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THE AUTOMORPHISM GROUPS OF GROUPS OF ORDER p^2q

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Abstract. We record for reference a detailed description of the automorphism groups of the groups of order p^2q, where p and q are distinct primes.

1. Introduction

Let p, q be distinct primes. O. Hölder classified the groups of order p^2q [7], and also the groups of square-free order [8]. H. Dietrich and B. Eick [4] gave a detailed description of the structure of groups of cube-free order, which was complemented by S. Qiao and C. H. Li [10]. Groups of order p^3q were classified by A.E. Western [12] and R. Laue [9]. B. Eick [5] has given an enumeration of the groups whose order factorises in at most 4 primes. B. Eick and T. Moede [6] have enumerated groups of order p^nq, for $n \leq 5$.

For our paper [2], we need a detailed description of the automorphism groups of the groups of order p^2q, where p and q are distinct primes. We have recorded these data for reference here.

2. The groups

Describing the groups G of order p^2q, where p and q are distinct primes, is an easy exercise about Sylow's theorems, which we now describe briefly, the basic point being that G has a normal Sylow subgroup.

If there are more than 1, and thus exactly q, Sylow p-subgroups, then $p \mid q - 1$.

- If the Sylow p-subgroups intersect pairwise trivially, counting p-elements show that then G has exactly one Sylow q-subgroup.

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- If there are two distinct Sylow p-subgroups P_1, P_2 that intersect non-trivially in a subgroup N of order p, then $N \trianglelefteq G$, so that G has a subgroup R of order pq, which is normal in G, as $p < q$. For the same reason, a Sylow q-subgroup of R is normal in R, and thus in G, so that G has a normal Sylow q-subgroup in this case as well.

We introduce some notation.

$\mathcal{C}_n$**:** denotes a cyclic group of order n.

$\rtimes$ **and** $\ltimes$**:** when they appear without subscripts, they denote the unique (up to isomorphism) non-direct, semidirect product that is possible in the given situation.

$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_S \mathcal{C}_q$**:** denotes a semidirect product where a generator of $\mathcal{C}_q$ acts as a non-identity scalar matrix.

$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$**:** denotes a semidirect product where a generator of $\mathcal{C}_q$ acts as a diagonal, non-scalar matrix with no eigenvalue 1, and determinant different from 1.

$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D1} \mathcal{C}_q$**:** is the same as above, but the non-scalar matrix has determinant 1 (and thus still no eigenvalue 1).

$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_C \mathcal{C}_q$**:** denotes a semidirect product where a generator of $\mathcal{C}_q$ acts as a suitable power of a Singer cycle; note that the determinant of the matrix of a generator of $\mathcal{C}_q$ acting on $\mathcal{C}_p \times \mathcal{C}_p$ is 1.

$\mathcal{C}_{p^2} \ltimes_1 \mathcal{C}_q$**:** denotes a semidirect product with trivial centre.

$\mathcal{C}_{p^2} \ltimes_p \mathcal{C}_q$**:** denotes a semidirect product with centre of order p.

Considering the possible actions on the normal Sylow subgroup of another Sylow subgroup, we obtain the following table. The automorphism groups are determined in Section 4, on the basis of results of G. L. Walls [11], J. N. S. Bidwell, M. J. Curran and D. J. McCaughan [1], and M. J. Curran [3], which we recall in Section 3.

Type	Conditions	G	$\mathrm{Aut}(G)$	Explanation
1		$\mathcal{C}_{p^2} \times \mathcal{C}_q$	$\mathcal{C}_p \times \mathcal{C}_{p-1} \times \mathcal{C}_{q-1}$	Cyclic groups
2	$p \mid q-1$	$\mathcal{C}_{p^2} \ltimes_p \mathcal{C}_q$	$\mathcal{C}_p \times \mathrm{Hol}(\mathcal{C}_q)$	Subs. 4.5
3	$p^2 \mid q-1$	$\mathcal{C}_{p^2} \ltimes_1 \mathcal{C}_q$	$\mathrm{Hol}(\mathcal{C}_q)$	Thm 3.4
4	$q \mid p-1$	$\mathcal{C}_{p^2} \rtimes \mathcal{C}_q$	$\mathrm{Hol}(\mathcal{C}_{p^2})$	Thm 3.4
5		$\mathcal{C}_p \times \mathcal{C}_p \times \mathcal{C}_q$	$\mathrm{GL}(2,p) \times \mathcal{C}_{q-1}$	Thm 3.1
6	$q \mid p-1$	$\mathcal{C}_p \times (\mathcal{C}_p \rtimes \mathcal{C}_q)$	$\mathcal{C}_{p-1} \times \mathrm{Hol}(\mathcal{C}_p)$	Thms 3.1, 3.4
7	$2 < q \mid p-1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_S \mathcal{C}_q$	$\mathrm{Hol}(\mathcal{C}_p \times \mathcal{C}_p)$	Subs. 4.1, 4.2
8	$3 < q \mid p-1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$	$\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p)$	Subs 4.1, 4.3
9	$2 < q \mid p-1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D1} \mathcal{C}_q$	$\mathcal{C}_2 \ltimes (\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p))$	Subs. 4.1, 4.3
10	$2 < q \mid p+1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_C \mathcal{C}_q$	$(\mathcal{C}_2 \ltimes \mathcal{C}_{p^2-1}) \ltimes (\mathcal{C}_p \times \mathcal{C}_p)$	Subs. 4.1, 4.4
11	$p \mid q-1$	$\mathcal{C}_p \times (\mathcal{C}_p \ltimes \mathcal{C}_q)$	$\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_q)$	Sub. 4.6

2.1. **Isomorphism.** It is immediate to see that all types in this table consist of exactly one isomorphism class of groups, with the exception of type 8. If G is a group of this type, we can give it a

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canonical form by choosing as generators first of all two eigenvectors with respect to distinct eigenvalues in the normal, elementary abelian Sylow p-subgroup V. If ζ is a fixed element of order q in the multiplicative group of the field with p elements, we can then choose as a third generator a suitable power a of a q-element, so that it has eigenvalues $\{\zeta, \zeta^s\}$ on V. The parameter $s \notin \{0, 1, -1\}$ determines G. If t is the inverse of s modulo p, then a^t has eigenvalues $\{\zeta^t, \zeta\}$ on V. It follows that the parameters s, t yield isomorphic groups, so that there are $(q-3)/2$ isomorphism classes of groups here.

3. Automorphisms of (semi)direct products

We collect here the results we need of [1, 11, 3]. We write (auto)morphisms as exponents.

Theorem 3.1. [1, Theorem 3.2]
Let $G = H \times K$, where H, K have no common direct factors.
Then $\mathrm{Aut}(G)$ can be described in the natural way via the set of matrices

$$\left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} : a \in \mathrm{Aut}(H), d \in \mathrm{Aut}(K), \right.$$
$$\left. b \in \mathrm{Hom}(K, Z(H)), c \in \mathrm{Hom}(H, Z(K)) \right\}.$$

Theorem 3.2. [3, Theorem 1]
Let $G = H \rtimes K$ be a semidirect product.
Then the subgroup of $\mathrm{Aut}(G)$ consisting of the automorphisms that leave H invariant can be described in a natural way via the set of matrices

$$\left\{ \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} : a \in \mathrm{Aut}(H), d \in \mathrm{Aut}(K), \right.$$
$$(h^k)^a = (h^a)^{k^b k^d}, \text{ for } h \in H, k \in K,$$
$$\left. b : K \to H, (xy)^b = x^b (y^b)^{x^d}, \text{ for } x, y \in K \right\}.$$

Remark 3.3. *If H is abelian, the condition*

$$(h^k)^a = (h^a)^{k^b k^d} = (h^a)^{k^d}$$

in Theorem 3.2 can be rewritten as

$$\iota(k)^a = a^{-1}\iota(k)a = \iota(k^d),$$

where

$$\iota : K \to \mathrm{Aut}(H)$$
$$k \mapsto (h \mapsto h^k).$$

If $\mathrm{Aut}(H)$ is abelian, we get $\iota(k) = \iota(k^d)$, that is, $[k, d] \in \mathcal{C}_K(H)$. In particular, if $\mathcal{C}_K(H) = 1$, then $d = 1$.

Theorem 3.4. [11, Theorem B], [3, Example 1]
Let $G = \mathcal{C}_n \rtimes \mathcal{C}_k$, with $Z(G) = 1$. Write $H = \mathcal{C}_n$, $K = \mathcal{C}_k$.
Then $H = G'$ is characteristic in G, and we have

$$\mathrm{Aut}(G) \cong \mathrm{Hol}(\mathcal{C}_n) = \mathcal{C}_n \rtimes \mathrm{Aut}(\mathcal{C}_n).$$

Remark 3.5. *In the matrix terms of Theorem 3.2, Theorem 3.4 can be reformulated as*

$$\mathrm{Aut}(G) = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} : a \in \mathrm{Aut}(H), b : K \to H \right.$$
$$\left. (xy)^b = x^b\, (y^b)^x, \text{ for } x, y \in K \right\}.$$

The b's can be described in terms of the image $b_0 \in H$ of a fixed generator of K: see Subsection 4.1 for the details.

Theorem 3.6. [3, Theorem 3 and Example 1]
Let $G = \mathcal{C}_n \rtimes \mathcal{C}_k$, with $Z(G)$ possibly non-trivial. Write $H = \mathcal{C}_n$, $K = \mathcal{C}_k$.
Assume $H = G'$.
Then

$$\mathrm{Aut}(G) \cong H \rtimes (\mathrm{Aut}(H) \times S),$$

where

$$S = \left\{ d \in \mathrm{Aut}(K) : [k, d] = k^{-1}k^d \in \mathcal{C}_K(H), \text{ for } k \in K \right\}.$$

In matrix terms, Theorem 3.6 states that

$$\mathrm{Aut}(G) = \left\{ \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} : a \in \mathrm{Aut}(H), d \in \mathrm{Aut}(K), \right.$$
$$[k, d] = k^{-1}k^d \in \mathcal{C}_K(H), \text{ for } k \in K,$$
$$\left. b : K \to H, (xy)^b = x^b(y^b)^x, \text{ for } x, y \in K \right\}$$

4. Automorphism groups

We appeal to the results of Section 3, whose notation we employ.

4.1. **Describing b.** We begin by collecting some facts that hold true for most cases.

Let us first consider the types 7, 8, 9, 10. Write

- $\mathcal{C}_q = \langle z \rangle$,
- Z for the linear map z induces on $H = \mathcal{C}_p \times \mathcal{C}_p$, and
- Y for the linear map induced by z^d on H.

First note that for each $b_0 \in H$ there exists a unique function b as in Theorem 3.2 such that $z^b = b_0$. In fact, one has for $j = 1, \dots, q-1$

$$(z^j)^b = b_0^{1+Y+\dots+Y^{j-1}}.$$

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In $\mathrm{End}(H)$ we have

$$0 = Y^q - 1 = (Y - 1)(1 + Y + \cdots + Y^{q-1}).$$

Now $Y - 1$ invertible, as Y has no eigenvalue 1, so that $1 + Y + \cdots + Y^{q-1} = 0$. It follows that

$$(z^q)^b = 1 = b_0^{1+Y+\cdots+Y^{q-1}},$$

is also satisfied.

A similar argument holds

- for the types 4, 3, 2,
- for the subgroup $\mathcal{C}_p \rtimes \mathcal{C}_q$ of type 6, and
- for the subgroup $\mathcal{C}_p \ltimes \mathcal{C}_q$ of type 11.

In these cases Y is an automorphism of order coprime to r of a cyclic group C of order a power of a prime r, so that $Y - 1$ is not nilpotent, and thus it is invertible, in $\mathrm{End}(C)$.

Note that conjugating

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

by

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

we get

$$\begin{bmatrix} 1 & 0 \\ d^{-1}ba & 1 \end{bmatrix},$$

so that if $d = 1$ we have $z^{ba} = b_0^a$, and thus the group

$$\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$$

is a split extension of H by the group of the a's.

4.1.1. **Between d and a.** Suppose $d : z \mapsto z^i$, with $0 < i < q$ and $\gcd(i, q) = 1$. For $h \in H$ we have

$$h^{a^{-1}Za} = h^{Z^i},$$

and thus

$$a^{-1}Za = Z^i. \tag{4.1}$$

4.2. **Type 7, $G = (\mathcal{C}_p \times \mathcal{C}_p) \rtimes_S \mathcal{C}_q$.** In this case, since Z is scalar, we have $Z = Z^i$, thus $q \mid i - 1$, that is, $i = 1$ and d is trivial. Since a is arbitrary, we obtain as the automorphism group the holomorph of $\mathcal{C}_p \times \mathcal{C}_p$, that is, the affine group in dimension 2 over $\mathbb{F}_p$.

4.3. **Type 8 and 9,** $G = (\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$ **or** $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D1} \mathcal{C}_q$**.** In this case

$$Z = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

with $\lambda \neq \mu$, $\lambda, \mu \neq 1$. Then (4.1) yields $\{\lambda, \mu\} = \{\lambda^i, \mu^i\}$. If $\lambda = \lambda^i$ and $\mu = \mu^i$, we obtain that $q \mid i - 1$, and thus $i = 1$ and $d = 1$. From (4.1) and Subsection 4.1, we obtain that a centralizes Z, and that the automorphism group contains

$$(\mathcal{C}_{p-1} \times \mathcal{C}_{p-1}) \ltimes (\mathcal{C}_p \times \mathcal{C}_p) = \mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p), \tag{4.2}$$

with $\mathcal{C}_{p-1} \times \mathcal{C}_{p-1}$ acting by diagonal matrices on $\mathcal{C}_p \times \mathcal{C}_p$, a typical element being

$$\begin{bmatrix} T & 0 \\ b & 1 \end{bmatrix} \tag{4.3}$$

with T diagonal.

If $\lambda = \mu^i$ and $\mu = \lambda^i$, then $\lambda = \lambda^{i^2}$, so that $q \mid (i-1)(i+1)$. When $q \mid i - 1$ we get again $d = 1$, whereas when $q \mid i + 1$ we get $z^d = z^{-1}$ and $\lambda = \mu^{-1}$. Thus this case only occurs when $\det(Z) = 1$, that is, when G is of type 9. The inversion d can then be paired with

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

to get $S^{-1} Z S = Z^{-1}$. In this case the automorphism group is the extension of the group (4.2) by the involution

$$\begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix}, \tag{4.4}$$

which acts on (4.3) as

$$\begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} T & 0 \\ b & 1 \end{bmatrix} \cdot \begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} STS & 0 \\ d^{-1} b S & 1 \end{bmatrix};$$

Now

$$z^{d^{-1} b S} = (z^{-1})^{bS} = (z^{q-1})^{bS} = (b_0^{1+Y+\dots+Y^{q-2}})^S = (b_0^{-Y^{q-1}})^S = b_0^{-Y^{-1} S},$$

where

$$-Y^{-1} S = \begin{bmatrix} 0 & -\lambda^{-1} \\ -\lambda & 0 \end{bmatrix}$$

is an involution, that acts by exchanging the two copies of $\mathrm{Hol}(\mathcal{C}_p)$.

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4.4. **Type 10,** $G = (\mathcal{C}_p \times \mathcal{C}_p) \rtimes_C \mathcal{C}_q$**.** Note first that the order $q \neq 2$ of Z divides $p+1$, so it does not divide $p-1$. It follows that $Z \in \mathrm{SL}(2,p)$, that is, $\det(Z) = 1$.

If $\lambda, \mu = \lambda^{-1}$ are the (distinct) eigenvalues of Z in the field $\mathbb{F}_{p^2}$, then $\{\lambda, \mu\} = \{\lambda^i, \mu^i\}$. If $\lambda = \lambda^i$ and $\mu = \mu^i$, we get once more $d = 1$. Thus in this case a lies in the centralizer of Z in $\mathrm{Aut}(H) = \mathrm{GL}(2,p)$

$$\mathcal{C}_{\mathrm{Aut}(H)}(Z) = \{u + vZ \neq 0 : u, v \in \mathbb{F}_p\},$$

which is cyclic, of order $p^2 - 1$.

If $\lambda = \mu^i$ and $\mu = \lambda^i$, then $\lambda = \lambda^{i^2}$, so that $q \mid (i-1)(i+1)$. When $q \mid i-1$ we get again $d = 1$, whereas when $q \mid i+1$ we get $z^d = z^{-1} = z^p$, as $p \equiv -1 \pmod{q}$.

In an appropriate basis of H we have

$$Z = \begin{bmatrix} 0 & 1 \\ -1 & t \end{bmatrix},$$

where $t = \lambda + \lambda^{-1} = \lambda + \lambda^p$.

The equation $\iota(z)^a = \iota(z^d)$ of Remark 3.3 has now become

$$Z^a = Z^{-1}.$$

One solution a for this is

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

as

$$S^{-1} Z S = \begin{bmatrix} t & -1 \\ 1 & 0 \end{bmatrix} = Z^{-1}.$$

All other solutions are obtained in the form $a = XS$, where $X \in \mathcal{C}_{\mathrm{Aut}(H)}(Z)$. So in this case $\mathrm{Aut}(G)$ is the extension of the subgroup determined by $a = d = 1$, which is isomorphic to $\mathcal{C}_p \times \mathcal{C}_p$, acted upon by the subgroup determined by $b = 0$. The latter subgroup has a normal subgroup

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_{\mathrm{Aut}(H)}(Z) & 0 \\ 0 & 1 \end{bmatrix},$$

which is cyclic, of order $p^2 - 1$ extended by the involution

$$D = \begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix},$$

where $z^d = z^{-1} = z^p$. Now $\mathcal{C}_{\mathrm{Aut}(H)}(Z)$ is the multiplicative group of the field with p^2 elements; conjugation by S induces an automorphism group of order 2, which is then the Frobenius map. Thus $\mathcal{C}_{\mathcal{C}}(D) = \mathcal{C}_{\mathrm{Aut}(H)}(S)$ has order $p-1$, and $g^S = g^p$ for $g \in \mathcal{C}$.

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4.5. **Type 2, $G = \mathcal{C}_{p^2} \ltimes_p \mathcal{C}_q$.** Here $\mathcal{C}_{p^2} = \langle\, x \,\rangle$ induces on $\mathcal{C}_q$ a group of automorphisms of order p, and thus the centraliser $\mathcal{C}_{\langle\, x \,\rangle}(\mathcal{C}_q) = \langle\, x^p \,\rangle$ has order p.

As per Remark 3.3, here

$$S = \left\{ d \in \operatorname{Aut}(\mathcal{C}_{p^2}) : [x, d] \in \mathcal{C}_{\langle\, x \,\rangle}(\mathcal{C}_q) = \langle\, x^p \,\rangle \right\}.$$

is a group of order p, generated by the automorphism $x \mapsto x^{1+p}$. According to Theorem 3.6, we get that the automorphism group is isomorphic to

$$\mathcal{C}_q \rtimes (\mathcal{C}_{q-1} \times \mathcal{C}_p) \cong \mathcal{C}_p \times \operatorname{Hol}(\mathcal{C}_q).$$

4.6. **Type 11, $G = \mathcal{C}_p \times (\mathcal{C}_p \ltimes \mathcal{C}_q)$.** According to Theorem 3.1 and Theorem 3.4, we have that the automorphism group has the form

$$\begin{bmatrix} \mathcal{C}_{p-1} & 0 \\ \mathcal{C}_p & \operatorname{Hol}(\mathcal{C}_q) \end{bmatrix}.$$

To see the structure, let us consider the conjugate

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

where $a \in \operatorname{Aut}(\mathcal{C}_p)$, $d \in \operatorname{Aut}(\mathcal{C}_p \ltimes \mathcal{C}_q) \cong \operatorname{Hol}(\mathcal{C}_q)$, and $b \in \operatorname{Hom}(\mathcal{C}_p \ltimes \mathcal{C}_q, \mathcal{C}_p)$. The conjugate equals

$$\begin{bmatrix} 1 & 0 \\ d^{-1}ba & 1 \end{bmatrix}.$$

Since d acts trivially on the quotient $(\mathcal{C}_p \ltimes \mathcal{C}_q)/\mathcal{C}_q$, we get that the automorphism group has structure

$$\mathcal{C}_p \rtimes (\mathcal{C}_{p-1} \times \operatorname{Hol}(\mathcal{C}_q)),$$

with $\mathcal{C}_{p-1}$ acting as $\operatorname{Aut}(\mathcal{C}_p)$ and $\operatorname{Hol}(\mathcal{C}_q)$ acting trivially, that is

$$\operatorname{Hol}(\mathcal{C}_p) \times \operatorname{Hol}(\mathcal{C}_q).$$

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