma 3.1. *Let $m \geq 3$ and $\Gamma = H(n, m)$. Let $x \in V(\Gamma)$, $\Gamma_i = \Gamma_i(x)$ and $v \in \Gamma_i$. Then we have,*

$$\bigcap_{w \in \Gamma_{i-1} \cap N(v)} (N(w) \cap \Gamma_i) = \{v\}.$$

Proof. It is obvious that

$$v \in \bigcap_{w \in \Gamma_{i-1} \cap N(v)} (N(w) \cap \Gamma_i).$$

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Let $x = (x_1, \ldots, x_n)$. Since the Hamming graph $H(n,m)$ is a distance-transitive graph [3], then we can assume that, $v = (x_1, \ldots, x_{n-i}, y_{n-i+1}, \ldots, y_n)$, where $y_j \in \mathbb{Z}_m - \{x_j\}$ for all $j = n-i+1, \ldots, n$. Let $w \in \Gamma_{i-1} \cap N(v)$. Then w, x differ in exactly $i-1$ coordinates and w, v differ in exactly one coordinate. Note that, if we change one coordinates x_j of v, where $j = 1, \ldots, n-i$, then we obtain a vertex u such that $d(u,x) \geq i+1$. Thus, w has a form as follows,

$$w = w_r = (x_1, \ldots, x_{n-i}, y_{n-i+1}, \ldots, y_{r-1}, x_r, y_{r+1}, \ldots, y_n).$$

We show that if $u \in \Gamma_i$ and $u \neq v$ and u is adjacent to some w_r, then there is some w_p such that u is not adjacent to w_p.
If $v \neq u \in \Gamma_i$ is adjacent to w_r then u has one of the following forms;
(i) $u_1 = (x_1, \ldots, x_{n-i}, y_{n-i+1}, \ldots, y_{r-1}, y, y_{r+1}, \ldots, y_n)$, where $y \in \mathbb{Z}_m$, and $y \neq y_r, x_r$ (note that since $m \geq 3$, hence there is such a y).
(ii) $u_2 = (x_1 \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n-i}, y_{n-i+1}, \ldots, y_{r-1}, x_r, y_{r+1}, \ldots, y_n)$, where $y \in \mathbb{Z}_m$, and $y \neq x_j$.
In the case (i), u_1 is not adjacent to w_t, for all possible t, $t \neq r$.
In the case (ii), it is obvious that u_2 is not also adjacent to w_t for all possible t, $t \neq r$.
Our argument shows that if $u \in \Gamma_i$, and $u \neq v$, then there exists some w_r such that u is not adjacent to w_r, in other words $u \notin N(w_r)$. Thus we have,

$$\bigcap_{w \in N(v) \cap \Gamma_{i-1}} (N(w) \cap \Gamma_i) = \{v\}.$$

□

Let $J = \{\gamma_1, \ldots, \gamma_n\}$ be a set, K be a group and $Fun(J,K)$ be the group of all functions from J into K (with the operation, $(fg)(\gamma) = f(\gamma)g(\gamma),\ f, g \in Fun(J,K),\ \gamma \in J$). Since J is finite, the group $Fun(J,K)$ is isomorphic to K^n (the direct product of n copies of K), by the isomorphism $f \mapsto (f(\gamma_1), \ldots, f(\gamma_n))$. Let H be a group and assume that H acts on the nonempty set J. Then, the wreath product of K by H with respect to this action is the semidirect product $Fun(J,K) \rtimes H$ where H acts on the group $Fun(J,K)$, by the following rule,

$$f^x(\gamma) = f(\gamma^{x^{-1}}),\ f \in Fun(J,K), \gamma \in J,\ x \in H.$$

We denote the latter group by $Kwr_J H$. It is clear that if J, K and H are finite sets, then $G = Kwr_J H$ is a finite group, and we have $|G| = |K|^{|J|}|H|$.

In the sequel, we need the following fact [5].

Theorem 3.2. *Let Γ be a graph with n connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, where Γ_i is isomorphic to Γ_1 for all $i \in [n] = \{1, \ldots, n\} = I$. Then we have $Aut(\Gamma) = Aut(\Gamma_1) wr_I Sym([n])$.*

Lemma 3.3. *Let $n \geq 2,\ m \geq 3$ be integers. Let v be a vertex of the Hamming graph $H(n,m)$. Then, $\Gamma_1 = \langle N(v) \rangle$, the induced subgraph of $N(v)$ in $H(n,m)$, is isomorphic with nK_{m-1}, where nK_{m-1} is the disjoint union of n copies of the complete graph K_{m-1}.*

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Proof. Let $v = (v_1, \ldots, v_n)$. Then, for all i, $i \in \{1, \ldots, n\}$, there are $m-1$ elements j, $j \in \mathbb{Z}_m - \{v_i\}$. Let $x_{ij} = (v_1, \ldots, v_{i-1}, j, v_{i+1}, \ldots, v_n)$. Then, $N(v) = \{x_{ij} : \ 1 \leq i \leq n, \ j \in \mathbb{Z}_m - \{v_i\}\}$. Let x_{ij}, x_{rs} be two vertices in $\Gamma_1 = \langle N(v) \rangle$. Then x_{ij} and x_{rs} are adjacent in Γ_1 if and only if $i = r$ (note that two vertices $(v_1, \ldots, v_{i-1}, j, v_{i+1}, \ldots, v_n)$ and $(v_1, \ldots, v_{i-1}, s, v_{i+1}, \ldots, v_n)$ differ in only one coordinate). It is obvious that the subgraph induced by the set $\{x_{ij}, j \in \mathbb{Z}_m - \{v_i\}\}$ is isomorphic with K_{m-1}, the complete graph of order $m-1$. Now it is easy to see that the subgraph induced by the set $N(v)$ is isomorphic with nK_{m-1}, the disjoint union of n copies of the complete graph K_{m-1}. Now, the result follows from Theorem 3.2. □

We now are ready to prove the main result of this paper.

Theorem 3.4. *Let $n \geq 2$, $m \geq 3$, and $\Gamma = H(n,m)$ be a Hamming graph. Then $Aut(\Gamma) \cong Sym([n]) wr_I Sym([m])$, where $I = [n] = \{1, 2, \ldots, n\}$.*

Proof. Let $G = Aut(\Gamma)$. Let $x \in V = V(\Gamma)$, and $G_x = \{f \in G \mid f(x) = x\}$ be the stabilizer subgroup of the vertex x in Γ. Let $\langle N(x) \rangle = \Gamma_1$ be the induced subgroup of $N(x)$ in Γ. If $f \in G_x$ then $f_{|N(x)}$, the restriction of f to $N(x)$ is an automorphism of the graph Γ_1. We define the mapping $\psi : G_x \longrightarrow Aut(\Gamma_1)$ by the rule $\psi(f) = f_{|N(x)}$. It is not hard to show that ψ is a group homomorphism. We show that $Ker(\psi)$ is the identity group. If $f \in Ker(\psi)$, then $f(x) = x$ and $f(w) = w$ for every $w \in N(x)$. Let Γ_i be the set of vertices of Γ which are at distance i from the vertex x. Since the diameter of the graph $\Gamma = H(n,m)$ is n, then $V = V(\Gamma) = \bigcup_{i=0}^{n} \Gamma_i$. We prove by induction on i that $f(u) = u$ for every $u \in \Gamma_i$. Let $d(u,x)$ be the distance of the vertex u from x. If $d(u,x) = 1$, then $u \in \Gamma_1$ and we have $f(u) = u$. Assume that $f(u) = u$, when $d(u,x) = i-1$. If $d(u,x) = i$, then by Lemma 3.1, we have,

$$\{u\} = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} (N(w) \cap \Gamma_i).$$

Note that if $w \in \Gamma_{i-1}$, then $d(w,x) = i-1$, and hence $f(w) = w$. Therefore,

$$\{f(u)\} = f(\bigcap_{w \in \Gamma_{i-1} \cap N(u)} (N(w) \cap \Gamma_i)) = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} (N(f(w) \cap \Gamma_i) = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} (N(w) \cap \Gamma_i) = \{u\}.$$

Thus, $f(u) = u$ for all $u \in V(\Gamma)$, hence we have $Ker(\psi) = \{1\}$. On the other hand, $\frac{G_x}{Ker(\psi)} \cong \psi(G_x) \leq Aut(\Gamma_1)$, hence $G_x \cong \psi(G_x) \leq Aut(\Gamma_1)$. Thus, $|G_x| \leq |Aut(\Gamma_1)|$. We know by Lemma 3.3, that $\Gamma_1 \cong nK_{m-1}$. Since $\mathrm{Aut}(K_{m-1}) \cong Sym([m-1])$, then by Theorem 3.2, we have,

$$|G_x| \leq |Aut(\Gamma_1)| = |Sym([m-1]) wr_I Sym([n])| = ((m-1)!)^n n!,$$

where $I = [n] = \{1, \ldots, n\}$. Since $\Gamma = H(n,m)$ is a vertex-transitive graph, hence $|V(\Gamma)| = \frac{|G|}{|G_x|}$, and therefore,

$$|G| = |G_x||V(\Gamma)| \leq |Aut(nK_{m-1})| m^n = ((m-1)!)^n n! m^n = (m!)^n n! \qquad (*)$$

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We have seen (in introduction of this paper) that if $\theta \in Sym([n])$, where $[n] = \{1, \ldots, n\}$, then

$$f_\theta : V(H(n,m)) \longrightarrow V(H(n,m)), f_\theta(x_1, \ldots, x_n) = (x_{\theta(1)}, \ldots, x_{\theta(n)}),$$

is an automorphism of the Hamming graph $H(n,m)$. Moreover, the mapping $\psi : Sym([n]) \longrightarrow Aut(H(n,m))$ defined by the rule, $\psi(\theta) = f_\theta$ is an injection. Therefore, the set $H = \{f_\theta \mid \theta \in Sym([n])\}$ is a subgroup of $Aut((H(n,m)))$, which is isomorphic with $Sym([n])$. Hence, we have $Sym([n]) \leq Aut(H(n,m))$. On the other hand, if $f \in Fun([n], Sym([m]))$, then we define the mapping,

$$A_f : V(\Gamma) \to V(\Gamma) \textit{ by the rule, } A_f(x_1, \ldots, x_n) = (f(1)(x_1), \ldots, f(n)(x_n)).$$

It is easy to see that the mapping A_f is an automorphism of the Hamming graph Γ, and hence the group $F = \{A_f \mid f \in Fun([n], Sym([m]))\}$ is a subgroup of the Hamming graph $\Gamma = H(n,m)$. Therefore, the subgroup which is generated by H and F is in the group $Aut(\Gamma)$, namely, $W = \langle H, F\rangle$ is a subgroup of $Aut(\Gamma)$. Note that $W = Sym([m])wr_I Sym([n])$, where $I = [n] = \{1, 2, \ldots, n\}$. Since the subgroup W has $(m!)^n n!$ elements, then by $(*)$ we conclude that,

$$Aut(\Gamma) = W = Sym([m])wr_I Sym([n]).$$

□

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