Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 10 No. 2 (2021), pp. 121-128.

www.toc.ui.ac.ir
www.ui.ac.ir

THE a-NUMBER OF JACOBIANS OF CERTAIN MAXIMAL CURVES

VAHID NOUROZI, SAEED TAFAZOLIAN* AND FARHAD RAHMATI

Abstract. In this paper, we compute a formula for the a-number of certain maximal curves given by the equation $y^q + y = x^{\frac{q+1}{2}}$ over the finite field $\mathbb{F}_{q^2}$. The same problem is studied for the maximal curve corresponding to $\sum_{t=1}^{s} y^{q/2^t} = x^{q+1}$ with $q = 2^s$, over the finite field $\mathbb{F}_{q^2}$.

1. Introduction

Let $\mathcal{C}$ be a geometrically irreducible, projective, and non-singular algebraic curve defined over the finite field $\mathbb{F}_\ell$ of order ℓ. Let $\mathcal{C}(\mathbb{F}_\ell)$ denotes the set of $\mathbb{F}_\ell$-rational points of $\mathcal{C}$. In the study of curves over finite fields, a fundamental problem is on the size of $\mathcal{C}(\mathbb{F}_\ell)$. The very basic result here is the Hasse-Weil bound which asserts that

$$\mid \#\mathcal{C}(\mathbb{F}_\ell) - (\ell + 1) \mid \leq 2g\sqrt{\ell},$$

where $g = g(\mathcal{C})$ is the genus of $\mathcal{C}$.

The curve $\mathcal{C}$ is called maximal over $\mathbb{F}_\ell$ if the number of elements of $\mathcal{C}(\mathbb{F}_\ell)$ satisfies

$$\#\mathcal{C}(\mathbb{F}_\ell) = \ell + 1 + 2g\sqrt{\ell}.$$

We only consider maximal curves of positive genus and hence ℓ will always be a square, say $\ell = q^2$.

Communicated by Dariush Kiani
Manuscript Type: Research Paper
MSC(2010): Primary: 11G20; Secondary: 11M38, 13
Keywords: a-number; Cartier operator; Super-singular Curves; Maximal Curves
Received: 31 August 2020, Accepted: 31 January 2021
∗Corresponding author.
http://dx.doi.org/10.22108/toc.2021.124678.1758

.

In [10], Ihara showed that if a curve $\mathcal{C}$ is maximal over $\mathbb{F}_{q^2}$, then

$$g \leq g_1 := \frac{q(q-1)}{2}.$$

In [6] authors showed that

$$\text{either} \qquad g \leq g_2 := \lfloor \frac{(q-1)^2}{4} \rfloor \qquad \text{or} \qquad g_1 = \frac{q(q-1)}{2}.$$

Rück and Stichtenoth [16] showed that, up to $\mathbb{F}_{q^2}$-isomorphism, there is just one maximal curve over $\mathbb{F}_{q^2}$ of genus $\frac{q(q-1)}{2}$, namely the so-called Hermitian curve over $\mathbb{F}_{q^2}$ which can be defined by the affine equation

$$y^q + y = x^{q+1}.$$

If q is odd, from [5] there is a unique maximal curve $\mathcal{X}$ over $\mathbb{F}_{q^2}$ of genus $g = \frac{(q-1)^2}{4}$ which can be defined by the affine equation

$$y^q + y = x^{\frac{q+1}{2}}. \tag{1.1}$$

For q even, from [1] there is a unique maximal curve $\mathcal{Y}$ over $\mathbb{F}_{q^2}$ of genus $g = \lfloor \frac{(q-1)^2}{4} \rfloor = \frac{q(q-2)}{4}$ which can be defined by the affine equation

$$\sum_{t=1}^{s} y^{q/2^t} = x^{q+1} \text{ with } q = 2^s, \tag{1.2}$$

provided that $q/2$ is a Weierstrass non-gap at some point of the curve. It is easy to see that a maximal curve $\mathcal{C}$ is supersingular, since all slopes of its Newton polygon are equal $1/2$. This fact implies that the Jacobin $X := \mathrm{Jac}(\mathcal{C})$ has no p-torsion points over $\bar{\mathbb{F}}_p$. A relevant invariant of the p-torsion group scheme of the Jacobian of the curve is the a-number.

Consider the multiplication by p-morphism $[p] : X \to X$ which is a finite flat morphism of degree p^{2g}. It factors as $[p] = V \circ F$. Here, $F : X \to X^{(p)}$ is the relative Frobenius morphism coming from the p-power map on the structure sheaf; and the Verschiebung morphism $V : X^{(p)} \to X$ is the dual of F. The kernel of multiplication-by-p on X, is defined by the group of $X[p]$. The important invariant is the a-number $a(\mathcal{C})$ of curve $\mathcal{C}$ defined by

$$a(\mathcal{C}) = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}(\alpha_p, X[p]),$$

where α_p is the kernel of the Frobenius endomorphism on the group scheme $\mathrm{Spec}(k[X]/(X^p))$. Another definition for the a-number is

$$a(\mathcal{C}) = \dim_{\bar{\mathbb{F}}_p}(\mathrm{Ker}(F) \cap \mathrm{Ker}(V)).$$

A few results on the rank of the Carteir operator (especially a-number) of curves is introduced by Kodama and Washio [11], Gonzlez [8], Pries and Weir [15], Yui [19] and Montanucci and Speziali [13].

In this paper, we determine the a-number of certain maximal curves. In the case $g = g_1$, the a-number of the Hermitian curves is computed by Gross in [9]. Here we compute the a-number of maximal curves over $\mathbb{F}_{q^2}$ with genus $g = g_2$ for infinitely many values of q.

In Section 3, we prove that the a-number of the curve $\mathcal{X}$ with Equation (1.1) is $a(\mathcal{X}) = \frac{p-1}{8}(p^{s-1}+1)(q-1)$, see Theorem 3.2. In Section 4, we prove that the a-number of the curve $\mathcal{Y}$ with Equation (1.2) is $\frac{q^2}{16}$, see Theorem 4.1. The proofs use directly the action of the Cartier operator on $H^0(\mathcal{C}, \Omega^1)$.

2. The Cartier operator

Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathcal{C}$ be a curve defined over k. The Cartier operator is a $1/p$-linear operator acting on the sheaf $\Omega^1 := \Omega^1_{\mathcal{C}}$ of differential forms on $\mathcal{C}$ in positive characteristic $p > 0$.

Let $K = k(\mathcal{C})$ be the function field of the curve $\mathcal{C}$ of genus g defined over k. A separating variable for K is an element $x \in K \setminus K^p$.

Definition 2.1. *(The Cartier operator). Let $\omega \in \Omega_{K/K_q}$. There exist $f_0, \ldots, f_{p-1}$ such that $\omega = (f_0^p + f_1^p x + \cdots + f_{p-1}^p x^{p-1})dx$. The Cartier operator $\mathfrak{C}$ is defined by*

$$\mathfrak{C}(\omega) := f_{p-1}dx.$$

The definition does not depend on the choice of x (see [17, Proposition 1]*).*

We refer the reader to [17, 2, 3, 18] for the proofs of the following statements.

Proposition 2.2. *(Global Properties of $\mathfrak{C}$). For all $\omega \in \Omega_{K/K_q}$ and all $f \in K$,*

- $\mathfrak{C}(f^p\omega) = f\mathfrak{C}(\omega)$;
- $\mathfrak{C}(\omega) = 0 \Leftrightarrow \exists h \in K, \omega = dh$;
- $\mathfrak{C}(\omega) = \omega \Leftrightarrow \exists h \in K, \omega = dh/h$.

If $\operatorname{div}(\omega)$ is effective then differential ω is holomorphic. The set $H^0(\mathcal{C}, \Omega^1)$ of holomorphic differentials is a g-dimensional k-vector subspace of Ω^1 such that $\mathfrak{C}(H^0(\mathcal{C}, \Omega^1)) \subseteq H^0(\mathcal{C}, \Omega^1)$. If $\mathcal{C}$ is a curve, then the a-number of $\mathcal{C}$ equals the dimension of the kernel of the Cartier operator $H^0(\mathcal{C}, \Omega^1)$ (or equivalently, the dimension of the space of exact holomorphic differentials on $\mathcal{C}$) (see [12, 5.2.8]).

The following theorem is due to Gorenstein; see [4, Theorem 12].

Theorem 2.3. *A differential $\omega \in \Omega^1$ is holomorphic if and only if it is of the form $(h(x,y)/F_y)dx$, where $H : h(X,Y) = 0$ is a canonical adjoint.*

Theorem 2.4. [13] *With the above assumptions,*

$$\mathfrak{C}(h\frac{dx}{F_y}) = (\frac{\partial^{2p-2}}{\partial x^{p-1}\partial y^{p-1}}(F^{p-1}h))^{\frac{1}{p}}\frac{dx}{F_y}$$

for any $h \in K(\mathcal{X})$.

The differential operator ∇ is defined by

$$\nabla = \frac{\partial^{2p-2}}{\partial x^{p-1}\partial y^{p-1}},$$

has the property

$$\nabla(\sum_{i,j} c_{i,j}X^iY^j) = \sum_{i,j} c_{ip+p-1,jp+p-1}X^{ip}Y^{jp}. \tag{2.1}$$

3. The a-number of Curve $\mathcal{X}$

In this section, we consider the curve $\mathcal{X}$ is given by the equation $y^q+y = x^{\frac{q+1}{2}}$ of genus $g(\mathcal{X}) = \frac{(q-1)^2}{4}$, with $q = p^s$ and $p > 2$ over $\mathbb{F}_{q^2}$. From Theorem 2.3, one can find a basis for the space $H^0(\mathcal{X}, \Omega^1)$ of holomorphic differentials on $\mathcal{X}$, namely

$$\mathcal{B} = \{x^iy^jdx \mid 1 \leq \frac{q+1}{2}i + qj \leq g\}.$$

Proposition 3.1. *The rank of the Cartier operator $\mathfrak{C}$ on the curve $\mathcal{X}$ equals the number of pairs (i,j) with $\frac{q+1}{2}i + qj \leq g$ such that the system of congruences mod p*

$$\begin{cases} kq + h - k + j \equiv 0, \\ (p-1-h)(\frac{(q+1)}{2}) + i \equiv p-1, \end{cases} \tag{3.1}$$

has a solution (h,k) for $0 \leq h \leq \frac{p-1}{2}, 0 \leq k \leq h$.

Proof. By Theorem 2.4, $\mathfrak{C}((x^iy^j/F_y)dx) = (\nabla(F^{p-1}x^iy^j))^{1/p}dx/F_y$. So, we apply the differential operator ∇ to

$$(y^q + y - x^{\frac{q+1}{2}})^{p-1}x^iy^j = \sum_{h=0}^{p-1}\sum_{k=0}^{h}\tbinom{p-1}{h}\tbinom{h}{k}(-1)^{h-k}x^{(p-1-h)(\frac{(q+1)}{2})+i}y^{kq+h-k+j} \tag{3.2}$$

for each i, j such that $\frac{q+1}{2}i + qj \leq g$.

From the Formula (2.1), $\nabla(y^q+y+x^{\frac{q+1}{2}})^{p-1}x^iy^j \neq 0$ if and only if for some (h,k), with $0 \leq h \leq \frac{p-1}{2}$ and $0 \leq k \leq h$, satisfies both the following congruences mod p:

$$\begin{cases} kq + h - k + j \equiv 0, \\ (p-1-h)(\frac{(q+1)}{2}) + i \equiv p-1. \end{cases} \tag{3.3}$$

Take $(i,j) \neq (i_0,j_0)$ in this situation both $\nabla(y^q + y + x^{\frac{q+1}{2}})^{p-1}x^iy^j$ and $\nabla(y^q + y + x^{\frac{q+1}{2}})^{p-1}x^{i_0}y^{j_0}$ are nonzero. We claim that they are linearly independent over k. To show independence, we prove that, for each (h,k) with $0 \leq h \leq p-1$ and $0 \leq k \leq h$ there is no (h_0,k_0) with $0 \leq h_0 \leq p-1$ and $0 \leq k_0 \leq h_0$ such that

$$\begin{cases} kq + h - k + j = k_0q + h_0 - k_0 + j_0, \\ (p-1-h)(\frac{(q+1)}{2}) + i = (p-1-h_0)(\frac{(q+1)}{2}) + i_0. \end{cases} \tag{3.4}$$

If $h = h_0$, then $j \neq j_0$ by $i = i_0$ from the second equation, therefore $k \neq k_0$. We may assume $k > k_0$. Then $j - j_0 = (q-1)(k-k_0) > q-1$, a contradiction as $j - j_0 \leq \frac{(q-1)^2}{4q}$. Similarly, if $k = k_0$, then

http://dx.doi.org/10.22108/toc.2021.124678.1758

$h \neq h_0$ by $(i,j) \neq (i_0, j_0)$. We assume that $h > h_0$. Then $i - i_0 = \frac{q+1}{2}(h - h_0) > \frac{q+1}{2}$, a contradiction as $i - i_0 \leq \frac{(q-1)^2}{2(q+1)}$. □

For the rest in this Section, $A_s := A(\mathcal{X})$ denotes the matrix representing the p-th power of the Cartier operator $\mathfrak{C}$ on the curve $\mathcal{X}$ with respect to the basis $\mathcal{B}$, where $q = p^s$. Now we are able to compute the a-number of curve $\mathcal{X}$.

Theorem 3.2. *If $q = p^s$ for $s \geq 1$ and $p > 2$, then the a-number of the curve $\mathcal{X}$ equals*

$$\frac{p-1}{8}(p^{s-1}+1)(q-1).$$

Proof. First we prove that, if $q = p^s, s \geq 1$, then $\operatorname{rank}(A_s) = \dfrac{p+1}{8}(p^s - 1)(p^{s-1} - 1)$. In this case, $\frac{q+1}{2}i + qj \leq g$ and System (3.1) mod p reads

$$\begin{cases} h - k + j \equiv 0, \\ -\frac{h}{2} - \frac{1}{2} + i \equiv p - 1. \end{cases} \tag{3.5}$$

First assume that $s = 1$, for $q = p$, we have $\frac{p+1}{2}i + pj \leq g$ and System (3.5) becomes

$$\begin{cases} j = k - h, \\ i = p + \frac{h}{2} - \frac{1}{2}, \end{cases}$$

in this case $\frac{p+1}{2}i + pj \leq g$ that is, $\frac{h(1-3p)}{4} + kp \leq \frac{-p^2-3p+2}{4}$ then $h \geq \frac{-p^2-3p+2}{1-3p}$, thus $h \geq \frac{3p+10}{9}$ a contradiction by Proposition 3.1. As a consequence, there is no pair (i,j) for which the above system admits a solution (h,k). Thus, $\operatorname{rank}(A_1) = 0$.

Let $s = 2$, so $q = p^2$. For $\frac{p^2+1}{2}i + p^2 j \leq g$, the above argument still works. Therefore, $\frac{(p-1)^2}{4} + 1 \leq \frac{p^2+1}{2}i + p^2 j \leq \frac{(p^2-1)^2}{4}$ and our goal is to determine for which (i,j) there is a solution (h,k) of the system mod p

$$\begin{cases} h - k + j \equiv 0, \\ -\frac{h}{2} - \frac{1}{2} + i \equiv p - 1. \end{cases}$$

Take $l, m \in Z_0^+$ so that

$$\begin{cases} j = lp + k - h, \\ i = mp + p + \frac{h}{2} - \dfrac{1}{2}. \end{cases}$$

In this situation, $i < \frac{2g}{p^2+1} = \frac{(p^2-1)^2}{2(p^2+1)}$, so $mp + p + \frac{h}{2} - \frac{1}{2} \leq \frac{(p^2-1)^2}{2(p^2+1)}$. Then $m \leq \frac{(p^2-1)^2}{2(p^2+1)}$. And $j < \frac{(p^2-1)^2}{4p^2}$, so $lp + k - h < \frac{(p^2-1)^2}{4p^2}$, Then $l < \frac{(p^2-1)^2}{4p^2}$. From this we can say that $\frac{p^2-1}{4} - 1 \leq l \leq \frac{p^2-1}{4}$, and $\frac{p^2-1}{2} \leq m \leq \frac{p^2-1}{2}$. In this way, $\dfrac{(p^2-1)^2}{8}$ suitable values for (i,j) are obtained, whence $\operatorname{rank}(A_2) = \dfrac{(p^2-1)^2}{8}$.

http://dx.doi.org/10.22108/toc.2021.124678.1758

For $s \geq 3$, $\mathrm{rank}(A_s)$ equals $\mathrm{rank}(A_{s-1})$ plus the number of pairs (i,j) with $\frac{(p^{s-1}-1)^2}{4}+1 \leq \frac{q+1}{2}i+qj \leq \frac{(p^s-1)^2}{4}$ such that the system mod p

$$\begin{cases} h-k+j \equiv 0, \\ -\frac{h}{2}-\frac{1}{2}+i \equiv p-1, \end{cases}$$

has a solution. With our usual conventions on l, m, a computation shows that such pairs (i,j) are obtained for $0 \leq l \leq \frac{(p^s-1)^2}{4p^{s+1}}$ from this we have $\frac{p^{s-2}(p^2-1)}{4}-1 \leq l \leq \frac{p^{s-2}(p^2-1)}{4}$, and $0 \leq m \leq \frac{(p^s-1)^2}{2(p^s+1)}$ from this we have $\frac{(p^{s-1}-1)(p+1)}{2}-1 \leq m \leq \frac{(p^{s-1}-1)(p+1)}{2}$. In this case we have

$$\frac{(p^{s-1}-1)(p+1)p^{s-2}(p^2-1)}{8}$$

choices for (h,k). Therefore we get

$$\mathrm{rank}(A_s) = \mathrm{rank}(A_{s-1}) + \frac{(p^{s-1}-1)(p+1)p^{s-2}(p^2-1)}{8}.$$

Now our claim on the rank of A_s follows by induction on s. Hence

$$\begin{aligned} a(\mathcal{X}) &= \frac{(p^s-1)^2}{4} - \frac{(p+1)(p^s-1)(p^{s-1}-1)}{8} \\ &= \frac{(p^s-1)}{8}(p^s+p-p^{s-1}-1) \\ &= \frac{(p^s-1)}{8}(p(p^{s-1}+1)-(p^{s-1}+1)) \\ &= \frac{(p^s-1)}{8}((p^{s-1}+1)(p-1)) \\ &= \frac{(p-1)}{8}((p^{s-1}+1)(q-1)). \end{aligned}$$

□

For the finite feild $\mathbb{F}_{q^2}$ let m be an integer number, such that m divides $(q+1)$. In this case the curve $y^q+y=x^m$ is maximal over $\mathbb{F}_{q^2}$. From this fact we are led to the following problem.

Problem 3.3. *What is the dimension of the space of exact holomorphic differentials of* $y^q+y=x^m$ *where* $m \mid (q+1)$

4. The a-number of Curve $\mathcal{Y}$

In this section, we consider the curve $\mathcal{Y}$ given by the equation $\sum_{t=1}^{s} y^{q/2^t} = x^{q+1}$ of genus $g(\mathcal{Y}) = \frac{q(q-2)}{4}$, with $q=2^s$ and $p=2$ over $\mathbb{F}_{q^2}$. With the simple computation, we have $\mathrm{div}_\infty(x) = q/2P_1$ and $\mathrm{div}_\infty(y) = (q+1)P_1$, so one can find a basis for the space $H^0(\mathcal{Y}, \Omega^1)$ of holomorphic differentials on $\mathcal{Y}$, namely

$$\mathcal{B}' = \{x^i y^j dx \mid (q+1)i + \frac{q}{2}j \leq 2g-2\}. \tag{4.1}$$

Theorem 4.1. *If* $q=2^s$ *for* $s \geq 1$, *then the* a*-number of the curve* $\mathcal{Y}$ *equals*

$$\frac{q^2}{16}.$$

Proof. In characteristic two, every 1-form $\omega \in H^0(\mathcal{Y}, \Omega^1)$ can be written as $\omega = (f^2 + g^2x)dx$. So we have

$$\mathfrak{C}((f^2 + g^2x)dx) = gdx \tag{4.2}$$

in characteristic two. By Equation (4.2), a-number of $\mathcal{Y}$ is the dimensional vector space of regular 1-forms of the form f^2dx. For each even integers i, j, we have $\mathfrak{C}(x^iy^jdx) = 0$. So we want to find (i, j), where i is an odd number and j is an even number. We know that $0 \le i \le \frac{2g-2}{2(q+1)}$ and $0 \le j \le \frac{2g-2}{2q}$. Therefore this follows from the fact that

$$\frac{q}{4} - 1 < \frac{2g-2}{2(q+1)} < \frac{q}{4}$$

there are $\frac{q}{4}$ choices of i and from the fact that

$$\frac{q}{4} - 1 < \frac{2g-2}{2q} < \frac{q}{4}$$

there are $\frac{q}{4}$ choices of j. Hence

$$a(\mathcal{Y}) = \frac{q^2}{16}$$

□

Acknowledgments

This paper was written while Vahid Nourozi was visiting Unicamp (Universidade Estadual de Campinas) supported by TWAS/Cnpq (Brazil) with fellowship number $314966/2018-8$. And the second author was supported by FAPESP/SP-Brazil grant $2018/24604-6$.

References

[1] M. Abdon and F. Torres, On maximal curves in characteristic two, *Manuscripta Math.*, **99** (1999) 39–53.

[2] P. Cartier, Une nouvelle opration sur les formes diffrentielles. *C. R. Acad. Sci. Paris*, **244** (1957) 426–428.

[3] P. Cartier, Questions de rationalit des diviseurs en gomtrie algbrique, *Bull. Soc. Math. France*, **86** (1958) 177–251.

[4] D. Gorenstein, An arithmetic theory of adjoint plane curves, *Trans. Am. Math. Soc.*, **72** (1952) 414–436.

[5] R. Fuhrmann, A. Garcia and F. Torres, On maximal curves, *J. Number Theory*, **67** (1997) 29–51.

[6] R. Fuhrmann and F. Torres, The genus of curves over finite fields with many rational points, *Manuscripta Math*, **89** (1996) 103–106.

[7] H. Friedlander, D. Garton, B. Malmskog, R. Pries and C. Weir, The a-number of Jacobians of Suzuki curves, *Proc. Amer. Math. Soc.*, **141** (2013) 3019–3028.

[8] J. Gonzlez, Hasse-Witt matrices for the Fermat curves of prime degree, *Tohoku Math. J.*, **49** (1997) 149–163.

[9] B. H. Gross, Group representations and lattices, *J. Am. Math. Soc.*, **3** (1990) 929–960.

[10] Y. Ihara, Some remarks on the number of rational points of algebraic curves over finite fields, *J. Fac. Sci. Tokyo*, **28** (1981) 721–724.

[11] T. Kodama and T. Washio, Hasse-Witt matrices of Fermat curves, *Manuscr. Math.*, **60** (1988) 185–195.

[12] K.-Z. Li and F. Oort, Moduli of Supersingular Abelian Varieties, *Lecture Notes in Mathematics*, **1680**, Springer-Verlag, Berlin, 1998, iv+116pp.

[13] M. Montanucci and P. Speziali, The a-numbers of Fermat and Hurwitz curves, *J. Pure Appl. Algebra*, **222** (2018) 477–488.

[14] V. Nourozi, F. Rahmati and S. Tafazolian, The a-number of certain hyperelliptic curves, ArXiv: 1902.03672v2, 2019.

[15] R. Pries and C. Weir, The Ekedahl-Oort type of Jacobians of Hermitian curves, *Asian J. Math.*, **19** (2015) 845–869.

[16] H. G. Rück and H. Stichtenoth, A characterization of Hermitian function fields over finite fields, *J. Reine Angew. Math.*, **457** (1994) 185–188.

[17] C. S. Seshadri, Loperation de Cartier, Applications, In Varietes de Picard, **4**, of Sminaire Claude Chevalley. Secrtariat Mathmatiques, Paris, 1958–1959.

[18] M. Tsfasman, S. Vladut and D. Nogin, *Algebraic geometric codes: basic notions*, **139**, of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007.

[19] N. Yui, On the Jacobian Varieties of Hyperelliptic Curves over Fields of Characteristic p, *J. Algebra*, **52** (1978) 378–410.

Vahid Nourozi

Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Ave., Tehran 15914, Iran

Email: nourozi@aut.ac.ir

Saeed Tafazolian

IMECC/UNICAMP, R. Sergio Buarque de Holanda, 651, Cidade Universitaria, Zeferino Vaz, 13083-859, Campinas, SP, Brazil

Email: tafazolian@ime.unicamp.br

Farhad Rahmati

Faculty of Mathematics and Computer Science, Amirkabir University of Technology, (Tehran Polytechnic), 424 Hafez Ave., Tehran 15914, Iran

Email: frahmati@aut.ac.ir

http://dx.doi.org/10.22108/toc.2021.124678.1758