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PARAMETERS OF THE COPRIME GRAPH OF A GROUP

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Abstract. There are many different graphs one can associate to a group. Some examples are the well-known Cayley graph, the zero divisor graph (of a ring), the power graph, and the recently introduced coprime graph of a group. The coprime graph of a group G, denoted Γ_G, is the graph whose vertices are the group elements with g adjacent to h if and only if $(o(g), o(h)) = 1$. In this paper we calculate the independence number of the coprime graph of the dihedral groups. Additionally, we characterize the groups whose coprime graph is perfect.

1. Introduction

One of the most beautiful things about mathematics is the interconnected nature of the discipline. In this paper we will explore a problem involving two different areas of mathematics, namely group theory and graph theory. Group theory has been around since the 1800's and, as such, finite groups have been studied extensively. Graph theory, a branch of discrete mathematics, has been around for a similar amount of time, however it wasn't formalized until the 1900's. Toward the end of the 1800's, Arthur Cayley connected graph theory and group theory by introducing the Cayley graph of a group. These graphs encode algebraic information about a group. More precisely, for a group G and a set $S \subseteq G$, one has the group elements as a vertex set for the Cayley graph and two vertices a, b are adjacent if $b = as$ for some $s \in S$. Observe that the chosen set S determines the Cayley graph and so it is possible to learn information about the group by choosing S wisely. For instance, the Cayley graph will be connected if and only if S is a generating set for the group G. Since this graph

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is interesting and gives a great deal of information about the group, it has been heavily studied and many results are known (see [4]).

While Cayley graphs are by far the most well-known graph associated with an algebraic structure, there are others. One such graph is the zero divisor graph of a ring. For a ring R the zero divisor graph has ring elements as vertices and two elements a, b are adjacent if and only if $ab = 0$. This is another generally well-known and well-studied graph. See [1] for a survey of these graphs.

While there are other notions of graphs associated to groups a recent example is the coprime graph of a group. The coprime graph of a group was introduced in [8] in 2014 and, as such, it is a relatively new area of study. For a finite group G, the coprime graph of a group G has the group elements as a vertex set and two vertices a, b are adjacent if and only if their orders in the group are relatively prime, (i.e. $(o(a), o(b)) = 1$). In this paper we will explore some graph parameters for the coprime graphs of different classes of finite groups and will determine which groups have perfect coprime graphs.

As one would expect, studying the coprime graph of a group requires knowledge of group theory, graph theory, and a bit of elementary number theory. To ensure the reader has the necessary background we will review a few definitions and facts regarding both graph theory and group theory that we plan to use throughout the paper in the following section. Following the background information we have a short section introducing the coprime graph of a group followed by our results. Our results are given in two different sections. The first gives the independence number of the coprime graph for the dihedral groups. The second gives a characterization for which groups have perfect coprime graphs. We close with some avenues for future research.

2. Background Information

2.1. **Graph Theory.** We will begin by recalling some basic graph theory. Here we will be studying only simple graphs. A *simple graph* G is a graph that has no multiple edges and no loops. A graph H is a *subgraph* of a graph G, denoted $H \leq G$, if and only if $V_H \subseteq V_G$ and $E_H \subseteq E_G$ where V_G and E_G represent the vertices and edges of G respectively. A subgraph H of G is called an *induced subgraph* if $V_H \subseteq V_G$ and for all $u, v \in V_H$ if $\{u, v\} \in E_G$ then $\{u, v\} \in E_H$. The *complement* of a graph G, denoted $\overline{G}$, is defined as the graph with $V_{\overline{G}} = V_G$ and $\{u, v\} \in E_{\overline{G}}$ if and only if $\{u, v\} \notin E_G$. Graphs G and H are *isomorphic*, denoted $G \cong H$, if and only if there exists a bijective function $f : V_G \to V_H$ such that $\{u, v\} \in E_G$ if and only if $\{f(u), f(v)\} \in E_H$. For an in-depth discussion of graph theory see [2].

In the study of graph theory, there are some commonly appearing graphs. One such family of graph is the *complete graph* on n vertices, denoted K_n which has vertex set V and edge set $E = \binom{V}{2}$. This means that the graph contains every possible edge. Another common graph family is the *cycle* on n vertices, denoted C_n. A cycle on n vertices is a connected graph that has the property $|V_G| = |E_G|$ and $d(v) = 2$ for all vertices v. A *bipartite graph* is a graph whose vertices can be partitioned into two sets V_1 and V_2 such that for all $u, v \in V_i$, we have $\{u, v\} \notin E_G$. A useful characterization for bipartite graphs is the following.

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Proposition 2.1. *A graph is bipartite if and only if it has no odd cycle as a subgraph.*

A *complete bipartite graph* is a bipartite graph in which all vertices of V_1 are adjacent to all vertices of V_2. A *tree* is a connected graph that has no cycles as a subgraph. By this definition, it is clear that all trees are bipartite graphs. A graph is called a *star* on n vertices if there is one vertex with degree $n-1$ and every other vertex has degree 1.

In addition to the different families of graphs, there are also several graph parameters we will consider. A *coloring* of a graph is a function $c : V \to S$, where V is the vertex set of the graph. A coloring is called a *proper coloring* if and only if for all $\{u, v\} \in E$, $c(u) \neq c(v)$. The *chromatic number* of a graph, denoted $\chi(G)$, is the smallest number of colors needed to properly color the graph. In complete graphs, since every vertex shares an edge with every other vertex, we have $\chi(K_n) = n$. It is also useful to notice that if $H \leq G$, we have $\chi(H) \leq \chi(G)$ because if G can be colored properly with $\chi(G)$ colors and $H \leq G$, then any proper coloring of G is certainly a proper coloring of H. The *clique number* of a graph G, denoted $\omega(G)$, is the number of vertices in the largest complete subgraph of G. An *independent set* is a set of vertices that give an empty induced subgraph.The *independence number* of a graph, denoted $\alpha(G)$, is size of the largest independent set. Since the complement of a graph has edges where the original doesn't, and vice-versa, we have $\omega(G) = \alpha(\overline{G})$ and $\omega(\overline{G}) = \alpha(G)$ for all graphs G. Also, since the clique number is the largest complete subgraph, and the chromatic number of a complete graph is the number of vertices in it, it follows that $\chi(G) \geq \omega(G)$. These facts will be useful going forward.

Lastly, we will define a type of graph that will be of particular interest in our own work below.

Definition 2.2. *If for every induced subgraph H of a graph G we have $\omega(H) = \chi(H)$, then G is called a* ***perfect graph****.*

Checking all induced subgraphs of a graph is a rather tedious and difficult task. In 1961 Berge conjectured a characterization of perfect graphs which was proved in 2006 by Chudnovksy, Robertson, Seymour, and Thomas in [3].

Theorem 2.3 (Strong Perfect Graph Theorem)**.** *A graph is perfect if and only if neither the graph nor its complement contain C_n as an induced subgraph, where n is odd and $n \geq 5$.*

We should also note that $\overline{G}$ having no induced C_{odd} is equivalent to G having no $\overline{C_{\text{odd}}}$. This fact will be useful in some of our proofs later.

2.2. **Group Theory.** In this section we review a few definitions and facts from group theory that will be relevant to our work. For an in-depth discussion of group theory see [7]. First recall that the *order of a group* is the number of elements in the group and is denoted $|G|$. We will be considering only finite groups in this paper. The *order of an element* a, denoted $o(a)$, is the smallest natural number m such that $a^m = e$. If no such integer exists, we say that a has infinite order. If G is an abelian group with elements a and b, then $o(ab) = \text{lcm}(o(a), o(b))$. These facts about orders of elements will be important as we move into finding the coprime graphs of groups. In addition to this, the following two theorems will be used multiple times throughout our work.

Theorem 2.4 (Lagrange's Theorem)**.** *For any finite group G and subgroup H, the order of H divides the order of G.*

We should note that Lagrange's Theorem implies that the order of any element of a group must also divide the order of the group. This is because for all $a \in G$, $H = \langle a \rangle := \{a^n \mid n \in \mathbb{Z}\}$ always forms a subgroup of G and $o(a) = |\langle a \rangle|$.

Theorem 2.5 (Cauchy's Theorem)**.** *If p divides $|G|$ and p is prime, then there exists an element $a \in G$ such that $o(a) = p$.*

Just as there are classes of graphs there are also classes of groups. In this paper we look at parameters of the coprime graph of cyclic groups, dihedral groups, and the alternating and symmetric groups.

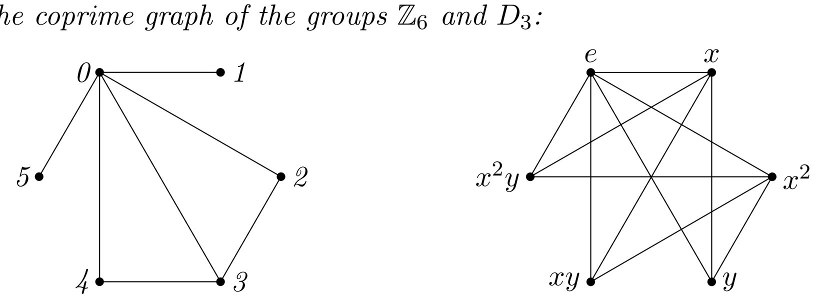
3. The Coprime Graph of a Group

Now that we have the necessary background we can begin work on our primary object of study, the coprime graph of a group. We will start by restating the definition of the coprime graph of a group. We will then look at some examples and state some known results before moving on to our own contributions.

Definition 3.1. *The **coprime graph of a group** G, denoted Γ_G, is the graph with $V(G) = G$ and $\{a,b\} \in E(G)$ if and only if $(o(a), o(b)) = 1$.*

One thing that follows immediately from the definition is that the coprime graph of any group is connected. This is because every group has an identity element e and $o(e) = 1$. As such, the identity will be adjacent to all other vertices in the coprime graph of the group. Let's look at a couple of examples.

Example 3.2. *The coprime graph of the groups $\mathbb{Z}_6$ and D_3:*



$\Gamma_{\mathbb{Z}_6}$ Γ_{D_3}

Notice that while $|D_3| = |\mathbb{Z}_6|$ the graphs above are clearly not isomorphic. This makes sense because these two groups are vastly different. However, what is interesting is the fact that both graphs above

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have the same chromatic and clique number. In fact, any two groups of the same order will have the same chromatic and clique number. We state this theorem below. We should note that the result for chromatic number was stated in [8] and shortly followed by the full proof in [5].

Theorem 3.3. *If $|G| = p_1^{e_1} p_2^{e_2} ... p_n^{e_n}$ where p_i is prime for $1 \leq i \leq n$, then $\omega(\Gamma_G) = n + 1 = \chi(\Gamma_G)$.*

Since the chromatic and clique number of the coprime graph are known for any group we began our work on the next natural graph parameter, the independence number.

4. Independence Number of Coprime Graphs

Recall that the *independence number* of a graph Γ, denoted $\alpha(\Gamma)$, is the size of the largest independent set. In other words $\alpha(\Gamma)$ is the size of the largest induced empty subgraph of Γ. While the chromatic and clique numbers depend only on the order of the group G, the independence number depends on the group structure as well. In particular, the arguments we use below rely on finding the number of elements of certain orders. As such, it is not clear if a broad approach can be used for a general group. The independence number for cyclic groups is known and can be found in [6] and [10]. Here we calculate the independence number for the dihedral groups.

Theorem 4.1. *Let $G = D_n$ with $n = 2^l m$, where m is odd and l is an integer. Then $\alpha(\Gamma_G) = 2n - m$.*

Proof. Let $G = D_n$ with $n = 2^l m$, where is m is odd and l is an integer. Let V_2 consist of all elements of even order. Since the order of every element in V_2 is even, the gcd of any two orders is at least 2. Therefore V_2 is an independent set. Let's count the size of V_2. First recall that $D_n = \{e, x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$ and $x^i y$ is a reflection for all i. Hence $o(x^i y) = 2$ for all $0 \leq i \leq n-1$. Now notice that $\langle x \rangle = \{e, x, \ldots, x^{n-1}\} \cong \mathbb{Z}_n$. As such we can count the number of elements of even order in $\mathbb{Z}_n$ where $n = 2^l m$ with m odd. Recall that the order of any element $a \in \mathbb{Z}_n$ is given by $o(a) = \frac{n}{\gcd(n,a)}$. It follows that an element a in $\mathbb{Z}_n$ can have odd order if and only if a is a multiple of 2^l. Hence there are precisely m elements of odd order giving $n - m$ elements of even order. Thus $|V_2| = n + n - m = 2n - m$.

Now let I be an independent set in Γ_G such that I contains an element of odd order. Notice that $x^a y \notin I$ since each such element has order 2. Also, $e \notin I$ since $o(e) = 1$. Therefore, $|I| \leq n - 1 < n \leq 2n - m$ because $m \leq n$. This implies that an independent set of maximum size must only contain elements of even order. Since V_2 consists of all elements of even order it is the unique maximum independent set. Therefore $\alpha(\Gamma_{D_n}) = 2n - m$. $\square$

5. Perfect Coprime Graphs

Recall that a graph is perfect when all of its induced subgraphs have equal chromatic and clique numbers. Notice that if G is a p-group with order p^a, then by Lagrange's Theorem, the order of every element of the group must divide p^a. Therefore the order of any two non-identity elements of G share a common factor of p. So the only edges that appear in Γ_G are ones containing the identity. Since

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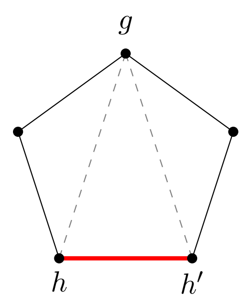
the identity is adjacent to every other element, we see that the graph of a p-group will always be a star, and hence perfect. In this section we give a partial classification of groups that have perfect coprime graph. Specifically, we give a complete classification for all abelian groups, all symmetric and alternating groups, and all dihedral groups.

Theorem 5.1. *If G is group whose order has at most four distinct prime factors, then Γ_G is perfect.*

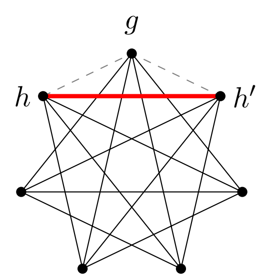
Proof. Suppose that G is a group such that $|G| = p^a q^b r^c s^d$ where p, q, r, s are primes. Note that we may assume $a, b \geq 1$ and $c, d \geq 0$ because if $b = c = d = 0$, then Γ_G is a star and hence perfect. Now let's consider possible types of elements in G. By Lagrange's Theorem, we know that the set of prime divisors for the order of any element in G is contained in $P = \{p, q, r, s\}$. For each $g \in G$, let $P_g = \{x \in P : x \mid o(g)\}$. By the Strong Perfect Graph Theorem, we know that Γ_G is perfect if and only if it does not contain an induced C_{2k+3} or $\overline{C_{2k+3}}$ with $k \geq 1$. Here, and for the duration of the argument, we assume $k \geq 1$. Let's first consider which types of vertices could be taken to give such induced subgraphs. Suppose that g is an element such that $|P_g| = 1$. Without loss of generality, say $P_g = \{p\}$. Notice that this vertex cannot be in an induced cycle of odd length of at least five. If it were, we would have at least two vertices that are not adjacent to g but are adjacent to each other. However, if two vertices are not adjacent to g this means that they share a common factor of p and as such cannot be adjacent to each other. Similarly, in the complement of an odd cycle of length 5 or more we must have two vertices that are not adjacent to g but are adjacent to each other, which cannot happen. See the figure below.

For C_5:



If $p \mid o(g)$, then $p \mid o(h)$ and $p \mid o(h')$

For $\overline{C_7}$:



If $p \mid o(g)$, then $p \mid o(h)$ and $p \mid o(h')$

Therefore, if there exists an induced C_{2k+3} or $\overline{C_{2k+3}}$ in Γ_G, every vertex g in the induced subgraph must satisfy $|P_g| \geq 2$. Observe that this means $|P_g|$ is exactly two for each such g because if $\{g, h\}$ is an edge in Γ_G then $P_h \cap P_g = \emptyset$. However, since each $|P_g| \geq 2$, by the pigeonhole principle, this is only possible if $|P_g| = 2$ and $|P_h| = 2$. Lastly, consider the induced subgraph of Γ_G on the elements $g \in$G with $|P_g| = 2$. Suppose first that $d = 0$. Note that by the pigeonhole principle any pair of subsets of size two from $\{p, q, r\}$ will share an element. Hence taking all the vertices in G whose order is divisible by exactly two primes yields an independent set (and so neither C_{2k+3} nor $\overline{C_{2k+3}}$ can be an induced subgraph of Γ_G). Now suppose $d > 0$ and again take the induced subgraph of Γ_G on the elements whose order has two prime divisors. This gives a bipartite graph as can be seen in the diagram below.

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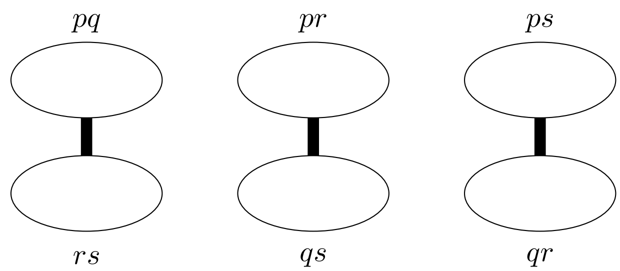


Each oval represents the set of elements within G whose order is divisible by exactly two primes. Note that each pod is an independent set and each pod forms a complete bipartite graph with exactly one other such pod. Hence, by Proposition 2.1 there can be no odd cycles as subgraphs and thus no odd cycles as induced subgraphs. To see that $\overline{C_{2k+3}}$ cannot appear as an induced subgraph note two things. First $C_5 \cong \overline{C_5}$. Now notice that for any $\overline{C_{2k+3}}$ for $k \geq 2$, we have $K_3 \leq \overline{C_{2k+3}}$. Thus in both cases the previous argument applies. □

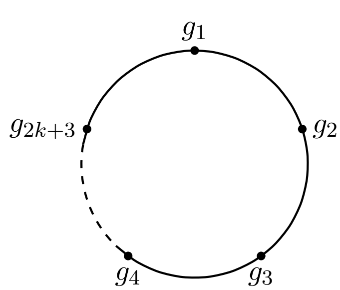
We move now to groups with at least five distinct prime divisors.

Theorem 5.2. *Suppose that G is a group with order n. Suppose n at least five distinct prime divisors and let $P = \{\text{primes } p : p \mid n\}$. For $k \geq 1$ let $A_1, A_2, \ldots, A_{2k+3} \subseteq P$ and set $V_i = \{g \in G : o(g)$ is the product of the primes in $A_i\}$. Consider the following two conditions:*

(1) $A_i \cap A_j = \emptyset$ iff $i - j \equiv \pm 1 \bmod n$
(2) $A_i \cap A_j \neq \emptyset$ iff $i - j \equiv \pm 1 \bmod n$

Then Γ_G is not perfect if and only if $V_i \neq \emptyset$ for all i and condition 1 or 2 holds.

Proof. Let G be a group of order n. Suppose that n has at least five prime divisors and let P, A_i, and V_i be as described above. First suppose that $V_i \neq \emptyset$ for all i and condition 1 holds. It follows that there exists group elements $g_i \in V_i$ such that $(o(g_i), o(g_j)) = 1$ iff $i - j \equiv \pm 1 \bmod n$. Hence the following will be an induced subgraph of Γ_G.



Since $k \geq 1$ this gives an induced odd cycle of length at least 5. Hence Γ_G is not perfect by the Strong Perfect Graph Theorem. Similarly condition 2 implies that Γ_G has an induced subgraph of $\overline{C_{2k+3}}$.

Now suppose that Γ_G is not perfect. By the Strong Perfect Graph Theorem we know that Γ_G must have C_{2k+3} or $\overline{C_{2k+3}}$ as an induced subgraph and $k \geq 1$. If Γ_G has C_{2k+3} as an induced subgraph this implies the existence of group elements $g_1, g_2, \ldots, g_{2k+3}$ such that $(o(g_i), o(g_j)) = 1$ iff $i - j \equiv \pm 1 \mod n$. This is precisely having each V_i non-empty and condition 1. Similarly, the existence of $\overline{C_{2k+3}}$ as an induced subgraph gives rise to condition 2. □

Corollary 5.3. *Let G be an abelian group. Then Γ_G is perfect if and only if the order of G has at most four prime factors.*

Proof. Let G be an abelian group. By Theorem 5.1 if $|G|$ has at most four prime factors, then Γ_G is perfect. So now suppose the order of G has at least five prime factors, say p, q, r, s, and t. By Cauchy's Theorem we know that there exist elements $a, b \in G$ such that $o(a) = p$ and $o(b) = q$. Since G is abelian we have $o(ab) = \text{lcm}(o(a), o(b)) = pq$. By similar reasoning, we can show that elements with order st, rt, pr, and qs also exist in G. Therefore Γ_G is not perfect by Theorem 5.2. □

Next we examine the coprime graph of the symmetric, alternating, and dihedral groups to determine when they are perfect. To do this we utilize the following fact about perfect graphs. Let Γ be a graph with H an induced subgraph of Γ. If Γ is perfect, then H is perfect. Similarly, if Γ has any induced subgraph H such that H is not perfect, then Γ is not perfect. This is easy to see since H being an induced subgraph of Γ implies any induced subgraph of H is also an induced subgraph of Γ. We now state an observation that will be used to prove the complete classification for symmertric, alternating, and dihedral groups.

Recall from above that $P = \{\text{primes } p : p \mid n\}$ and for $g \in G$, $P_g = \{p \in P \mid p \mid o(g)\}$.

Observation 5.4. *If H is an induced subgraph of Γ_G and if there exists $h \in H$ and a prime p such that*

(1) $p \in P_h$,
(2) for all $h' \in H \setminus \{h\}$, $p \notin P_{h'}$,
(3) there exists $g \in G$ such that $o(g) = \frac{1}{p}o(h)$,

then $(H \setminus \{h\}) \cup \{g\} \cong H$.

This observation states that if a prime p appears in P_h for exactly one element h we can remove h, replace it with a group element of order $\frac{1}{p}o(h)$, and have an isomorphic copy of the graph we started with. This is because removing a factor of p from the order of h will not affect the common factors, and hence adjacency, it shares with other vertices in H.

Theorem 5.5. *Let G be the symmetric group, S_n. Then Γ_G is perfect if and only if $n \leq 13$.*

Proof. Let $G = S_{13}$ and suppose that Γ_G has an induced subgraph H with $H \cong C_{2k+3}$ or $H \cong \overline{C_{2k+3}}$, $k \geq 1$. Recall from the proof of Theorem 5.1, that if $g \in V(H)$, we must have $|P_g| \geq 2$. For primes p, the only elements in S_p whose order is divisible by p are p-cycles, i.e. elements of exactly order p. It follows that for any $g \in V(H)$, $13 \notin P_g$. Now let's consider elements whose order is divisible by 11. In S_{13}, the only elements whose order is divisible by 11 are 11-cycles and disjoint products of a transposition with an 11-cycle. Hence the only elements whose order is divisible by 11 are elements of

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order 11 and 22. We know that elements of order 11 can not appear in H so the only elements that can appear are those of order 22. However, notice that no two elements in H can have the same order because no two vertices have the same neighborhood. It follows that there can be only one element, $h \in V(H)$ such that $11 \in P_h$ and $o(h) = 22$. But now by Observation 5.4, we know that any induced subgraph containing h must be isomorphic to an induced subgraph with vertex g in place of h with $o(g) = 2$. But then $|P_g| = 1$, a contradiction. It follows that there can be no vertices in H whose order is divisible by 11. But now the prime factors of any vertex in H must come from $P = \{2, 3, 5, 7\}$, a set containing only four primes. As seen in the proof of Theorem 5.1, this can not happen. Hence the coprime graph of S_{13} is perfect. Since S_m is an induced subgraph of S_n for $m \leq n$, this shows that Γ_{S_n} is perfect for $n \leq 13$. Now let $n \geq 14$. It follows that S_n contains the following elements:

$$\alpha_1 = (1,2)(3,4,\ldots,13) \quad \alpha_2 = (1,2\cdots,5)(6,7,\ldots,12) \quad \alpha_3 = (1,2,3)(4,5,\ldots,14)$$
$$\alpha_4 = (1,2)(3,4,\ldots,9) \quad \alpha_5 = (1,2,3)(4,5,6,7,8)$$

Recall that cycles of length k have order k and that the order of a product of disjoint cycles is the least common multiple of their orders. Hence $o(\alpha_1) = 2 \cdot 11$, $o(\alpha_2) = 5 \cdot 7$, $o(\alpha_3) = 3 \cdot 11$, $o(\alpha_4) = 2 \cdot 7$, and $o(\alpha_5) = 3 \cdot 5$. Therefore the coprime graph is not perfect by Theorem 5.2. □

Corollary 5.6. *Let G be the alternating group, A_n. Then Γ_G is perfect if and only if $n \leq 14$.*

Proof. First notice that Γ_{A_n} is an induced subgraph of Γ_{S_n} for all n. It follows from Theorem 5.5 that Γ_{A_n} is perfect if $n \leq 13$. Now consider A_{14} and suppose that H is an induced subgraph of A_{14} with $H \cong C_{2k+3}$ or $H \cong \overline{C_{2k+3}}$, $k \geq 1$. Note that the only elements in A_{14} divisible by 13 are 13-cycles, which can not appear in H. Recall that a permutation is in A_n if it can be written as a product of an even number of transpositions. For a cycle this implies that the length must be odd. Hence for a permutation written as a product of disjoint cycles to be in A_n it must contain an even number of even length cycles or any number of odd cycles. There are elements in S_{14} whose order is 22 but these are disjoint products of a transposition and an 11-cycle which do not appear in A_{14}. So the only elements in A_{14} whose order is divisible by 11 are 11-cycles and disjoint products of a 3-cycle and an 11-cycle. As in the proof of Theorem 5.5 since no two elements in H can have the same order we can only have one element whose order is divisible by 11 and this element has order 33. By Observation 5.4 this implies that there can be no vertex in H whose order is divisible by 11. But then we are down to a set of only four available primes, namely $P = \{2, 3, 5, 7\}$, as in the proof of Theorem 5.5. Again by the proof of Theorem 5.1, the coprime graph of A_{14} is perfect.

Let $n \geq 15$. Consider the following elements:

$$\alpha_1 = (1,2)(3,4)(5,6,\ldots,15) \quad \alpha_2 = (1,2,3)(4,5,\ldots,8) \quad \alpha_3 = (1,2)(3,4)(5,6,\ldots,11)$$
$$\alpha_4 = (1,2,3)(4,5,\ldots,14) \quad \alpha_5 = (1,2,\ldots,5)(6,7,\ldots,12)$$

Notice that $\alpha_i \in A_n$ for all i. Furthermore, $o(\alpha_1) = 2 \cdot 11$, $o(\alpha_2) = 3 \cdot 5$, $o(\alpha_3) = 2 \cdot 7$, $o(\alpha_4) = 3 \cdot 11$, and $o(\alpha_5) = 5 \cdot 7$. Therefore Γ_{A_n} is not perfect by Theorem 5.2. □

Theorem 5.7. *The coprime graph of D_n is perfect if and only if n has at most four prime factors.*

Proof. Let $G = D_n$ where n has 5 or more distinct prime divisors. Consider the induced subgraph of Γ_G with vertices from $\langle x \rangle$; call it H. The graph H is isomorphic to $\Gamma_{\mathbb{Z}_n}$, since $\langle x \rangle \cong \mathbb{Z}_n$ as groups. Using Corollary 5.3, we have that H is not perfect and since H is an induced subgraph of Γ_G, Γ_G is also not perfect. Now suppose that n has at most 4 prime factors. If $2 \mid n$ then $|D_n|$ has at most four prime factors and Γ_{D_n} is perfect by Theorem 5.1. Suppose $2 \nmid n$. Then Γ_{D_n} is the join of an independent set of size n (consisting of the n reflections of order 2) and an isomorphic copy of $\Gamma_{\mathbb{Z}_n}$. But since n has at most 4 prime factors we know that $\Gamma_{\mathbb{Z}_n}$ is perfect by Theorem 5.1. Hence Γ_{D_n} must also be perfect. □

6. **Future Work**

In the 1870's, Leopold Kronecker proved the structure theorem for finite abelian groups. This theorem states that any finite abelian group is isomorphic to a direct product of cyclic groups, $\mathbb{Z}_n$, with $n = p^a$ for some prime p. This motivates us to investigate the coprime graph of a direct product of groups. For two groups G and H, we would like to see if there is a relationship between $\Gamma_{G\times H}$ and the pair of graphs Γ_G, Γ_H. Namely is there a graph product $*$ such that $\Gamma_{G\times H} \cong \Gamma_G * \Gamma_H$? This could give a different approach to finding graph parameters for coprime graphs of groups for finite abelian groups.

Additionally, we would like to explore relationships between parameters of the coprime graph of a group and the order divisor graph of a group. The order divisor graph of a group was introduced in [9] and is the graph with vertices given by group elements and two vertices a, b are adjacent if and only if $o(a) \mid o(b)$ (or vice versa). While this graph is not the complement of the coprime graph it is very clearly related to the complement. As such it would be interesting to find and explore relationships between parameters for both graphs.

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