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BOUNDEDLY FINITE CONJUGACY CLASSES OF TENSORS

RAIMUNDO BASTOS AND CARMINE MONETTA*

Abstract. Let n be a positive integer and let G be a group. We denote by $\nu(G)$ a certain extension of the non-abelian tensor square $G \otimes G$ by $G \times G$. Set $T_{\otimes}(G) = \{g \otimes h \mid g, h \in G\}$. We prove that if the size of the conjugacy class $\left|x^{\nu(G)}\right| \leq n$ for every $x \in T_{\otimes}(G)$, then the second derived subgroup $\nu(G)''$ is finite with n-bounded order. Moreover, we obtain a sufficient condition for a group to be a BFC-group.

Dedicated to Pavel Shumyatsky on the occasion of his 60th birthday

1. Introduction

The non-abelian tensor square $G \otimes G$ of a group G as introduced by R. Brown and J. L. Loday [4] is defined to be the group generated by all symbols $g \otimes h$, $g, h \in G$, subject to the relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$$

for all $g, g_1, h, h_1 \in G$. In the same paper, R. Brown and J.-L. Loday presented a topological significance for the non-abelian tensor square of groups (cf. [4, Section 3]). The study of the non-abelian tensor square of groups from a group theoretic point of view was initiated by R. Brown, D. L. Johnson and E. F. Robertson [3]. If x and y are group-elements, we denote by $x^y = y^{-1}xy$ the conjugate of x by y, while the commutator of x and y is the element $[x, y] = x^{-1}x^y$. We observe that the defining relations of the tensor square can be viewed as abstractions of commutator relations; thus in [10] the

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*Corresponding author.
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following construction is considered. Let G be a group and $\varphi : G \to G^{\varphi}$ an isomorphism (G^{φ} is an isomorphic copy of G, where $g \mapsto g^{\varphi}$, for all $g \in G$). Define the group $\nu(G)$ to be

$$\nu(G) := \langle G \cup G^{\varphi} \mid [g_1, {g_2}^{\varphi}]^{g_3} = [{g_1}^{g_3}, ({g_2}^{g_3})^{\varphi}] = [g_1, {g_2}^{\varphi}]^{{g_3}^{\varphi}}, \ \ g_i \in G\rangle.$$

The motivation for studying $\nu(G)$ is the commutator connection: indeed, the map $\Phi : G \otimes G \to [G, G^{\varphi}]$, defined by $g \otimes h \mapsto [g, h^{\varphi}]$, for all $g, h \in G$, is an isomorphism (N. R. Rocco, [10, Proposition 2.6]). Therefore, from now on we identify the non-abelian tensor square $G \otimes G$ with the subgroup $[G, G^{\varphi}]$ of $\nu(G)$. An element $\alpha \in \nu(G)$ is called a *tensor* if $\alpha = [a, b^{\varphi}]$ for suitable $a, b \in G$. We write $T_{\otimes}(G)$ to denote the set of all tensors (in $\nu(G)$). In particular, $T_{\otimes}(G)$ is a commutator closed subset of the group $\nu(G)$, where a subset X of a group is called commutator closed if $[x, y] \in X$ for any $x, y \in X$.

It is well-known that the set of all tensors $T_{\otimes}(G)$ affects the structure of the non-abelian tensor square $G \otimes G$, and of related constructions (see [1, 2]). For instance, in [1] and in [2], it was proved that if the set $T_{\otimes}(G)$ is finite (resp. has exactly n elements), then the non-abelian tensor square $[G, G^{\varphi}]$ is finite (resp. finite with n-bounded order). Here and throughout the article we use the expression $\{a, b, \ldots\}$-bounded to mean that a quantity is bounded by a certain number depending only on the parameters $a, b, \ldots$.

Recall that a group G is a BFC-group if every conjugacy class of G contains at most n elements, for a positive integer n. In [8], B. H. Neumann proved that the group G is a BFC-group if and only if the derived subgroup G' is finite, and this occurs if and only if G contains only finitely many commutators. Later, J. Wiegold proved a quantitative version of Neumann's result: if G contains exactly m commutators, then the order of the derived subgroup G' is finite with m-bounded order [12, Theorem 4.7], and the best known bound was obtained in [7]. Recall that $x^G = \{x^g \mid g \in G\}$ denote the conjugacy class of x in G. Recently, in [6], G. Dierings and P. Shumyatsky proved that if $|x^G| \leq n$ for every commutator x in G, then the second derived subgroup G'' is finite with n-bounded order. Subsequently, E. Detomi, M. Morigi and P. Shumyatsky obtain an interesting version for higher commutator subgroups (cf. [5]).

In the present paper we consider non-abelian tensor square of groups (and related construction) in which tensors have a bounded number of conjugates, obtaining the following result.

Theorem 1.1. *Let n be a positive integer and G a group. Suppose that the size of the conjugacy class $|x^{\nu(G)}| \leq n$ for every $x \in T_{\otimes}(G)$. Then the second derived subgroup $\nu(G)''$ is finite with n-bounded order.*

It is worth to mention that we manage to adapt the proof of [6, Theorem 1.1] in the settings of tensors using the tools presented in [11].

Furthermore, we also examine the structure of G in terms of some finiteness properties of the set of tensors $T_{\otimes}(G)$, obtaining a sufficient condition for a group G to be a BFC-group.

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Corollary 1.2. *Let n be a positive integer. Let G be a group in which the derived subgroup G' is finitely generated. Assume that the size of the conjugacy class $|x^{\nu(G)}| \leq n$ for every $x \in T_{\otimes}(G)$. Then G is a BFC-group.*

Finally, we show, by means of two counterexamples, that Corollary 1.2 is no longer true neither if we discard the hypothesis about G' to be finitely generated (see Example 3.7, below) nor if we consider bounded conjugacy classes of commutators instead of tensors (see Example 3.8, below).

The paper is organized as follows. In the next section we collect some preliminary results about the non-abelian tensor square, while Section 3 contains the proofs of our main results.

2. The group $\nu(G)$

It is well known that the finiteness of the non-abelian tensor square $[G, G^{\varphi}]$, does not imply that G is a finite group. For instance, the Prüfer group C_{p^∞} is an example of an infinite group such that the non-abelian tensor square $[C_{p^\infty}, (C_{p^\infty})^{\varphi}]$ is trivial (and so, finite). This is the case for all torsion, divisible abelian groups. A useful result, due to Parvizi and Niroomand [9], provides a sufficient condition for a group to be finite.

Lemma 2.1. *Let G be a finitely generated group. Suppose that the non-abelian tensor square $[G, G^{\varphi}]$ is finite. Then G is finite.*

The following basic properties are consequences of the defining relations of $\nu(G)$ and commutator rules (see [10, Section 2] for more details).

Lemma 2.2. *The following relations hold in $\nu(G)$, for all $g, h, x, y \in G$.*

(a) $[g, h^{\varphi}]^{[x, y^{\varphi}]} = [g, h^{\varphi}]^{[x,y]}$;
(b) $[g, h^{\varphi}, x^{\varphi}] = [g, h, x^{\varphi}] = [g, h^{\varphi}, x] = [g^{\varphi}, h, x^{\varphi}] = [g^{\varphi}, h^{\varphi}, x] = [g^{\varphi}, h, x]$;
(c) $[[g, h^{\varphi}], [x, y^{\varphi}]] = [[g, h], [x, y]^{\varphi}]$.

Given a group G there exists an epimorphism

$$\rho : \nu(G) \to G,$$

given by $g \mapsto g$, $h^{\varphi} \mapsto h$. In the notation of [11, Section 2], let $\Theta(G)$ denote the kernel of ρ. The following lemma collects some results related to the subgroup $\Theta(G)$ which can be found in [11, Section 2].

Lemma 2.3.

(a) *The quotient group $\nu(G)/\Theta(G)$ is isomorphic to G;*
(b) *The subgroup $\Theta(G)$ centralizes $[G, G^{\varphi}]$.*

The next lemma is a crucial observation, that we will use several times in our proofs.

Lemma 2.4. *Let $\alpha, \beta \in \nu(G)$.*

(a) $[x, y^\varphi]^\alpha = [x, y^\varphi]^{\rho(\alpha)} = [x, y^\varphi]^{\rho(\alpha)^\varphi}$, *for any* $x, y \in G$;

(b) *There exists an element* $\theta \in \Theta(G)$ *such that*

$$[\alpha, \beta] = [u, v^\varphi]\theta,$$

where $u = \rho(\alpha)$ *and* $v = \rho(\beta)$.

Proof. (a) It holds by defining relations of $\nu(G)$.

(b) Note that

$$\rho([\alpha, \beta]) = [u, v] = \rho([u, v^\varphi]).$$

Since $\Theta(G)$ is the kernel of ρ, it follows that there exists an element $\theta \in \Theta(G)$ such that

$$[\alpha, \beta] = [u, v^\varphi]\theta,$$

as well. The proof is complete. □

Given an element $\alpha \in [G, G^\varphi]$, we define $l_\otimes(\alpha)$ to be the smallest positive integer such that α may be written as a product of $l_\otimes(\alpha)$ tensors. We call $l_\otimes(\alpha)$ the length of α with respect to $T_\otimes(G)$. The following result is an immediate consequence of [6, Lemma 2.1].

Lemma 2.5. *Let* C *be a subgroup of finite index* m *in* $[G, G^\varphi]$. *Then for every* $b \in [G, G^\varphi]$, *the coset* Cb *contains an element* h *such that* $l_\otimes(h) \le m - 1$.

3. **Proofs**

We will now fix some notation and hypothesis.

Hypothesis 3.1. *Let* G *be a group. Suppose that* $C_{\nu(G)}(x)$ *has finite index at most* n *in* $\nu(G)$ *for each* $x \in T_\otimes(G)$. *Let* m *be the maximum of indices of* $C_{[G,G^\varphi]}(x)$ *in* $[G, G^\varphi]$, *where* $x \in T_\otimes(G)$. *Then consider* $a \in T_\otimes(G)$ *such that* $C_{[G,G^\varphi]}(a)$ *has index precisely* m *in* $[G, G^\varphi]$. *By Lemma 2.5 we can choose* $b_1, \dots, b_m \in [G, G^\varphi]$ *such that* $l_\otimes(b_i) \le m - 1$ *and* $a^H = \{a^{b_i};\ i = 1, \dots, m\}$. *Finally set* $U = C_{\nu(G)}(\langle b_1, \dots, b_m \rangle)$.

Remark 3.2. *Notice that the subgroup* U *has finite* n*-bounded index in* $\nu(G)$ *because* $l_\otimes(b_i) \le m - 1$ *and* $C_{\nu(G)}(x)$ *has index at most* n *in* $\nu(G)$ *for every* $x \in T_\otimes(G)$. *Moreover, from Lemma 2.3 (b) it follows that the subgroup* $\Theta(G)$ *is contained in* U.

Since Lemma 2.2 (a) shows that the commutator of tensors is still a tensor, from [6, Lemma 2.3] we have directly the following.

Lemma 3.3. *Assume Hypothesis 3.1. Then the subgroup* $[[G, G^\varphi], x]$ *has finite* m*-bounded order for any* $x \in T_\otimes(G)$.

The next lemma is somewhat analogous to [12, Lemma 4.5].

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Lemma 3.4. *Assume Hypothesis 3.1. If $u \in U$ and $ua \in T_{\otimes}(G)$, then $[[G, G^{\varphi}], u] \leq [[G, G^{\varphi}], a]$.*

Proof. For every $i = 1, \ldots, m$ $(ua)^{b_i} = ua^{b_i}$. Thus the elements ua^{b_i} form the conjugacy class $(ua)^{[G,G^{\varphi}]}$ because $|(ua)^{[G,G^{\varphi}]}| \leq m$. Therefore, whenever we consider an element $h \in [G, G^{\varphi}]$ there exists $i \in \{1, \ldots, m\}$ such that $(ua)^h = ua^{b_i}$ and so $u^h a^h = ua^{b_i}$. Hence,

$$[u, h] = a^{b_i} a^{-h} = [b_i, a^{-1}][a^{-1}, h] \in [[G, G^{\varphi}], a].$$

The lemma follows. □

Proposition 3.5. *Assume Hypothesis 3.1 and write $a = [d, e^{\varphi}]$ for suitable $d, e \in G$. Then there exists a subgroup $U_1 \leq U$ with the following properties.*

1. *The index of U_1 in $\nu(G)$ is n-bounded;*
2. $[[G, G^{\varphi}], U_1'] \leq [[G, G^{\varphi}], a]^{d^{-1}}$*;*
3. $[[G, G^{\varphi}], [U_1, d]] \leq [[G, G^{\varphi}], a]$*.*

Proof. Set

$$U_1 = U \cap U^{d^{-1}} \cap U^{d^{-1}e^{-1}}.$$

Since the index of U in $\nu(G)$ is n-bounded, we conclude that the index of U_1 in $\nu(G)$ is n-bounded as well.

Now, for every $h_1, h_2 \in U_1$ we have

$$[\rho(h_1)d, e^{\varphi}\rho(h_2)^{\varphi}] = [\rho(h_1), \rho(h_2)^{\varphi}]^d [d, \rho(h_2)^{\varphi}][\rho(h_1), e^{\varphi}]^{dh_2}[d, e^{\varphi}]^{h_2}$$

and so

$$[\rho(h_1)d, e^{\varphi}\rho(h_2)^{\varphi}]^{h_2^{-1}} = [\rho(h_1), \rho(h_2)^{\varphi}]^{dh_2^{-1}}[d, \rho(h_2)^{\varphi}]^{h_2^{-1}}[\rho(h_1), e^{\varphi}]^d[d, e^{\varphi}].$$

Set $u = [\rho(h_1), \rho(h_2)^{\varphi}]^{dh_2^{-1}}[d, \rho(h_2)^{\varphi}]^{h_2^{-1}}[\rho(h_1), e^{\varphi}]^d$. Thus, on the left hand side of the above equation we have a tensor, while the right hand side coincides with ua. Now we show that $u \in U$.

By Lemma 2.4, there exist elements $\alpha, \beta, \gamma \in \Theta(G)$ such that

(i) $[\rho(h_1), \rho(h_2)^{\varphi}] = [h_1, h_2]\alpha$;
(ii) $[d, \rho(h_2)^{\varphi}] = [d, h_2]\beta$;
(iii) $[\rho(h_1), e^{\varphi}] = [h_1, e]\gamma$.

From (i) we have $[\rho(h_1), \rho(h_2)^{\varphi}]^{dh_2^{-1}} = ([h_1, h_2]\alpha)^{dh_2^{-1}} \in U_1^{dh_2^{-1}}\Theta(G) \leq U$. Point (ii) implies that $[d, \rho(h_2)^{\varphi}]^{h_2^{-1}} = ([d, h_2]\beta)^{h_2^{-1}} \in U$. Finally, (iii) shows that $[\rho(h_1), e^{\varphi}]^d = ([h_1, e]\gamma)^d \in U_1^d U_1^{ed}\Theta(G) \leq U$. Hence $u \in U$. By Lemma 3.4, $[[G, G^{\varphi}], u] \leq [[G, G^{\varphi}], a]$. Since $\Theta(G)$ centralizes $[G, G^{\varphi}]$, it follows that

$$[[G, G^{\varphi}], [h_1, h_2]^{dh_2^{-1}}[d, h_2]^{h_2^{-1}}[h_1, e]^d] \leq [[G, G^{\varphi}], a],$$

for any choice of $h_1, h_2 \in U_1$. In particular, taking $h_1 = 1$ we have $[[G, G^{\varphi}], [d, h_2]^{h_2^{-1}}] \leq [[G, G^{\varphi}], a]$, while taking $h_2 = 1$ it follows that $[[G, G^{\varphi}], [h_1, e]^d] \leq [[G, G^{\varphi}], a]$. Hence we can conclude that

$$[[G, G^{\varphi}], [h_1, h_2]^{dh_2^{-1}}] \leq [[G, G^{\varphi}], a].$$

Since $[[G, G^\varphi], a]$ is normal in $[G, G^\varphi]$, we have $[[G, G^\varphi], [h_1, h_2]] \leq [[G, G^\varphi], a]^{d^{-1}}$ and so $[[G, G^\varphi], U_1'] \leq [[G, G^\varphi], a]^{d^{-1}}$, which proves that U_1 has property 2.

Finally, consider again the inclusion $[[G, G^\varphi], [d, h_2]^{h_2^{-1}}] \leq [[G, G^\varphi], a]$. Since $[[G, G^\varphi], a]$ is normal in $[G, G^\varphi]$, it follows that $[[G, G^\varphi], [U_1, d]] \leq [[G, G^\varphi], a]$. Therefore U_1 has property 3. as well, and we are done. □

The following result is an immediate consequence of [6, Theorem 1.1].

Proposition 3.6. *Let n be a positive integer and G a group. Suppose that the size of the conjugacy class $\left|x^{\nu(G)}\right| \leq n$ for every $x \in T_\otimes(G)$. Then the second derived subgroup G'' is finite with n-bounded order.*

Proof. Let $w = [a, b]$ be a commutator of G. Since $\rho : \nu(G) \to G$ is an epimorphism, we deduce that the commutator w has at most as many conjugate in G as the tensor $[a, b^\varphi]$ has in $\nu(G)$. Consequently, every commutator of G has n-boundedly finite conjugacy class in G. Therefore [6, Theorem 1.1] shows that $|G''|$ is finite and n-bounded. □

We are in a position to give a proof of our main result.

Proof of Theorem 1.1. Denote the non-abelian tensor square $[G, G^\varphi]$ by H. Let m be the maximum of indices of $C_H(x)$ in H, where $x \in T_\otimes(G)$. Then consider $a \in T_\otimes(G)$ such that $C_H(a)$ has index precisely m in H. By Lemma 2.5 we can choose $b_1, \ldots, b_m \in H$ such that $l_\otimes(b_i) \leq m - 1$ and $a^H = \{a^{b_i};\ i = 1, \ldots, m\}$. Set $U = C_{\nu(G)}(\langle b_1, \ldots, b_m \rangle)$. Note that the index of U in $\nu(G)$ is n-bounded. Applying Proposition 3.5, we find a subgroup U_1, of n-bounded index, such that $[H, U_1'] \leq \langle [H, a]^{\nu(G)} \rangle$. Since the index of U_1 in $\nu(G)$ is n-bounded, so $H/H \cap U_1$ is. Therefore, we can find n-boundedly many tensors $t_1, \ldots, t_s \in T_\otimes(G)$ such that $H = \langle t_1, \ldots, t_s, H \cap U_1 \rangle$. Let T be the normal closure in $\nu(G)$ of the product of the subgroups $[H, a]$ and $[H, t_i]$ for $i = 1, \ldots, s$. By Lemma 3.3 each of these subgroups has n-bounded order. Moreover, they have at most n conjugates. Thus, T is a product of n-boundedly many finite subgroups, normalizing each other and having n-bounded order. We conclude that T has finite n-bounded order. Therefore it is sufficient to show that the second derived group of the quotient group $\nu(G)/T$ has finite n-bounded order.

Notice that the derived subgroup of HU_1 is contained in $Z(H)$ modulo T. Indeed, HU_1 is generated by $t_1, \ldots, t_s$ and U_1. Thus $[H, t_i] \leq T$ and $[H, U_1'] \leq \langle [H, a]^{\nu(G)} \rangle \leq T$, that is, $t_1, \ldots, t_s \in Z(H)$ and $U_1' \leq Z(H)$ modulo T. So we pass to the quotient $\nu(G)/T$ and to avoid complicated notation the images of $\nu(G)$, H and $T_\otimes(G)$ will be denoted by the same symbols.

Let $\mathcal{F}$ denote the family of subgroups $S \leq \nu(G)$ with the following properties.

(1) $\nu(G)' \leq S$;
(2) $S' \leq Z(H)$;
(3) S has finite index in $\nu(G)$.

First of all observe that $\mathcal{F}$ is not empty because HU_1 belongs to $\mathcal{F}$. Then let $F \in \mathcal{F}$ of minimal possible index j in $\nu(G)$. Thus j is n-bounded because the index of U_1 in $\nu(G)$ is n-bounded. Now, we argue by induction on j. If $j = 1$, then $F = \nu(G)$ and $\nu(G)' \leq Z(H)$. So $\nu(G)'' = 1$ and we have nothing to prove. Thus, assume that $j \geq 2$.

Let $a_0 \in T_{\otimes}(G)$ such that $C_H(a_0)$ has maximal possible index in H and write $a_0 = [d, e^{\varphi}]$ for suitable $d, e \in G$. Firstly assume that both d and e^{φ} belong to F. Then $a_0 \in F' \leq Z(H)$, and we conclude that H is abelian, that is $H' = 1$. Moreover, by Proposition 3.6 it follows that G'' is finite on n-bounded order. Therefore [10, Theorem 3.3] implies that $\nu(G)'' = [G', (G')^{\varphi}]G''(G'')^{\varphi}$ is finite.

Thus, we may assume that at least one among d and e^{φ} does not belong to F, say d. By Proposition 3.5 it follows that there exists a subgroup V of n-bounded index in $\nu(G)$ such that $[H, [V, d]] \leq [H, a_0]$. Without loss of generality we may assume that $V \leq F$, otherwise we can replace V by $V \cap F$. Let $L = F\langle d \rangle$. Note that $L' = F'[F, d]$. Let $1 = g_1, \ldots, g_t$ be a full system of representatives of the right cosets of V in F. Then standard commutator identities show that $[F, d]$ is generated by $[V, d]^{g_1}, \ldots, [V, d]^{g_t}$ and $[g_1, d], \ldots, [g_t, d]$. Denote by R the normal closure in $\nu(G)$ of the product of the subgroups $[H, a_0]^{g_i}$ and $[H, [g_i, d]]$ for $i = 1, \ldots, t$. For every $i = 1, \ldots, t$, Lemma 2.4 implies that $[g_i, d]$ is a tensor modulo $\Theta(G)$. Hence, applying Lemma 3.3, $[H, a_0]^{g_i}$ and $[H, [g_i, d]]$ have finite n-bounded order. Moreover, the hypothesis implies that each of them has at most n conjugates. Thus, R is the product of n-boundedly many finite subgroups, normalizing each other and having n-bounded orders. We conclude that R has finite n-bounded order. Since $F' \leq Z(H)$, $[H, L'] = [H, [F, d]] \leq R$. This means that $L' \leq Z(H)$ modulo R. Moreover, since $d \notin F$, the index of L in $\nu(G)$ is strictly smaller than j. Therefore, by induction on j, the second derived group of $\nu(G)/R$ is finite with n-bounded order. Taking into account that also R is finite with n-bounded order, we deduce that $\nu(G)''$ is finite with n-bounded order. The proof is now complete. □

Proof of Corollary 1.2. By Neumann's result [8, Theorem 3.1], it is sufficient to prove that the derived subgroup G' is finite.

By Theorem 1.1, the second derived subgroup $\nu(G)''$ is finite. Since $\nu(G)'' = [G', (G')^{\varphi}](G'')(G'')^{\varphi}$, it follows that the non-abelian tensor square $[G', (G')^{\varphi}]$ is finite. By Lemma 2.1, the derived subgroup G' is finite. The proof is complete. □

3.1. **Examples.** The next example shows that Corollary 1.2 no longer holds if we get rid of the hypothesis of G' to be finitely generated.

Example 3.7. *Let p be a prime. We define the semi-direct product $G = A \rtimes C_2$, where $C_2 = \langle d \mid d^2 = 1\rangle$, $A = C_{p^\infty}$ is the Prüfer group and*

$$a^d = a^{-1},$$

for every $a \in A$. Then the group G is not a BFC-group, whose commutator subgroup G' is not finitely generated, such that $|x^{\nu(G)}| \leq 4$ for every $x \in T_{\otimes}(G)$.

Since $G' = A$ is a Prüfer group, it follows that G' is not finitely generated, and G is not a BFC-group by Neumann's result [8, Theorem 3.1]. *Now, for all $g, h \in A$, as C_2 is generated by d, we have*

- $[g, d^{\varphi}]^h = [g, (d^h)^{\varphi}] = [g, (dh^2)^{\varphi}] = [g, (h^2)^{\varphi}][g, d^{\varphi}]^{h^2} = [g, d^{\varphi}]^{h^2}$;

- $[d, g^{\varphi}]^h = [d^h, g^{\varphi}] = [dh^2, g^{\varphi}] = [d, g^{\varphi}]^{h^2}[h^2, g^{\varphi}] = [d, g^{\varphi}]^{h^2}$;
- $[g, d^{\varphi}]^d = [g^{-1}, d^{\varphi}]$ *and* $[d, g^{\varphi}]^d = [d, (g^{-1})^{\varphi}]$.

In particular, $[g, d^{\varphi}]^h = [g, d^{\varphi}]$ *and* $[d, g^{\varphi}]^h = [d, g^{\varphi}]$*, for all* $g, h \in A$*. If* $\alpha, \beta \in G$*, there exist* $a, b \in A$ *and* $i, j \in \{0, 1\}$ *such that* $\alpha = ad^i$ *and* $\beta = bd^j$*. Therefore,*

$$\begin{aligned} [\alpha, \beta^{\varphi}] &= [ad^i, (bd^j)^{\varphi}] \\ &= ([a, (d^j)^{\varphi}][a, b^{\varphi}]^{d^j})^{d^i}([d^i, (d^j)^{\varphi}][d^i, b^{\varphi}]^{d^j}) \\ &= [a, (d^j)^{\varphi}]^{d^i}[d^i, b^{\varphi}]^{d^j}[d, d^{\varphi}]^{ij} \\ &= [a^{\varepsilon_1}, (d^j)^{\varphi}][d^i, (b^{\varepsilon_2})^{\varphi}]z, \end{aligned}$$

where $z = [d, d^{\varphi}]^{ij} \in Z(\nu(G))$ *and* $\varepsilon_k \in \{1, -1\}$*,* $k = 1, 2$*. Moreover, for every* $w \in \nu(G)$ *there exist* $\gamma \in G$*,* $c \in A$ *and* $l \in \{0, 1\}$ *such that* $\rho(w) = \gamma = cd^l$*. Now, by Lemma 2.2 (d),* $[\alpha, \beta^{\varphi}]^w = [\alpha, \beta^{\varphi}]^{\rho(w)}$ *and*

$$[\alpha, \beta^{\varphi}]^{\rho(w)} = [\alpha, \beta^{\varphi}]^{\gamma} = ([a^{\varepsilon_1}, (d^j)^{\varphi}][d^i, (b^{\varepsilon_2})^{\varphi}]z)^{cd^l} = ([a^{\varepsilon_1}, (d^j)^{\varphi}][d^i, (b^{\varepsilon_2})^{\varphi}]z)^{d^l}.$$

It follows that, the conjugacy class

$$[\alpha, \beta^{\varphi}]^{\nu(G)} \subseteq \{[a^{\varepsilon_1}, (d^j)^{\varphi}][d^i, (b^{\varepsilon_2})^{\varphi}]z \mid \varepsilon_i \in \{-1, 1\}\},$$

and $|[\alpha, \beta^{\varphi}]^{\nu(G)}| \leq 4$*, for any* $\alpha, \beta \in G$*.*

Finally, notice that in Corollary 1.2 we cannot replace the hypothesis "$|x^G| \leq n$ for every $x \in T_{\otimes}(G)$" by "$|x^G| \leq n$ for every x commutator of G", as the following example shows.

Example 3.8. *Let* $A = \langle a \rangle$ *be an infinite cyclic group and let* $C_2 = \langle d \mid d^2 = 1 \rangle$ *be a cyclic group of order* 2*. Then consider the semi-direct product* $G = A \rtimes C_2$ *where* $a^d = a^{-1}$ *(infinite dihedral group). Then for every commutator* x *of* G *we have* $|x^G| \leq 2$*. However, the derived subgroup* G' *is an infinite subgroup of* A*, so* G *is not a BFC-group.*

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Raimundo Bastos

Departamento de Matemática, Universidade de Brasília, Brasilia-DF Brazil

Email: bastos@mat.unb.br

Carmine Monetta

Dipartimento di Matematica, Università di Salerno, Salerno, Italy

Email: cmonetta@unisa.it

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