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EXACT BOUNDS FOR (λ, n)–STABLE 0-1 MATRICES.

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In memory of Robert Silverman

Abstract. Consider a $v \times v$ $(0,1)$ matrix A with exactly k ones in each row and each column. A is (λ, n)–stable, if it does not contain any $\lambda \times n$ submatrix with exactly one 0. If A is (λ, n)–stable, $\lambda, n \geq 2$, then under suitable conditions on A, $v \geq \frac{k}{n} \frac{k(n-1)+(\lambda-2)}{\lambda-2}$. The case of equality leads to new and substantive connections with block designs. The previous bound and characterization of $(\lambda, 2)$–stable matrices follows immediately as a special case.

1. **Introduction**

The objects under consideration are square $(0,1)$ matrices of size v with *exactly* k ones in each row and each column with $k < v$.

A given matrix is $(\lambda, 2)$–stable if it *does not* contain a $\lambda \times n$ submatrix with *exactly one* zero where λ and n are positive integers *not* exceeding k.

The following elementary monotone remark concerning stable matrices follows by definition.

Remark 1. *If A is an (a, b)–stable matrix then A is (a', b')–stable whenever $a' \geq a$ and $b' \geq b$.*

These (λ, n)–stable matrices arise in various contexts including group theory and projective geometry as described in the examples below. Investigation of $(0,1)$ matrices lacking certain submatrices or combinatorial configurations has been pursued by Ryser [8] and more recently by Anstee [1, 2].

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Bruen-Bruen-Silverman [7] gave an exact bound for $(\lambda, 2)$–stable $(0, 1)$ matrices and characterization in the case of equality was determined.

In Section 3, 4, 5 and 6, we study the case of (λ, n)–stable matrices and obtain generalizations of the results obtained with $(\lambda, 2)$–stable matrices. By examining (λ, n)–stable matrices, new associations with block designs are revealed. In particular, the characterization of $(\lambda, 2)$–stable matrices arises as a consequence of the new design theoretic perspective.

A general construction of (λ, n)–stable matrices is also given while the existence of some of these matrices is shown to be equivalent to the existence of particular block designs. The new associations of (λ, n)–stable matrices with block designs are described in Sections 4 to 6.

1.1. **Examples.** In this section we give some motivating examples of stable matrices that arise in various contexts are reviewed.

Example 1.

Let G be any finite group and H a subgroup of G, where $1 < |H| < |G|$. Let $A = (a_{ij})$ be a $|G| \times |G|$ square matrix whose rows and columns are indexed by elements of G, where:

$$a_{ij} = \begin{cases} 1 \text{ if } g_i = hg_j \text{ for some } h \text{ in} H \\ 0 \text{ otherwise} \end{cases}$$

Then A is $(2, 2)$–stable since any two (right) cosets are either disjoint or identical. Furthermore, A is also $(\lambda, 2)$–stable for $\lambda \geq 2$ by Remark 1.

Example 2.

Let U and V, with $|U| = |V|$, be two sets of pairwise skew lines in $PG(3, F)$ which cover the same points, but have no lines in common. Let $A = (a_{uv})$ be a square matrix whose rows and columns are indexed by the lines of U and V. Put

$$a_{uv} = \begin{cases} 1 \text{ if } l_u \in U \text{ intersects } l_v \in V \\ 0 \text{ otherwise} \end{cases}$$

Then A is $(4, 4)$–stable and also $(\lambda, 4)$–stable for $\lambda \geq 4$. This follows from the "regulus" or Gallucci's theorem [4, 5].

2. Some More Definitions

An $m \times n$ $(0, 1)$ matrix A is considered as an element of $M_{m,n}(F)$, with entries consisting only of 0 and 1 where F is a field of numbers of characteristic 0 such as $\mathbf{R}$ or $\mathbf{Q}$. The set of matrices $M_{m,m}(F)$ is shortened to $M_m(F)$. $J_{m,n}$ is the $m \times n$ matrix whose elements are one. The matrix $J_{n,n}$ is abbreviated to J_n. The matrix 0 of size $m \times n$ is denoted by $0_{m,n}$. The matrix $0_{n,n}$ is abbreviated to 0_n.

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Given $A \in M_{m,n}(F)$ and $B \in M_{p,q}(F)$, then the tensor or Kronecker product of A and B, denoted by $A \otimes B$, is a matrix of size $mp \times nq$ consisting of mn matrix blocks, $a_{11}B, a_{12}B, \ldots, a_{mn}B$. The (i,j) matrix block is the matrix $a_{ij}B$, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

An extension of the standard dot product to $n \geq 3$ vectors is defined as follows. Given three vectors $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$, $a, b, c \in F^k$ the dot (or scalar) product of a, b, c is defined as $\bullet[a, b, c] = a_1b_1c_1 + a_2b_2c_2 + \cdots + a_kb_kc_k$ [6]. The dot product of n vectors is defined in a similar way giving a mapping $\bullet : F^n \to F$, $n \geq 1$.

Consider $A \in M_{m,n}(F)$. Let $A_{(i)}$ denote the i^{th} row of A and $A^{(j)}$ the j^{th} column of A.

Let $M_m^k(F)$ consist of all the $(0,1)$ matrices with *exactly* k ones in each row and each column. Equivalently, $A \in M_m^k(F)$ if and only if both A and A^T have $(1, \ldots, 1)$ as an eigenvector with associated eigenvalue k. Consider $A \in M_m^k(F)$. Let $\mathcal{C}_l$ denote the set of all columns whose dot product with column $A^{(l)}$ is k. A column $A^{(j)}$ is in $\mathcal{C}_l$ if (and only if) $A^{(j)} = A^{(l)}$. If $A^{(j)} \neq A^{(l)}$ (as vectors), then $A^{(j)}$ and $A^{(l)}$ are said to be *unequal* or *distinct*. The set of all columns *unequal* (or *distinct*) from $A^{(l)}$ is denoted by $\overline{\mathcal{C}_l}$ and so $|\mathcal{C}_l| + |\overline{\mathcal{C}_l}| = m$. A row $A_{(i)}$ (resp. column $A^{(j)}$) is *incident* with a column $A^{(j)}$ (resp. row $A_{(i)}$) if $(a_{ij}) = 1$. Let $\mathcal{R}_l$ denote the set of all rows that are incident with column $A^{(l)}$. Then $|\mathcal{R}_l| = k$.

3. **Exact lower bound**

The first result for this section generalizes of the paper [7, Theorem 3.1]. A *regularity* assumption on the columns is made along with an *incidence* assumption. Although both conditions are used in the characterization (Section 6), they *may* be replaced by the condition that there is a column $A^{(j)}$ such that $n - 1 \leq |\mathcal{C}_j| \leq \frac{k}{n}$.

Theorem 3.1. *Let $A \in M_v^k(F)$ be a (λ, n)–stable matrix with $\lambda, n \geq 2$. Suppose that:*

(1) $|\mathcal{C}_j| \geq n - 1$ *for every column* $A^{(j)}$

(2) $1 \leq \bullet[A^{(j_1)}, \ldots, A^{(j_n)}]$ *for some set $\mathcal{S}$ consisting of n mutually distinct columns* $A^{(j_1)}, \ldots, A^{(j_n)}$.

Then $v \geq \frac{k}{n}\frac{k(n-1)+(\lambda-2)}{\lambda-2}$ holds. Moreover, equality holds if and only if $|\mathcal{C}_j| = \frac{k}{n}$ and $A^{(j)} \bullet A^{(l)} = \lambda - 2$ for any two distinct columns.

Proof. Let $A^{(j)} \in \mathcal{S}$. Then $n - 1 \leq |\mathcal{C}_j|$ as $|\mathcal{C}_j|$ is at least $n-1$ by (1). Moreover, since $A^{(j)} \in \mathcal{S}$ with $n \geq 2$ there is at least one other column $A^{(l)}$ such that $1 \leq A^{(l)} \cdot A^{(j)} < k$ which implies that $|\mathcal{C}_j| < k$.

Consider the total incidences of $\mathcal{R}_j$ and $\overline{\mathcal{C}_j}$. Every $A_{(l)} \in \mathcal{R}_j$ is incident with $|\mathcal{C}_j|$ columns of $\mathcal{C}_j$ and hence $k - |\mathcal{C}_j|$ columns in $\overline{\mathcal{C}_j}$. Yet, *any* column in $\overline{\mathcal{C}_j}$ is incident with *at most* $(\lambda - 2)$ rows of $\mathcal{R}_j$ *since* A is (λ, n)–stable (with $|\mathcal{C}_j| \geq n-1$ by (1)) and $A^{(j)} \bullet A^{(l)} < k$ whenever $A^{(j)}$ and $A^{(l)}$ are distinct (see Figure 1). Then as $|\overline{\mathcal{C}_j}| = v - |\mathcal{C}_j|$, by the two incidence counts of $\mathcal{R}_j$ and $\overline{\mathcal{C}_j}$, the following inequality holds:

$$(k)(k - |\mathcal{C}_j|) \leq (v - |\mathcal{C}_j|)(\lambda - 2) \tag{3.1}$$

with equality if $A^{(j)} \bullet A^{(l)} = \lambda - 2$ for every other unequal column $A^{(l)} \neq A^{(j)}$

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$$\mathcal{R}_j \left\{ \begin{bmatrix} \dots & 0 & \overbrace{\mathbf{1} & \dots & \mathbf{1}}^{\mathcal{C}_j} & \overbrace{\textcircled{\mathbf{0}}}^{A^{(l)}} & \dots \\ & & & \vdots & & & \\ \dots & 0 & \mathbf{1} & \dots & \mathbf{1} & \boxed{\mathbf{1}} & \dots \\ & & & \vdots & & & \\ \dots & 0 & \mathbf{1} & \dots & \mathbf{1} & \boxed{\mathbf{1}} & \dots \\ & & 0 & \dots & 0 & & \\ & & & \vdots & & & \\ \dots & & \underbrace{0}_{A^{(j)}} & \dots & 0 & & \dots \end{bmatrix} \right\} (\leq \lambda - 2)$$

FIGURE 1. Inequality (3.1) illustrated with distinct columns $A^{(j)}$ and $A^{(l)}$ (up to permutation of rows and columns)

Rearranging gives:

$$v \geq \frac{|\mathcal{C}_j|(\lambda - 2 - k) + k^2}{\lambda - 2}. \tag{3.2}$$

If $|\mathcal{C}_j| \leq \frac{k}{n}$, as $\lambda - 2 - k < 0$ by *definition*, then using $-|\mathcal{C}_j| \geq -\frac{k}{n}$,

$$v \geq \frac{k((n-1)k + \lambda - 2)}{n(\lambda - 2)} \tag{3.3}$$

with strict inequality if $|\mathcal{C}_j| < \frac{k}{n}$. Then equality holds if $A^{(j)} \bullet A^{(l)} = \lambda - 2$ for any other unequal column $A^{(l)} \neq A^{(j)}$ and $|\mathcal{C}_j| = \frac{k}{n}$. By assumption (2), $A^{(j)} \in \mathcal{S}$, with $1 \leq \bullet[A^{(j)}, A^{(l_1)}, \dots, A^{(l_{n-1})}]$ which implies that there is some row $A_{(m)}$ such that $a_{mj}a_{ml_1ml_{n-1}} = 1$. Then $A_{(m)}$ is incident with $A^{(j)}$ and $(n-1)$ other columns $A^{(l_1)}, \dots, A^{(l_{n-1})}$ distinct from $A^{(j)}$. Then,

$$1 \leq |\mathcal{C}_j| + |\mathcal{C}_{l_1}| + |\mathcal{C}_{l_2}| + \cdots + |\mathcal{C}_{l_{n-1}}| \leq k$$

which implies that at least one of $|\mathcal{C}_j| \leq \frac{k}{n}$, $|\mathcal{C}_{l_1}| \leq \frac{k}{n}$,$\dots$,$|\mathcal{C}_{l_{n-1}}| \leq \frac{k}{n}$ giving inequality (3.3). Finally, suppose that $v = \frac{k(k(n-1)+\lambda-2)}{n(\lambda-2)}$. Then $|\mathcal{C}_j| = \frac{k}{n}$ by the arguments given above. Suppose there is some column $A^{(l)}$ such that $A^{(j)} \bullet A^{(l)} < \lambda - 2$. Then by counting incidences between $\mathcal{R}_j$ and $\overline{C_j}$ in two different ways as in inequality (3.1), while here considering column $A^{(l)}$ separately, gives

$$\begin{aligned}(k)(k - |\mathcal{C}_j|) = (k)(k - \frac{k}{n}) &\leq (A^{(j)} \bullet A^{(l)}) + (v - |\mathcal{C}_j| - 1)(\lambda - 2) \\ &= (A^{(i)} \bullet A^{(l)}) + (v - \frac{k}{n} - 1)(\lambda - 2)\end{aligned}$$

Rearranging gives

$$v > \frac{k}{n}\frac{(k(n-1)+\lambda-2)}{(\lambda-2)}.$$

Then $v \geq \frac{k}{n}\frac{k(n-1)+(\lambda-2)}{\lambda-2}$ with equality if and only if $|\mathcal{C}_j| = \frac{k}{n}$ and $A^{(j)} \bullet A^{(l)} = (\lambda-2)$ holds for any j and $l \neq j$.

□

If $n = 2$, Theorem 3.1 is considerably simpler as condition (1) holds trivially and condition (2) reduces to the dot product of *any* two distinct columns. The following Corollary, a re-statement of [7, Theorem 3.1], follows immediately.

Corollary 3.2. *Let $A \in M_v^k(F)$ be a $(\lambda, 2)$–stable matrix. Assume $\lambda \geq 3$ and $1 \leq A^{(j)} \bullet A^{(l)}$ holds for some distinct $A^{(j)}$ and $A^{(l)}$. Then $v \geq \frac{k}{2}\frac{k+(\lambda-2)}{\lambda-2}$ with equality if and only if the dot product of any two unequal columns is $(\lambda-2)$ and $|\mathcal{C}_j| = \frac{k}{2}$ for any column $A^{(j)}$.*

4. Block Designs

In order to address the case of equality of Theorem 3.1 and determine a characterization, the concept of a design is introduced. Moreover, to distinguish the design parameters from the matrix parameters, a * will be affixed to the design parameters.

Definition A t-(v^*, k^*, λ^*) [or t-$(v^*, k^*, \lambda^*, b^*, r^*)$] design is defined as a relation $I \subseteq V \times B$ where V consists of the v^* *points*, B consists of the b^* *blocks*. A block C is *incident* with a point p if $(p, C) \in I$. Also, every block is incident with *exactly* k^* points and conversely every point is incident with *exactly* r^* blocks. Moreover, *every* t-subset of points is incident with *exactly* λ^* shared blocks. Each parameter is a positive integer here.

The case when $t = 2$ is of particular importance and is occasionally referred to exclusively as a block design. A design without repeated blocks is referred to as a *simple* design. No requirement of simplicity is made on any of the designs described below, unless stated otherwise.

A t-$(v^*, k^*, \lambda^*, b^*, r^*)$ design can be associated with a $b^* \times v^*$ $(0,1)$ matrix A. Consider $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 \text{ if the } i^{th} \text{ block is } \textit{incident} \text{ with } j^{th} \text{point} \\ 0 \text{ otherwise} \end{cases}$$

The $b^* \times v^*$ $(0,1)$ matrix A contains k^* ones in each row and r^* ones in each column. Every t subset of columns (points) is *incident* with exactly λ^* rows (blocks). The matrix A is called an *incidence matrix* for the design. The following equality holds for any design and is easily shown by a counting argument (see [3]).

Remark 2. *Consider a t-(v^*, k^*, λ^*) [t-$(v^*, k^*, \lambda^*, b^*, r^*)$] design. Then $b^*k^* = v^*r^*$.*

Similarly, the following property relates collections of subsets of set $\mathcal{S}$ with designs.

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Remark 3. *Every t-$(v^*, t, 1)$ design can be described as the collection of all t–subsets of a set of size v^*.*

The next property relates a given t design in terms of a s design, where s is less than t.

Remark 4. *Let D be a t-(v^*, k^*, λ^*) design and $s < t$ a positive integer. Then D is also a s-$(v^*, k^*, \lambda^\dagger)$ design where $\lambda^\dagger = \binom{v-s}{t-s}$.*

5. Constructions

Theorem 5.1 stated below shows that every 2-$(v^*, k^*, \lambda^*, b^*, r^*)$ block design can be associated with a $(\lambda, 2)$–stable $(0, 1)$ matrix, which by Remark 1 is (λ, n)–stable for $n \geq 2$.

In this section, more $(\lambda, 2)$ and $(\lambda, 3)$–stable matrices are given and then the intricacies of the $(\lambda, 3)$–stable case is illustrated. A Corollary to Theorem 5.1 is also proved which provides a general construction for (λ, n)–stable matrices that achieve the lower bound of Theorem 3.1.

Theorem 5.1. *Consider a 2-$(v^*, k^*, \lambda^*, b^*, r^*)$ block design D, with incidence matrix A. Then $B = A \otimes J_{\frac{k}{r^*}, \frac{k}{k^*}}$ is a (λ, n)–stable matrix in $M_v^k(F)$, for $n \geq 2$, with $v = v^* \frac{k}{k^*}$, and $\lambda = \lambda^*(\frac{k}{r^*}) + 2$ provided $r^*|k$ and $k^*|k$, for a given positive integer k. Moreover, B satisfies:*

1. *For every row $B_{(j)}$ there is a set $\mathcal{S}$ of $n = k^*$ mutually unequal columns incident with $B_{(j)}$ such that $1 \leq \bullet[B^{(j_1)}, \ldots, B^{(j_n)}]$.*
2. *For every column $B^{(j)}$ there are exactly $\frac{k}{k^*} - 1$ other columns $B^{(l)}$ such that $B^{(j)} \bullet B^{(l)} = k$ (i.e. $|\mathcal{C}_j| = \frac{k}{k^*}$).*

Proof. Consider $B = A \otimes J_{\frac{k}{r^*}, \frac{k}{k^*}}$. Then B is a $b^*(\frac{k}{r^*}) \times v^*(\frac{k}{k^*})$ matrix. As $b^*k^* = v^*r^*$ by Remark 2, $b^*(\frac{k}{r^*}) = v^*(\frac{k}{k^*})$ which is equal to v. Moreover B has $\frac{k}{r^*}r^* = k$ 1^{s} in each column and $\frac{k}{k^*}k^* = k$ 1^{s} in each row.

Since D is a 2-(v^*, k^*, λ^*) block design, no two columns of A are equal. Moreover, since every row of A (and hence B) is incident with k^* unequal columns, condition (1) is satisfied. In addition, since $B = A \otimes J_{\frac{k}{r^*}, \frac{k}{k^*}}$, then for every column of $B^{(j)}$, $|\mathcal{C}_j| = \frac{k}{k^*}$, which shows that condition (2) is satisfied.

Moreover, B is $(\lambda, 2)$–stable as otherwise there would be two columns $A^{(j)}$ and $A^{(l)}$ where $\lambda - 1 \leq A^{(j)} \bullet A^{(l)} < k$. However, any two columns are either identical or have scalar product $\lambda^* \frac{k}{r^*}$, as D is a 2-(v^*, k^*, λ^*) block design. Since $\lambda^* \frac{k}{r^*}$ equals $\lambda - 2$ by definition, then B is $(\lambda, 2)$–stable and hence (λ, n)–stable for $n \geq 2$ by Remark 5.1.

□

The (λ, n)–stable matrices constructed with Theorem 5.1 can be analyzed in terms of the design parameter k^*. The particular case where $k^* = 2$ is an essential component of the $(\lambda, 2)$–stable matrix characterization [7].

Example 1:

Consider a 2-$(v^*, 2, 1)$ bock design D with $v^* = \frac{k+\lambda-2}{\lambda-2}$, k even, and A as an associated incidence matrix. By Remark 3, D can be described as the collection of all 2-subsets of a set of size v^*. Then in particular, $r^* = \frac{k}{\lambda-2}$ as every point (column) is incident with $v^* - 1$ blocks (rows).

By Theorem 5.1, $B = A \otimes J_{\lambda-2,\frac{k}{2}}$ is $(\lambda, 2)$–stable, with every row incident with exactly two unequal columns. Here,

$$B = \begin{pmatrix} a_{11}J_{\lambda-2,\frac{k}{2}} & \cdots & a_{1v^*}J_{\lambda-2,\frac{k}{2}} \\ \cdots & \cdots & \cdots \\ a_{\binom{v^*}{2}1}J_{\lambda-2,\frac{k}{2}} & \cdots & a_{\binom{v^*}{2}v^*}J_{\lambda-2,\frac{k}{2}} \end{pmatrix}$$

where $a_{ij}J_{\lambda-2,\frac{k}{2}}$ is a $(\lambda-2)\times\frac{k}{2}$ block of 0^{s} or 1^{s}. This result was obtained as Theorem 4.1 previously, with A described as the incidence matrix of all 2-subsets of a set of size v^* [7]. The matrix B is used to characterize the case of equality of Corollary 3.2 for $(\lambda, 2)$–stable matrices in Corollary 6.2.

The following two examples are of particular interest in considering $(\lambda, 3)$–stable matrices.

Example 2a:

Consider the *Fano* plane (Figure 2). The Fano plane can be described as a 2-$(7, 3, 1)$ block design by *identifying* the lines and points of the *plane* with blocks and points of the *design*, respectively. An incidence matrix matrix A is given as follows:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

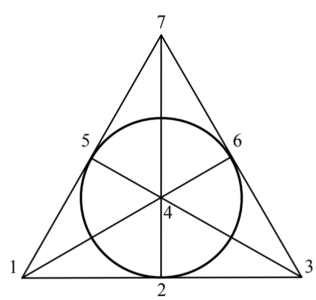

FIGURE 2. Fano Plane

Then as $k^* = 3 = r^*$, $A \otimes J_{\frac{k}{3},\frac{k}{3}}$ is $(\lambda, 2)$–stable $v \times v$ square matrix with $v = \frac{7k}{3}$ provided $3|k$ with $\lambda = \frac{k}{3} + 2$ by Theorem 5.1.

For instance, if $k = 45$, then $B = A \otimes J_{15,15}$ is $(\lambda, 2)$–stable $v \times v$ square matrix with $v = 7(15) = 105$ and $\lambda = \frac{45}{3} + 2 = 17$. Example 2b:

Consider all the 3-subsets of a set of size 7 which is a 3-$(7, 3, 1)$ design D with incidence matrix matrix $A^{'}$ (Remark 3). The incidence matrix matrix A' is:

$$A' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

with exactly 3 1's in each row. Furthermore, every point is incident with $\binom{6}{2} = 15$ blocks.

Then D is a 2-$(7, 3, 5, 35, 15)$ block design, as every two points are incident with exactly 5 blocks. Hence $B' = A' \otimes J_{\frac{k}{15}, \frac{k}{3}}$ is a $(\lambda, 2)$–stable $v \times v$ square matrix with $v = \frac{7k}{3}$ provided $15|k$ with $\lambda = \frac{k}{3} + 2$ by Theorem 5.1.

Note that both $B = A \otimes J_{15,15}$ and $B' = A' \otimes J_{3,15}$ are $(0, 1)$ matrices of size 105×105 with $k = 45$. B and B' are not only $(17, 2)$–stable but also $(17, 3)$–stable (Remark 1). Then Theorem 3.1 gives that $v \geq \frac{k}{n}\frac{k(n-1)+(\lambda-2)}{\lambda-2} = 15(\frac{2(45)+(15)}{15}) = (15)(7) = 105$, with conditions that apply to both B and B'.

$(\lambda, 3)$ characterization:

Then B and B' both achieve equality of the bound for v given in Theorem 3.1, but they are notably different, since they arise from very different designs (Fano plane and 3-subsets respectively). These cases show that the characterization of $(\lambda, 3)$–stable matrices is more complicated than the characterization of $(\lambda, 2)$–stable matrices.

(λ, n)–stable matrices:

The following Corollary to Theorem 5.1 generalizes to the construction of (λ, n)–stable matrices seen in Examples 1 and 2b. In particular, the construction achieves the bound for v of Theorem 3.1.

Corollary 5.2. *Let A be an incidence matrix for all the n-subsets of a set of size $v^* = (1 + \frac{(k)(n-1)}{\lambda-2})$. Then $A \otimes J_{\nu,\iota}$ is a $v \times v$ (λ, n)–stable matrix with $v = \frac{k}{n}\frac{k(n-1)+(\lambda-2)}{\lambda-2}$ such that conditions* (1) *and* (2) *of Theorem 5.1 are satisfied whenever $\nu = (\lambda - 2)/\binom{v^*-2}{n-2}$ and $\iota = \frac{k}{n}$ are integral.*

Proof. A is the incidence matrix matrix of a n-$(v^*, n, 1)$ design by Remark 3 and is a 2-$(v^*, n, \binom{v^*-2}{n-2}, \binom{v^*}{n}, \binom{v^*-1}{n-1})$ block design by Remark 4. The rest follows from Theorem 5.1 and noting that $\frac{k}{\binom{v^*-1}{n-1}} = \frac{\lambda-2}{\binom{v^*-2}{n-2}}$. □

6. Characterization

The following result shows that the existence of (λ, n)–stable matrix A that achieves the bound of Theorem 3.1 is equivalent to the existence of a specific block design D.

Theorem 6.1. *Consider $A \in M_v^k(F)$, a (λ, n)–stable matrix, $\lambda, n \geq 2$ with $v = \frac{k}{n}\frac{k(n-1)+(\lambda-2)}{\lambda-2}$. Suppose that:*

1. *For every column $A^{(j)}$ there are $(n-2)$ other columns $A^{(l)}$ such that $A^{(j)} \cdot A^{(l)} = k$ (i.e. $|\mathcal{C}_j| \geq n-1$).*
2. *For every column $A^{(j)}$ there are $(n-1)$ other unequal columns $A^{(j_1)}, \ldots, A^{(j_{n-1})}$ such that $1 \leq \cdot[A^{(j)}, A^{(j_1)}, \ldots, A^{(j_{n-1})}]$.*

Then $A = B \otimes J_{1,\frac{k}{n}}$ if and only if B is the incidence matrix matrix of a 2-$(v^*, n, \lambda-2, v, k)$ block design, with $v^* = \frac{k(n-1)+\lambda-2}{\lambda-2}$ (up to a permutation of the columns of A).

Proof. Suppose A satisfies conditions (1) and (2), with $v = \frac{k}{n}\frac{k(n-1)+(\lambda-2)}{\lambda-2}$. Then for any column $A^{(j)}$, $|\mathcal{C}_j| = \frac{k}{n}$ and the scalar product of $A^{(j)}$ with any other unequal column $A^{(l)}$ is $\lambda - 2$ by Theorem 3.1. There are $\frac{v}{\frac{k}{n}} = v^*$ unequal columns of A. Order v^* unequal columns arbitrarily as $A^{j_1}, A^{j_2}, \ldots, A^{j_{v^*}}$ and permute the columns of A so that the first $\frac{k}{n}$ columns consist of $\mathcal{C}_{j_1}$, the next $\frac{k}{n}$ columns consist of $\mathcal{C}_{j_2}$, and so forth.

Write A as $B \otimes J_{1,\frac{k}{n}}$ where B is a $v \times v^*$ matrix with n 1^{s} in each row and k 1^s in each column. Identify the points of a design D with the v^* columns of B and the blocks of D with the v rows of B. Then D has k blocks incident with each point, n points incident with each block and v blocks. D is a 2-$(v^*, n, \lambda-2, v, k)$ block design as any two columns of B have scalar product $\lambda - 2$ which by definition is the number of shared blocks of any two points. Conversely, suppose B is an incidence matrix of a 2-$(v^*, n, \lambda-2, v, k)$ block design, with $v^* = \frac{k(n-1)+\lambda-2}{\lambda-2}$. By Theorem 5.1, $B \otimes J_{1,\frac{k}{n}}$ is a (λ, n)–stable matrix with $v = \frac{k}{n}\frac{k(n-1)+(\lambda-2)}{\lambda-2}$ which satisfies (1) and (2). □

The following Corollary gives a characterization of a $(\lambda, 2)$–stable matrix. The characterization essentially depends on collapsing repeated blocks of the block design from Theorem 6.1. Corollary 6.2 implies Theorem 6.2 of a previous paper ([7]).

Corollary 6.2. *Let $A \in M_v^k(F)$ be a $(\lambda, 2)$–stable matrix with $v = \frac{k}{2}(\frac{k+(\lambda-2)}{\lambda-2})$ where there are two unequal columns $A^{(i)}$ and $A^{(j)}$ with $1 \leq A^{(i)} \cdot A^{(j)} < k$. Then $A = C \otimes J_{\lambda-2,\frac{k}{2}}$ up to a permutation of the rows and columns of A, where C is the incidence matrix matrix of a 2-$(v^*, 2, 1)$ block design with $v^* = (\frac{k+(\lambda-2)}{\lambda-2})$.*

Proof. By Corollary 3.2 if *any* two columns of A have a non-trivial scalar product, then *every* two unequal columns of A have a non-trivial scalar product of $\lambda - 2$. Then $A = B \otimes J_{1,\frac{k}{2}}$ by Theorem 6.1 following a permutation of the columns of A where B is the incidence matrix matrix of a 2-$(v^*, 2, \lambda-2, v, k)$ block design with $v^* = \frac{k}{\lambda-2} + 1$.

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However, any column of B, in particular $B^{(1)}$, has scalar product of $\lambda - 2$ with every other column. Then permute the rows of B so that $B^{(1)}$ and $B^{(2)}$ are *both* incident with *each* of the first $\lambda - 2$ rows, $B^{(1)}$ and $B^{(3)}$ are both incident with the next $\lambda - 2$ rows, and so forth, until $B^{(n)}$ and $B^{(n-1)}$ are both incident with each of the last $\lambda - 2$ rows.

The same permutation can be applied to the rows of A which can then be written as $A = C \otimes J_{\lambda-2,\frac{k}{2}}$. The matrix C has two ones in each row and by construction the scalar product of any two distinct columns equals one. Then C is the incidence matrix matrix of 2-$(v^*, 2, 1)$ block design D which can be seen by identifying the v^* columns of C with the points of the design and the rows of C with the blocks of the design. □

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References

[1] R. P. Anstee, Properties of $(0, 1)$–Matrices with no triangles, *Journal of Combinatorial Theory Series A*, **29** (1980) 186–198.

[2] R. P. Anstee and N. Kamoosi, Small Forbidden Configurations III, *Electronic Journal of Combinatorics*, **14** (2007).

[3] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Bibliographisches Institut, 1985.

[4] A. A. Bruen, Some new replaceable translation nets, *Canadian Journal of Mathematics*, **29** (1977) 225–237.

[5] A. A. Bruen and R. Silverman, Switching sets in $PG(3, q)$, *Proceedings American Math Society*, **43** (1974) 176–180.

[6] A. A. Bruen, T. C. Bruen and R. Silverman, *Incidence matrices with forbidden configurations*, AMS Press, Providence Rhode Island, 2010.

[7] A. A. Bruen and T. C. Bruen and R. Silverman, A characterization of $(\lambda, 2)$ stable matrices, *Linear Algebra and its Applications*, **436** (2012) 814–819.

[8] H. J. Ryser, Combinatorial configurations, *SIAM Journal in Applied Mathematics*, **17** (1969) 593–602.

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