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**International Journal of Group Theory**  
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669  
Vol. 10 No. 2 (2021), pp. 75-88.  
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## ON INFINITE GROUPS WHOSE FINITE QUOTIENTS HAVE RESTRICTED PRIME DIVISORS

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**ABSTRACT.** The effect of restricting the set of primes dividing the orders of the finite quotients of a group is investigated. Particular attention is paid to abelian, soluble, locally soluble and locally finite groups. The connection with the extraction of roots is explored.

### 1. Introduction

The present work is a study of groups in which limits are placed on the prime divisors of the orders of the finite quotients. If  $\pi$  is a set of primes, let

$$\mathcal{F}(\pi)$$

denote the class (or property) of groups in which every finite quotient is a  $\pi$ -group.<sup>1</sup> If  $\pi$  is empty,  $\mathcal{F}(\pi)$  is the class of groups without proper subgroups of finite index, while if  $\pi$  is the set of all primes,  $\mathcal{F}(\pi)$  is the class of all groups. Of course,  $\mathcal{F}(\pi)$  is really a restriction on the quotient group by the finite residual of a group, and clearly it becomes stronger the smaller the set  $\pi$  is. The property  $\mathcal{F}(\pi)$  is related to  $\pi'$ -radicability, i.e. the condition that every element of a group is the  $p$ th power of some (not necessarily unique) group element for all  $p \in \pi'$ . Here, as usual,  $\pi'$  denotes the complement of the set of primes  $\pi$ . In fact for hypercentral groups these two properties coincide (Proposition 2.4 below).

Our aim here is to study the effect of imposing the condition  $\mathcal{F}(\pi)$  on various classes of infinite groups. We first characterize the abelian groups with  $\mathcal{F}(\pi)$ , i.e., those which are  $\pi'$ -divisible. (Usually

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Communicated by Patrizia Longobardi

Manuscript Type: Research Paper

MSC(2010): Primary: 20E26; Secondary: 20F16.

Keywords: Finite quotients, Prime divisors.

Received: 06 September 2019, Accepted: 06 March 2020.

<http://dx.doi.org/10.22108/ijgt.2020.119054.1569>

<sup>1</sup>The author thanks Professor Heguo Liu of Hubei University and Dr. Jun Liao of Peking University for drawing his attention to this class of groups.

we will write abelian groups additively and speak of divisibility rather than radicability). This is done in Theorem 3.1, a result that may be known to some group theorists.

In many of our results a prominent role is played by the ring of  $\pi'$ -adic rational numbers

$$\mathbb{Q}_{\pi'}$$

which consists of all rational numbers of the form  $\frac{m}{n}$  where  $m, n \in \mathbb{Z}$  and  $n$  is a  $\pi'$ -number. The sharpest results tend to occur when  $\pi$  consists of a single prime. Our characterization of  $p'$ -divisible abelian groups runs as follows.

**Theorem 3.3.** *Let  $A$  be an abelian group and  $p$  a prime. Then  $A$  is  $p'$ -divisible if and only if  $A = D \oplus A_0$  where  $D$  is a divisible  $p'$ -group and  $A_0$  has a subgroup  $A_1 = S \oplus P$  such that  $S$  is a free  $\mathbb{Q}_{p'}$ -module,  $P$  is a  $p$ -group and  $A_0/A_1$  is a rational vector space.*

We turn next to soluble groups, a class of groups that is too wide to repay study in general. So we concentrate on soluble groups with finite abelian ranks (FAR-groups). Recall that a *soluble FAR-group* is a group with a series of finite length in which each factor is abelian with finite  $p$ -rank for  $p = 0$  or a prime: for an account of this extensive class of infinite soluble groups see [9, 5.1, 5.2, 5.3]. The effect of imposing the condition  $\mathcal{F}(\pi)$  on a soluble FAR-group where  $\pi$  is a proper subset of the primes is striking.

**Theorem 4.2.** *Let  $G$  be a soluble group with finite abelian ranks and let  $\pi$  be a proper subset of the primes. If  $G$  satisfies  $\mathcal{F}(\pi)$ , there are normal subgroups  $L \geq M \geq N$  of  $G$  such that  $G/L$  is a finite  $\pi$ -group,  $L/M$  is a torsion-free,  $\pi'$ -radicable nilpotent group,  $M/N$  is a residually finite  $\pi$ -group and  $N$  is a periodic radicable abelian group. Conversely, any group with this structure satisfies  $\mathcal{F}(\pi)$ .*

For example, whereas by a theorem of A.I. Mal'cev [9, 5.2.2] torsion-free soluble FAR-groups are nilpotent-by-abelian-by-finite, those groups which satisfy  $\mathcal{F}(\pi)$  for some proper set of primes  $\pi$  are nilpotent-by-finite. The result that underlies Theorem 4.2 is an interesting theorem of V.S. Čarin [3]: *a group with a series of finite length whose factors are non-cyclic rational groups, (i.e., subgroups of the additive group of rational numbers), is nilpotent-by-finite.* For a short proof of Čarin's theorem see [9, 5.2.10]. In addition the locally soluble groups with finite Prüfer rank that satisfy a condition  $\mathcal{F}(\pi)$  are characterized in Theorem 4.4.

Torsion-free nilpotent (and even hypercentral) radicable groups have been widely studied, for example by S.N. Černikov [4], [5] and G. Baumslag [1, 9.2]. Here we establish several new results in the nilpotent case. For example:

**Theorem 5.1.** *Let  $G$  be a torsion-free nilpotent group of finite rank  $r > 0$  and let  $p$  be a prime. Assume that  $G$  is  $p'$ -radicable. Then the following statements hold.*

- (i) There is a series  $H = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_\ell = G$  such that  $H$  has a central series of length  $r$  with exclusively  $\mathbb{Q}_{p'}$ -factors, while each factor  $H_{i+1}/H_i$  is a  $p^\infty$ -group. Moreover, the integer  $\ell$  is an invariant of  $G$ .
- (ii) There are subgroups  $U_0, U_1, \dots, U_{r-1}$ , each isomorphic with  $\mathbb{Q}_{p'}$ , such that  $H = U_{r-1} \cdots U_1 U_0$ .

In the final section we examine locally finite groups satisfying the property  $\text{min-}p$  for various primes  $p$  and determine which of them have the property  $\mathcal{F}(\pi)$ . The main conclusion is as follows.

**Theorem 6.1.** *Let  $G$  be a locally (finite soluble) group and denote by  $\sigma(G)$  the set of primes  $p$  for which  $G$  does not satisfy  $\text{min-}p$ . Let  $\pi$  be a set of primes that contains  $\sigma(G)$ . Then  $G$  satisfies  $\mathcal{F}(\pi)$  if and only if there are normal subgroups  $L \geq M$  such that  $G/L$  is a  $\pi$ -group,  $L/M$  is a radicable abelian  $\pi'$ -group and  $M$  is a  $\sigma(G)$ -group.*

The most interesting case is where  $\sigma(G)$  is empty and  $G$  has  $\text{min-}p$  for all  $p$ . In this situation the hypothesis of local solubility may be omitted.

**Corollary 6.2.** *Let  $G$  be a locally finite group satisfying  $\text{min-}p$  for all primes  $p$  and let  $\pi$  be any set of primes. Then  $G$  has the property  $\mathcal{F}(\pi)$  if and only if it is an extension of a radicable abelian  $\pi'$ -group by a  $\pi$ -group.*

**List of notation.**

- (i)  $\pi'$ : the complement of a set of primes  $\pi$ .
- (ii)  $\mathcal{F}(\pi)$ : the group theoretical property that every finite quotient is a  $\pi$ -group.
- (iii)  $\mathbb{Q}_{\pi'}, \mathbb{Q}_{p'}$ : the additive groups of  $\pi'$ -adic and  $p'$ -adic rational numbers.
- (iv)  $Z(G), Z_i(G)$ : terms of the upper central series.
- (v)  $G^{(n)}$ : a term of the derived series.
- (vi)  $\text{min-}p$ : the minimal condition for  $p$ -subgroups.
- (vii)  $\text{HP}(G)$ : the Hirsch-Plotkin radical.

**2. Elementary results**

In this section we record some elementary facts, mainly pertaining to radicableity, which will be used in the sequel.

**Lemma 2.1.** *Let  $G$  be a group with finite residual  $R$  and let  $\pi$  be a set of primes. Then:*

- (i)  $G$  has  $\mathcal{F}(\pi)$  if and only if  $G/R$  has  $\mathcal{F}(\pi)$ ;
- (ii) if  $G/R$  is a periodic group, then  $G$  has  $\mathcal{F}(\pi)$  if and only if  $G/R$  is a  $\pi$ -group.

Here (i) is obvious. As for (ii), if  $G$  has  $\mathcal{F}(\pi)$ , the quotient  $G/R$  is residually a finite  $\pi$ -group and, because it is periodic, it must be a  $\pi$ -group. The converse is clear.

**Lemma 2.2.** *The property  $\mathcal{F}(\pi)$  is inherited by quotients, extensions, unions of chains, and subgroups of finite index.*

The truth of each of these statement is easily established. Notice however that if  $\pi$  is not empty, the property  $\mathcal{F}(\pi)$  is not inherited by subgroups, as is shown by  $\mathbb{Z} < \mathbb{Q}$ .

The close connection between  $\mathcal{F}(\pi)$  and  $\pi'$ -radicality for groups with some degree of solubility is the topic of the next two results.

**Lemma 2.3.** *Let  $G$  be a soluble group and  $\pi$  a set of primes. If  $G$  has  $\mathcal{F}(\pi)$ , then  $G = G^p$  for all  $p \in \pi'$ .*

*Proof.* Let  $p \in \pi'$ . Then  $G/G^p$  is a soluble  $p$ -group of finite exponent, so if it is non-trivial,  $G$  has a non-trivial finite  $p$ -quotient. It follows that  $G$  does not have  $\mathcal{F}(\pi)$ , a contradiction.  $\square$

For hypercentral groups much more can be said regarding  $\mathcal{F}(\pi)$  and  $\pi'$ -radicality, as the next result shows. Most of this can be found in [10, Theorem 9.23]. When  $\pi$  is the empty set, it is largely due to Černikov [4, Theorem 10] and [5, Theorem 10]; see also [1, Theorem 14.1].

**Proposition 2.4.** *If  $G$  is a hypercentral group and  $\pi$  is a set of primes, then the following statements are equivalent.*

- (i)  $G$  satisfies  $\mathcal{F}(\pi)$ ;
- (ii)  $G = G^p$  for all  $p \in \pi'$ ;
- (iii)  $G$  is  $\pi'$ -radicable, i.e., each element of  $G$  is a  $p$ th power for all  $p \in \pi'$ ;
- (iv)  $G^{ab}$  satisfies  $\mathcal{F}(\pi)$ .

*Proof.* The implications (i)  $\Rightarrow$  (iv) and (iii)  $\Rightarrow$  (ii) are obvious. To show that (iv)  $\Rightarrow$  (iii) assume that  $G^{ab}$  has  $\mathcal{F}(\pi)$ ; thus  $G^{ab}$  is  $\pi'$ -radicable by Lemma 2.3. Let  $G$  be non-abelian and choose  $z \in Z_2(G) \setminus Z(G)$ . The assignment  $xG' \mapsto [x, z]$  determines a non-trivial homomorphism from  $G^{ab}$  onto  $[G, z] \leq Z(G)$ ; therefore  $[G, z]$  is  $\pi'$ -radicable. Iteration of this argument produces an ascending central series  $\{G_\alpha \mid \alpha < \beta\}$  in  $G$  such that each  $G_{\alpha+1}/G_\alpha$  is  $\pi'$ -radicable.

Let  $g \in G$  and  $p \in \pi'$ . It will be shown that  $g$  is a  $p$ th power, so assume this is false. Now  $g \in G_{\alpha_1+1} \setminus G_{\alpha_1}$  where  $\alpha_1 < \beta$ . Since  $G_{\alpha_1+1}/G_{\alpha_1}$  is  $\pi'$ -radicable, there exist  $g_1 \in G_{\alpha_1+1}$  and  $h_1 \in G_{\alpha_1}$  such that  $g = g_1^p h_1$ . Next  $h_1 \neq 1$ , so  $h_1 \in G_{\alpha_2+1} \setminus G_{\alpha_2}$  where  $\alpha_2 < \alpha_1$ . Write  $h_1 = g_2^p \bar{h}_2$  where  $g_2 \in G_{\alpha_2+1}$  and  $\bar{h}_2 \in G_{\alpha_2}$ . Since  $G_{\alpha_2+1}/G_{\alpha_2} \leq Z(G/G_{\alpha_2})$ , we have  $g_1^p g_2^p \equiv (g_1 g_2)^p \pmod{G_{\alpha_2}}$ . Hence  $g = (g_1 g_2)^p h_2$  where  $h_2 \in G_{\alpha_2}$ . Again  $h_2 \neq 1$ , so that  $h_2 \in G_{\alpha_3+1} \setminus G_{\alpha_3}$  where  $\alpha_3 < \alpha_2$ , and so on. Since this process cannot terminate, it leads to an infinite decreasing sequence of ordinals, which establishes the claim.

It remains to show that (ii)  $\Rightarrow$  (i). Assume that (ii) is valid, yet  $G$  has a finite quotient  $\bar{G}$  with order divisible by some  $p \notin \pi$ . Since  $\bar{G}$  is nilpotent,  $\bar{G} > \bar{G}^p$ , which implies that  $G > G^p$ , in contradiction to (ii).  $\square$

**Corollary 2.5.** *If  $G$  is a hypercentral  $\pi'$ -radicable group, then  $G/Z(G)$  has no non-trivial  $\pi'$ -elements. Also each factor and term of the upper central series of  $G$  is  $\pi'$ -radicable.*

*Proof.* Suppose that  $G/Z(G)$  has a non-trivial  $\pi'$ -element. Then there exists  $x \in Z_2(G) \setminus Z(G)$  such that  $x^p \in Z(G)$  where  $p \in \pi'$ . For any  $g \in G$  we have  $1 = [x^p, g] = [x, g^p]$  because  $[x, g] \in Z(G)$ . Since  $G = G^p$ , it follows that  $x \in Z(G)$ , a contradiction.

Next let  $z \in Z(G)$  and  $p \in \pi'$ ; then  $z = y^p$  for some  $y \in G$ . By the previous paragraph  $y \in Z(G)$  and thus  $Z(G)$  is  $\pi'$ -radicable. This argument establishes the fact that all the upper central factors of  $G$  are  $\pi'$ -radicable, which implies that each  $Z_\alpha(G)$  is  $\pi'$ -radicable.  $\square$

A related property considered in [10, §9] is that of being *finite- $\pi'$ -perfect*, i.e., there are no non-trivial finite  $\pi'$ -quotients. Evidently  $\mathcal{F}(\pi)$  implies finite- $\pi'$ -perfect and the converse holds for hypercentral groups. But the converse fails in general, as the example of  $S_3$  and  $\pi = \{2\}$  shows.

### 3. Abelian groups

In this section we begin the study of abelian groups with the property  $\mathcal{F}(\pi)$ , i.e., those that are  $\pi'$ -divisible. Throughout this section most groups are abelian and are written additively. A basic case is that of torsion-free abelian groups of rank 1. Such groups are isomorphic with subgroups of the additive group of rational numbers  $\mathbb{Q}$  and are often referred to as *rational groups*.

A rational group is determined up to isomorphism by its *type*, which is a set of equivalent *height vectors*. The entries of a height vector correspond to primes and are either non-negative integers or  $\infty$ . For a detailed account of the theory of rational groups, which is due to R. Baer, see [8, §85].

Let  $A$  be a rational group and  $\pi$  a set of primes. It follows immediately from the definitions that  $A$  is  $\pi'$ -divisible if and only if in any height vector belonging to the type of  $A$  there is an entry  $\infty$  for each  $p \in \pi'$ . A fundamental example is the group of  $\pi'$ -adic rationals  $\mathbb{Q}_{\pi'}$ . It is divisible only by the primes in  $\pi'$  and has a height vector with entry  $\infty$  for each prime in  $\pi'$  and all other entries 0.

In principle the  $\pi'$ -divisible groups are determined by the following result.

**Theorem 3.1.** *Let  $A$  be an abelian group and  $\pi$  a set of primes. Then  $A$  is  $\pi'$ -divisible if and only if  $A = D \oplus B$  where  $D$  is a divisible  $\pi'$ -group and  $B$  is an extension of a  $\pi$ -group by a group with an ascending series whose factors are  $\pi'$ -divisible rational groups.*

*Proof.* Assume that  $A$  is  $\pi'$ -divisible and let  $T$  denote the torsion subgroup of  $A$ . Since  $A/T$  is torsion-free,  $T$  is  $\pi'$ -divisible. Therefore its  $\pi'$ -component  $T_{\pi'}$  is divisible and hence is a direct summand of  $A$ . Factor out by  $T_{\pi'}$ , so that  $T$  becomes a  $\pi$ -group. Now factor out by  $T$ , so  $A$  is torsion-free.

Let  $0 \neq a_1 \in A$  and denote the torsion subgroup of  $A/\langle a_1 \rangle$  by  $A_1/\langle a_1 \rangle$ . Thus  $A_1$  is a rational group and, since  $A/A_1$  is torsion-free,  $A_1$  is  $\pi'$ -divisible. Repetition of this argument generates an ascending

series  $\{A_\alpha \mid \alpha < \beta\}$  in  $A$  wherein each factor  $A_{\alpha+1}/A_\alpha$  is a  $\pi'$ -divisible rational group. Consequently  $A$  has the form claimed.

Conversely, if  $A$  has this structure, it has the property  $\mathcal{F}(\pi)$  because a finite quotient of a  $\pi'$ -divisible rational group is a  $\pi$ -group. Hence  $A$  is  $\pi'$ -divisible.  $\square$

**Corollary 3.2.** *If  $\pi$  is a finite set of primes, each factor of the ascending series in Theorem 3.1 is isomorphic with  $\mathbb{Q}_{\sigma'}$  for some  $\sigma \subseteq \pi$ .*

*Proof.* Let  $F$  be a factor of the ascending series and consider a height vector for  $F$ . Suppose that the finite entries occur for a set of primes  $\sigma$ . Since  $F$  is  $\pi'$ -divisible,  $\sigma \subseteq \pi$  and, because  $\pi$  is finite, all the  $\sigma$ -entries in the height vector may be taken as 0. Therefore  $F \simeq \mathbb{Q}_{\sigma'}$ .  $\square$

**The case of a single prime.** The criterion for a torsion-free abelian group to be  $\pi'$ -divisible provided by Theorem 3.1 shows that rational groups are inescapably involved. Moreover, the description of the structure features an ascending series with rational factors, and thus a possibly infinite number of extensions by rational groups. Extensions of even one rational group by another can be numerous and have complex structure – see for example [7]. However, when  $\pi$  consists of a single prime, the structural description in Theorem 3.1 takes a form involving a single extension.

**Theorem 3.3.** *Let  $A$  be an abelian group and  $p$  a prime. Then  $A$  is  $p'$ -divisible if and only if  $A = D \oplus A_0$  where  $D$  is a divisible  $p'$ -group and  $A_0$  has a subgroup  $A_1 = S \oplus P$  such that  $S$  is a free  $\mathbb{Q}_{p'}$ -module,  $P$  is a  $p$ -group and  $A_0/A_1$  is a rational vector space.*

We approach the proof via an auxiliary lemma.

**Lemma 3.4.** *Let  $A$  be a torsion-free abelian group and  $p$  a prime. If  $A$  is  $p'$ -divisible, then it is an extension of a free  $\mathbb{Q}_{p'}$ -module by a rational vector space.*

*Proof.* By Theorem 3.1 and Corollary 3.2 there is an ascending series  $\{A_\alpha \mid \alpha < \beta\}$  of  $A$  such that  $A_{\alpha+1}/A_\alpha \simeq \mathbb{Q}_{p'}$  or  $\mathbb{Q}$ . We aim to show that  $A_\alpha$  has a free  $\mathbb{Q}_{p'}$ -submodule  $F_\alpha$  such that  $A_\alpha/F_\alpha$  is a vector space over  $\mathbb{Q}$ . Assume this true for  $A_\alpha$ . If  $A_{\alpha+1}/A_\alpha \simeq \mathbb{Q}$ , then  $A_{\alpha+1}/F_\alpha$  is a  $\mathbb{Q}$ -space and we define  $F_{\alpha+1}$  to be  $F_\alpha$ .

Suppose that  $A_{\alpha+1}/A_\alpha \simeq \mathbb{Q}_{p'}$ . Since  $A_\alpha/F_\alpha$  is divisible, we can write  $A_{\alpha+1}/F_\alpha = (A_\alpha/F_\alpha) \oplus (F_{\alpha+1}/F_\alpha)$  where  $F_{\alpha+1}/F_\alpha \simeq \mathbb{Q}_{p'}$ . Note that  $F_{\alpha+1}$  is a free  $\mathbb{Q}_{p'}$ -module which has  $F_\alpha$  as a direct summand. Furthermore  $A_{\alpha+1}/F_{\alpha+1} \simeq A_\alpha/F_\alpha$  is a  $\mathbb{Q}$ -space. Consequently, the claim holds for  $A_{\alpha+1}$ .

Now let  $\alpha$  be the least ordinal such that the claim fails for  $A_\alpha$ . By the last two paragraphs  $\alpha$  must be a limit ordinal. For  $\gamma < \alpha$  we have a free  $\mathbb{Q}_{p'}$ -module  $F_\gamma$  which is a direct summand of  $F_{\gamma+1}$ . It follows that  $F_\alpha = \bigcup_{\gamma < \alpha} F_\gamma$  is a free  $\mathbb{Q}_{p'}$ -module since it has a  $\mathbb{Q}_{p'}$ -basis. It remains to prove that  $A_\alpha/F_\alpha$  is a  $\mathbb{Q}$ -space.

First of all we show that  $A_\alpha/F_\alpha$  is torsion-free. Let  $a \in A_\alpha$  and assume that  $ma \in F_\alpha$  for some  $m > 0$ . Then  $a \in A_\gamma$  and  $ma \in F_\gamma$  for some  $\gamma < \alpha$ . Hence  $a \in F_\gamma \leq F_\alpha$  since  $A_\gamma/F_\gamma$  is torsion-free. It follows that  $A_\alpha/F_\alpha$  is torsion-free.

Next  $A_\alpha/F_\alpha$  is divisible. For, if  $a \in A_\alpha$  and  $n > 0$ , we have  $a \in A_\gamma$  where  $\gamma < \alpha$ , and  $a \equiv nb \pmod{F_\gamma}$  for some  $b \in A_\gamma$  since  $A_\gamma/F_\gamma$  is divisible. Hence  $a + F_\alpha = n(b + F_\alpha)$ , so that  $A_\alpha/F_\alpha$  is divisible and thus is a  $\mathbb{Q}$ -space.  $\square$

*Proof of Theorem 3.3.* Assume that  $A$  is  $p'$ -divisible. We may assume that the  $p'$ -component of the maximum divisible subgroup of  $A$  is trivial. By Theorem 3.1 the group  $A$  is an extension of a  $p$ -group  $P$  by a torsion-free group. Applying Lemma 3.4 to  $A/P$  we obtain a free  $\mathbb{Q}_{p'}$ -module  $A_1/P$  such that  $A/A_1$  is a  $\mathbb{Q}$ -space. Notice that  $A_1$  is a  $\mathbb{Q}_{p'}$ -module, so it splits over  $P$ , say as  $A_1 = P \oplus S$  where  $S$  is a free  $\mathbb{Q}_{p'}$ -module-module, as required.  $\square$

**Some examples.** We present next two examples to show the limitations to what can be proved about the structure of abelian groups with a property  $\mathcal{F}(\pi)$ . The first one is undoubtedly well known.

(i) *There is a torsion-free abelian  $p'$ -divisible group of rank 2 which is directly indecomposable.*

Let  $p$  be any prime; it is routine to show that  $\text{Ext}(\mathbb{Q}, \mathbb{Q}_{p'}) \neq 0$ . (This also follows from [7, Theorem 2.2]). Hence there is a non-split extension  $A$  of  $\mathbb{Q}_{p'}$  by  $\mathbb{Q}$ , let us say with a subgroup  $B \simeq \mathbb{Q}_{p'}$  such that  $A/B \simeq \mathbb{Q}$ . Clearly  $A$  is  $p'$ -divisible. Suppose that  $A = C \oplus D$  where  $C$  and  $D$  both have rank 1. Notice that these subgroups are  $p'$ -divisible and hence are isomorphic with  $\mathbb{Q}_{p'}$  or  $\mathbb{Q}$ . If  $B \cap C \neq 0$ , then, since  $A/C$  is torsion-free,  $B \leq C$  and  $B = C$ . But then  $A$  splits over  $B$ . Therefore  $B \cap C = 0$  and for the same reason  $B \cap D = 0$ .

If  $C \simeq \mathbb{Q}$ , then  $A = B \oplus C$ , so it follows that  $C \simeq \mathbb{Q}_{p'} \simeq D$ . But then  $A$  fails to have  $\mathbb{Q}$  as a quotient. Therefore  $A$  is indecomposable.

(ii) The second example demonstrates that splitting over the torsion subgroup may fail in the case of two primes. Let  $\pi = \{p, q\}$  where  $p$  and  $q$  are distinct primes, and write

$$T = \bigoplus_{i=1,2,\dots} \mathbb{Z}_{q^{2i}}.$$

We claim that  $\text{Ext}(\mathbb{Q}_{p'}, T) \neq 0$ . Since  $\mathbb{Q}_q \leq \mathbb{Q}_{p'}$ , there is an exact sequence

$$\text{Ext}(\mathbb{Q}_{p'}, T) \rightarrow \text{Ext}(\mathbb{Q}_q, T) \rightarrow 0.$$

Now there is a non-split extension of  $T$  by  $\mathbb{Q}_q$  which can be realized as a subgroup of the complete direct sum  $\prod_{i=1,2,\dots} \mathbb{Z}_{q^{2i}}$  containing  $T$ : for this see [8, §100, Example 3]. Thus  $\text{Ext}(\mathbb{Q}_q, T) \neq 0$  and hence  $\text{Ext}(\mathbb{Q}_{p'}, T) \neq 0$ . Consequently there is a non-split extension  $T \twoheadrightarrow A \twoheadrightarrow \mathbb{Q}_{p'}$ . Each finite quotient of  $A$  is a  $\pi$ -group, so  $A$  is  $\pi'$ -divisible.

Finally in this section, we extend our study to FC-groups, i.e., groups in which every element has finitely many conjugates. The following result reduces the description of FC-groups with  $\mathcal{F}(\pi)$  to the abelian case, for which Theorem 3.1 is available.

**Theorem 3.5.** *Let  $G$  be an FC-group and  $\pi$  a set of primes. Then  $G$  has the property  $\mathcal{F}(\pi)$  if and only if  $Z(G)$  is  $\pi'$ -radicable and  $G/Z(G)$  is a  $\pi$ -group.*

*Proof.* Assume that  $G$  has  $\mathcal{F}(\pi)$  and put  $Z = Z(G)$ . If  $x \in G$ , then  $|G : C_G(x)|$  is finite and  $G/C(x)$  is a finite  $\pi$ -group where  $C(x)$  denotes the core of  $C_G(x)$  in  $G$ . Since  $Z = \bigcap_{x \in G} C(x)$ , we deduce that  $G/Z$  is residually a finite  $\pi$ -group. In addition a well known result of Baer, (see [10, Theorem 4.32]), asserts that  $G/Z$  is locally finite. Hence  $G/Z$  is a  $\pi$ -group.

Suppose that  $Z$  is not  $\pi'$ -radicable, i.e., it does not have  $\mathcal{F}(\pi)$ . Then there must exist a non-trivial quotient  $Z/Y$  which is a  $p$ -group where  $p \in \pi'$ . Write  $\bar{G} = G/Y$  and  $\bar{Z} = Z/Y$ . Since  $\bar{G}/\bar{Z}$  is a locally finite  $\pi$ -group,  $\bar{G}'$  has this property too by a theorem of I. Schur, (see [10, Theorem 4.12]). It follows that the  $\pi$ -elements of  $\bar{G}$  form a subgroup  $\bar{P}$ . As a consequence  $\bar{G} = \bar{P} \times \bar{Z}$ . However, this means that  $G$  has a quotient isomorphic with  $\bar{Z}$ , a  $p$ -group. By this contradiction  $Z$  is  $\pi'$ -radicable. The converse is clearly true.  $\square$

#### 4. Soluble groups with finite abelian ranks

We cannot hope to give a precise description of the structure of arbitrary soluble groups with a property  $\mathcal{F}(\pi)$ . This is indicated by the following observation: *every soluble group can be embedded in a soluble group of the same derived length which has no proper subgroups of finite index.* The proof is a routine exercise using wreath products.

On the other hand, imposition of a condition  $\mathcal{F}(\pi)$  can have a marked effect in the right circumstances. For example, there is the following easy result.

**Proposition 4.1.** *If  $\pi$  is a proper set of primes, a finitely generated soluble group satisfying  $\mathcal{F}(\pi)$  is finite.*

*Proof.* Let  $G$  be finitely generated soluble group with  $\mathcal{F}(\pi)$ . If  $G$  is abelian, it cannot have an infinite cyclic quotient group since  $\pi$  is a proper set of primes; thus  $G$  must be finite. Next let  $G$  have derived length  $d > 1$  and set  $A = G^{(d-1)}$ . Induction on  $d$  shows that  $G/A$  is finite and thus  $A$  is finitely generated. By Lemma 2.2 the subgroup  $A$  also has  $\mathcal{F}(\pi)$ . Therefore  $A$ , and hence  $G$ , is finite.  $\square$

It is a more challenging problem to determine the structure of soluble groups with finite abelian ranks (soluble FAR-groups) which satisfy a condition  $\mathcal{F}(\pi)$ .

**Theorem 4.2.** *Let  $G$  be a soluble group with finite abelian ranks and let  $\pi$  be a proper subset of the primes. If  $G$  satisfies  $\mathcal{F}(\pi)$ , there are normal subgroups  $L \geq M \geq N$  of  $G$  such that  $G/L$  is a finite  $\pi$ -group,  $L/M$  is a torsion-free,  $\pi'$ -radicable nilpotent group,  $M/N$  is a residually finite  $\pi$ -group and  $N$  is a periodic radicable abelian group. Conversely, any group with this structure satisfies  $\mathcal{F}(\pi)$ .*

*Proof.* Assume that  $G$  has  $\mathcal{F}(\pi)$  and let  $M$  denote the maximum periodic normal subgroup of  $G$ . By a theorem of A.I. Mal'cev, (see [9, 5.2.1]),  $G/M$  has a normal series of finite length in which every infinite factor is torsion-free abelian of finite rank. Let  $N$  denote the finite residual of  $M$ . Then  $N$  is a periodic radicable nilpotent group by [9, 5.3.1], so it is abelian. Factor out by  $N$ , so that  $M$  is residually finite.



Consider a finite quotient  $M/P$  of order  $m$ ; then  $M/M^m$  is finite and  $M^m \leq P$ . Thus, replacing  $P$  by  $M^m$ , we can assume that  $P \triangleleft G$ . Evidently  $G/P$  has a series of finite length whose infinite factors are torsion-free abelian. It follows that  $G/P$  possesses a torsion-free normal subgroup of finite index – see [9, 5.2.5]. Therefore  $M/P$  is isomorphic with a subgroup of some finite quotient of  $G$ . Since  $G$  has  $\mathcal{F}(\pi)$ , we conclude that  $M/P$  is a  $\pi$ -group, and because  $M$  is residually finite and periodic, it follows that  $M$  must be a  $\pi$ -group. At this point we factor out by  $M$ . As a consequence  $G$  has a normal series of finite length whose infinite factors are torsion-free abelian of finite rank.

By [9, 5.2.5, 5.2.4] the group  $G$  has a normal subgroup  $L$  of finite index which has a  $G$ -invariant series of finite length in which every factor is a torsion-free abelian group of finite rank, say  $1 = L_0 \triangleleft L_1 \triangleleft \dots \triangleleft L_{k-1} \triangleleft L_k = L$ .

Since  $G/L$  is finite,  $L$  satisfies  $\mathcal{F}(\pi)$ . We claim that each  $L_{i+1}/L_i$  is  $\pi'$ -radicable. Indeed, if  $m > 0$  is a  $\pi'$ -number,  $L_1/L_1^m$  is finite and hence is isomorphic with a subgroup of a finite quotient of  $L$  by an argument used above. Hence  $L_1/L_1^m$  is a  $\pi$ -group and  $L_1 = L_1^m$ , showing that  $L_1$  is  $\pi'$ -radicable. The claim follows by induction on  $k$ .

By Theorem 3.1 each  $L_{i+1}/L_i$  has a series of finite length whose factors are  $\pi'$ -radicable rational groups. Thus we can refine the series of  $L_i$ 's to a series of finite length in which each factor is a  $\pi'$ -radicable rational group. Recall that  $\pi$  is a *proper* set of primes, so that no factor of the refined series can be infinite cyclic. We are therefore in a position to apply a theorem of Čarin [3] – see also [9, 5.2.10] – to show that  $L$  is nilpotent-by-finite. Replacing  $L$  by a suitable power, we may assume that  $L$  is nilpotent. Note that  $G/L$  is a finite  $\pi$ -group and  $L$  is torsion-free and  $\pi'$ -radicable. Thus  $G$  has the required structure. Conversely, it is clear that a finite quotient of a group with this structure is a  $\pi$ -group. □

When the set of primes is finite, Theorem 4.2 takes a simpler form.

**Corollary 4.3.** *Let  $G$  be a soluble group with finite abelian ranks and let  $\pi$  be a finite set of primes. If  $G$  satisfies  $\mathcal{F}(\pi)$ , then it has normal subgroups  $H \geq N$  such that  $G/H$  is a finite  $\pi$ -group,  $H/N$  is a torsion-free  $\pi'$ -radicable nilpotent group and  $N$  is a periodic radicable abelian group.*

*Proof.* We employ the notation of Theorem 4.2 with the normal subgroups  $L, M, N$ . Thus  $M/N$  is a residually finite  $\pi$ -group; moreover, since  $\pi$  is finite,  $M/N$  is actually finite by [6, 3.2.3]. Hence  $L/N$  has a  $G$ -invariant series whose infinite factors are torsion-free abelian, which shows that there is a torsion-free subgroup  $H/N$  with finite index in  $L/N$  and  $H \triangleleft G$ . Clearly  $H/N$  is nilpotent. As  $G/H$  is finite,  $H/N$  has  $\mathcal{F}(\pi)$ , i.e., it is  $\pi'$ -radicable, while  $N$  is periodic radicable abelian. □

**Example.** Corollary 4.3 is not valid when  $\pi$  is a proper infinite set of primes. To see this let  $p$  be an arbitrary prime and write  $\pi = p' = \{q_1, q_2, \dots\}$ . We first construct infinitely many distinct primes  $p_1, p_2, \dots$  in  $\pi$  such that  $q_i \mid p_i - 1$ . Assume this has been done for  $j \leq i$ . By Dirichlet's theorem

there are infinitely many primes  $r$  such that  $r \equiv 1 \pmod{q_{i+1}}$ . Choose  $p_{i+1}$  to be such a prime  $r$  that is different from  $p, p_1, p_2, \dots, p_i$ .

Let  $A_i = \mathbb{Z}_{p_i}$  and  $Q = \mathbb{Q}_{\pi'} = \mathbb{Q}_p$ . There are natural maps  $Q \rightarrow \mathbb{Z}_{q_i} \hookrightarrow \text{Aut}(A_i)$  that allow us to make  $A_i$  into a  $Q$ -module. Write  $A = A_1 \oplus A_2 \oplus \dots$ , which is also a  $Q$ -module. Now set  $G = Q \ltimes A$ . This is a metabelian FAR-group with finite Prüfer rank <sup>2</sup> which has the property  $\mathcal{F}(\pi) = \mathcal{F}(p')$ . It is easy to see that  $G$  has no non-trivial radicable abelian subgroups. Also, since  $C_G(A) = A$ , there are no non-trivial torsion-free normal subgroups. Thus Corollary 4.3 fails for  $G$ .

The final result in this section pertains to locally soluble groups of finite Prüfer rank with a property  $\mathcal{F}(\pi)$ .

**Theorem 4.4.** *Let  $G$  be a locally soluble group of finite Prüfer rank and let  $\pi$  be a proper set of primes. If  $G$  satisfies  $\mathcal{F}(\pi)$ , then  $G$  has the structure indicated in Theorem 4.2.*

To prove this theorem we will need an auxiliary result.

**Proposition 4.5.** *Let  $G$  be a locally soluble group of finite Prüfer rank. Then  $G$  has a periodic locally nilpotent normal subgroup  $T$  such that  $G/T$  is soluble and has finite Sylow subgroups.*

*Proof.* Let  $G$  have Prüfer rank  $r$  and consider a finitely generated subgroup  $X$  of  $G$ . Then  $X$  is a soluble FAR-group; therefore it has an ascending normal series whose factors are elementary abelian  $p$ -groups, radicable abelian  $p$ -groups or torsion-free abelian groups, in each case with rank at most  $r$ . By a well known theorem of H. Zassenhaus on soluble linear groups, (see [9, 3.1.10]), there exists  $m = m(r) > 0$  such that  $X^{(m)}$  centralizes every factor of the series. Consequently  $X^{(m)}$  is a hypercentral group, and hence is locally nilpotent. Therefore  $G^{(m)}$  is locally nilpotent.

Let  $T$  denote the torsion subgroup of the Hirsch-Plotkin radical  $\text{HP}(G)$ . Now  $G^{(m)} \leq \text{HP}(G)$  and furthermore  $\text{HP}(G)/T$  is nilpotent. The reason is that, as a torsion-free locally nilpotent group of finite rank, it is nilpotent by a theorem of Mal'cev, (see [10, 6.36]). Therefore  $G/T$  is soluble and in consequence its finite residual  $R/T$  is a radicable nilpotent group (see [9, 5.3.1]). Let  $S/T$  be the torsion subgroup of  $R/T$ . Then  $S \triangleleft G$  and  $S$  is periodic, while  $\text{HP}(S) = \text{HP}(G) \cap S = T$  since  $\text{HP}(G)/T$  is torsion-free. Thus we can apply a result of Kargapolov (see [6, 3.2.3]) to show that the Sylow subgroups of  $S/T$  are finite. But  $S/T$  is radicable, whence it follows that  $S = T$ . Finally,  $p$ -subgroups of  $G/R$  are finite, so the Sylow subgroups of  $G/T$  are finite.  $\square$

*Proof of Theorem 4.4.* By Proposition 4.5 there exists  $T \triangleleft G$  such that  $T$  is periodic and locally nilpotent, while  $G/T$  is soluble with finite Sylow subgroups. Applying Theorem 4.2 to the group  $G/T$ , we obtain normal subgroups  $L \geq M \geq N_0$  such that  $G/L$  is a finite  $\pi$ -group,  $L/M$  is torsion-free  $\pi'$ -radicable nilpotent,  $M/N_0$  is residually finite- $\pi$  and  $N_0/T$  is a periodic radicable abelian group. Since the Sylow subgroups of  $G/T$  are finite,  $N_0 = T$  and thus  $M/T$  is residually finite- $\pi$ .

<sup>2</sup>A group has finite Prüfer rank  $r$  if every finitely generated subgroup can be generated by  $r$  elements and  $r \geq 0$  is the least such integer.

Factor out by  $N$ , the maximum radicable subgroup of  $T$ . Since  $T$  is now a direct product of finite  $p$ -groups for various primes  $p$ , the subgroup  $M$  has finite Sylow subgroups. We claim that  $M$  is residually finite. Indeed,  $M/T_{p'}$  is periodic soluble with finite Sylow subgroups, since  $T_p$  is soluble. It follows that  $M/T_{p'}$  is a residually finite, soluble FAR-group since it cannot have subgroups of Prüfer type. Thus  $M$  is residually finite because  $\bigcap_p T_{p'} = 1$ .

Finally, if  $U \triangleleft G$  and  $M/U$  is finite, then, since  $G/U$  is virtually torsion-free, the usual argument applies to show that  $M/U$  is a  $\pi$ -group. Therefore  $M$  is a  $\pi$ -group and the desired result is proven.  $\square$

It is worth noting that the groups characterized in Theorem 4.4 need not be soluble. Indeed, for every prime  $p$  there exists a finite  $p$ -group  $G(p)$  of boundedly finite Prüfer rank and unbounded derived length. Let  $q$  be a fixed prime. The direct product of the  $G(p)$  taken over all  $p \neq q$  is a locally soluble with finite Prüfer rank and has the property  $\mathcal{F}(q')$ . But it is not soluble. For details see [10, vol. 2, p.179].

### 5. Torsion-free nilpotent groups of finite rank

As Theorem 4.2 shows, a principal component of a soluble FAR-group with  $\mathcal{F}(\pi)$  is a torsion-free nilpotent group of finite rank that is  $\pi'$ -radicable. Thus such groups merit attention, and in fact some information about them is already given in Proposition 2.4 and Corollary 2.5. In particular, if  $G$  is a torsion-free nilpotent,  $\pi'$ -radicable group, each upper central factor is also torsion-free and  $\pi'$ -radicable, so its structure is given by Theorem 3.1. If in addition the rank is finite, there is a central series of finite length in which each factor is a  $\pi'$ -radicable rational group.

As usual the most precise information available is when the set  $\pi$  consists of a single prime. In this case each factor in the series is isomorphic with  $\mathbb{Q}_{p'}$  or  $\mathbb{Q}$ . One might hope that the  $\mathbb{Q}$ -factors could be moved up the series leaving as lower factors those of type  $\mathbb{Q}_{p'}$ . This is certainly the case if the group is abelian by Theorem 3.3.

However, this is not possible in general. For consider a non-split central extension of the form  $\mathbb{Q} \rightarrow G \rightarrow \mathbb{Q}_{p'} \oplus \mathbb{Q}_{p'}$ . Here  $G' \leq Z(G) \simeq \mathbb{Q}$  and  $G' \simeq \mathbb{Q}_{p'}$ . Hence  $Z(G)/G'$  is a  $p^\infty$ -group, which shows that  $G$  cannot have  $\mathbb{Q}$  as a quotient. Thus the  $\mathbb{Q}$ -factor cannot be moved to the top of the central series.

Despite this example it turns out that  $\mathbb{Q}$ -factors can be eliminated entirely from the series at the expense of introducing  $p^\infty$ -factors at the top.

**Theorem 5.1.** *Let  $G$  be a torsion-free nilpotent group of finite rank  $r > 0$  and let  $p$  be a prime. Assume that  $G$  is  $p'$ -radicable. Then the following statements hold.*

- (i) *There is a series  $H = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_\ell = G$  such that  $H$  has a central series of length  $r$  with exclusively  $\mathbb{Q}_{p'}$ -factors, while each factor  $H_{i+1}/H_i$  is a  $p^\infty$ -group. Moreover, the integer  $\ell$  is an invariant of  $G$ .*
- (ii) *There are subgroups  $U_0, U_1, \dots, U_{r-1}$ , each isomorphic with  $\mathbb{Q}_{p'}$ , such that  $H = U_{r-1} \cdots U_1 U_0$ .*

During the proof we will have occasion to use the following elementary result: in this a group is said to be  $p^\infty$ -free if it has no sections of type  $p^\infty$ .

**Lemma 5.2.** *Let  $G$  be a nilpotent group and  $p$  a prime. Assume that  $G$  is generated by finitely many homomorphic images of  $\mathbb{Q}_{p'}$ . Then  $G$  is  $p^\infty$ -free.*

*Proof.* Let  $G = \langle H_i \mid i = 1, 2, \dots, n \rangle$  where  $H_i$  is an image of  $\mathbb{Q}_{p'}$ : note that that  $H_i$  is a finite  $p$ -group, a radicable  $p'$ -group or a group of type  $\mathbb{Q}_{p'}$ . Assume first that  $G$  is abelian and let  $H/K$  be a section of  $G$  of type  $p^\infty$ . Then  $H/K$  is a direct factor of  $G/K$  and hence  $G$  has a quotient of type  $p^\infty$ , say  $G/N$ . But each  $H_iN/N$  must be a finite  $p$ -group, leading to the contradiction that  $G/N$  is finite.

Now assume that  $G$  has nilpotent class  $c > 1$ . By induction on  $c$  the group  $G/\gamma_c(G)$  is  $p^\infty$ -free. Since  $G^{ab}$  is generated by images of  $\mathbb{Q}_{p'}$ , it is  $p'$ -radicable, as well as  $p^\infty$ -free. By Theorem 3.3 the group  $G^{ab}$  is a direct product of finite  $p$ -groups, radicable  $p'$ -groups and rational groups of type  $\mathbb{Q}_{p'}$ . Next there is a surjective homomorphism from  $\underbrace{G^{ab} \otimes \dots \otimes G^{ab}}_c$  to  $\gamma_c(G)$ , which shows that  $\gamma_c(G)$  is the direct product of a finite  $p$ -group and a free  $\mathbb{Q}_{p'}$ -module. It follows that  $\gamma_c(G)$  is  $p^\infty$ -free, as is  $G$ .  $\square$

*Proof of Theorem 5.1.* By refining the upper central series of  $G$  we obtain a central series  $1 = G_0 < G_1 < \dots < G_r = G$  such that each  $G_{i+1}/G_i$  is isomorphic with either  $\mathbb{Q}_{p'}$  or  $\mathbb{Q}$ . By Lemma 2.2 each  $G_i$  is  $p'$ -radicable.

Choose  $x_i \in G_{i+1} \setminus G_i$ . Enumerate the  $p'$ -numbers  $> 1$  as  $\ell_1, \ell_2, \dots$ . Since there is (unique) extraction of  $q$ th roots in  $G_{i+1}$  for  $q \neq p$ , there are elements  $y_{ij} \in G_{i+1}$  satisfying  $y_{i0} = x_i$  and  $y_{ij} = y_{ij+1}^{\ell_j}$  for  $j = 0, 1, \dots$ . Put  $Y_i = \langle y_{ij} \mid j = 0, 1, 2, \dots \rangle$ . Then  $Y_i$  is a rational group since it is the union of a chain of infinite cyclic groups. In addition  $Y_i/\langle x_i \rangle$  is a  $p'$ -group, from which it follows that  $Y_i \simeq \mathbb{Q}_{p'}$ . Furthermore  $Y_i \cap G_i = 1$  since  $G_{i+1}/G_i$  is torsion-free.

Suppose that  $G_{i+1}/G_i \simeq \mathbb{Q}_{p'}$ , which implies that  $|G_{i+1} : Y_i G_i|$  is a power of  $p$ . Assuming that  $Y_i G_i \neq G_{i+1}$ , choose  $u \in G_{i+1} \setminus Y_i G_i$ . Therefore  $\langle u, x_i \rangle \equiv \langle \bar{x}_i \rangle \pmod{G_i}$  for some  $\bar{x}_i \in G_{i+1} \setminus Y_i G_i$ . Apply the procedure for extracting  $p'$ -roots used above to  $\bar{x}_i$  and form the corresponding subgroup  $\bar{Y}_i$ . Then we have  $Y_i \leq \bar{Y}_i G_i$  and hence  $Y_i G_i < \bar{Y}_i G_i \leq G_{i+1}$ . Now replace  $x_i$  by  $\bar{x}_i$ . Continuing in this manner, we will eventually find an element  $x_i$  for which  $G_{i+1} = Y_i G_i$  where  $Y_i \simeq \mathbb{Q}_{p'}$ . If, on the other hand,  $G_{i+1}/G_i \simeq \mathbb{Q}$ , then  $G_{i+1}/Y_i G_i \simeq \mathbb{Q}/\mathbb{Q}_{p'} \simeq p^\infty$ .

Define

$$H = \langle Y_0, Y_1, \dots, Y_{r-1} \rangle$$

and write  $H_i = H G_i$ . Then either  $H_i = H_{i+1}$  or  $H_{i+1}/H_i \simeq p^\infty$ . Therefore we have a series  $H = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_\ell = G$  such that  $\ell \leq r$  and  $H_{i+1}/H_i \simeq p^\infty$  after some relabelling of the  $H_i$ .

Write  $V_i = H \cap G_i$ , noting that  $V_i < V_{i+1}$  since  $H_{i+1}/H_i$  is of type  $p^\infty$ . Thus  $1 = V_0 < V_1 < \dots < V_r = H$  is a central series of  $H$ . Since  $G$  is  $p'$ -radicable and each  $H_{i+1}/H_i$  is a  $p$ -group,  $H$  is  $p'$ -radicable, as is  $V_{i+1}/V_i$  because  $H/V_{i+1}$  is torsion-free. Next

$$V_{i+1}/V_i \simeq V_{i+1} G_i / G_i \leq G_{i+1} / G_i \simeq \mathbb{Q}_{p'} \text{ or } \mathbb{Q}.$$

However,  $H$  is  $p^\infty$ -free by Lemma 5.2, so in fact  $V_{i+1}/V_i \simeq \mathbb{Q}_{p'}$ . Notice that  $r$  is the rank of  $G$ , and that by the Schreier Refinement Theorem the number  $\ell$  of  $p^\infty$ -factors is an invariant of  $G$ . Thus (i) is established.

At this point we can use the argument of the first part of the proof to show that there are subgroups  $U_i \simeq \mathbb{Q}_{p'}$  for  $i = 0, 1, \dots, r - 1$  such that  $V_{i+1} = U_i \rtimes V_i$ . Hence  $H = U_{r-1} \cdots U_1 U_0$  and the proof is complete.  $\square$

**Remark.** One might hope that in Theorem 5.1 the subgroup  $H$  can be chosen to be normal in  $G$ , so that  $G/H$  would be a radicable abelian  $p$ -group. However, this is not the case.

To see this let  $p$  be an arbitrary prime and observe that there is a non-split central extension  $\mathbb{Q} \rightarrow G \rightarrow \mathbb{Q} \oplus \mathbb{Q}_{p'}$ . Then  $G$  has subgroups  $K \simeq \mathbb{Q}$  and  $L \simeq \mathbb{Q}_{p'}$  such that  $G = \langle K, L \rangle$  and hence  $G' = [K, L] = Z(G) \simeq \mathbb{Q}$ . Suppose there exists  $N \triangleleft G$  such that  $G/N$  is a radicable abelian  $p$ -group and  $N$  has rank 2; thus  $N \simeq \mathbb{Q}_{p'} \oplus \mathbb{Q}_{p'}$ . Then  $G' \leq N$ , which is impossible.

### 6. Locally finite groups

It is a consequence of Theorem 4.2 that if  $\pi$  is a proper set of primes, a periodic soluble FAR-group with  $\mathcal{F}(\pi)$  is an extension of a radicable abelian  $\pi'$ -group by a  $\pi$ -group. In fact this result can be extended to much wider classes of locally finite groups. Our principal conclusions are as follows.

**Theorem 6.1.** *Let  $G$  be a locally (finite soluble) group and denote by  $\sigma(G)$  the set of primes  $p$  for which  $G$  does not satisfy min- $p$ . Let  $\pi$  be a set of primes that contains  $\sigma(G)$ . Then  $G$  satisfies  $\mathcal{F}(\pi)$  if and only if there are normal subgroups  $L \geq M$  such that  $G/L$  is a  $\pi$ -group,  $L/M$  is a radicable abelian  $\pi'$ -group and  $M$  is a  $\sigma(G)$ -group.*

*Proof.* Write  $\sigma = \sigma(G)$  and assume that  $G$  has  $\mathcal{F}(\pi)$ . Let  $R$  denote the finite residual of  $G$ ; thus  $G/R$  is a  $\pi$ -group. Form a chief series of general order type in  $G$ . A theorem of Mal'cev asserts that a chief factor of a locally soluble group is abelian (see [10, 5.27]). Hence each factor  $F$  of the chief series in  $G$  is an elementary abelian  $p$ -group for some prime  $p$ . Furthermore, if  $p \in \sigma'$ , then  $F$  is finite since  $G$  satisfies min- $p$ . Hence  $G/C_G(F)$  is finite and  $R \leq C_G(F)$ . Consequently, by intersecting the terms of the chief series with  $R$ , we obtain a  $G$ -invariant series in  $R$  in which each factor is an elementary abelian  $p$ -group for some  $p$  and if  $p \in \sigma'$ , the factor is central in  $R$ .

Let  $H$  be a finite subgroup of  $R$ . Then  $H$  has a normal series in which each factor is an elementary abelian  $p$ -group and if  $p \in \sigma'$ , the factor is central in  $H$ . This is well known to imply that  $H/O_\sigma(H)$  is a nilpotent  $\sigma'$ -group. Hence, writing  $M = O_\sigma(R)$ , we deduce that  $R/M$  is a locally nilpotent  $\sigma'$ -group. Also  $R/M$  satisfies min- $p$  for all  $p$ , so its primary components are Černikov groups.

Let  $p \in \pi'$ , so that  $p \in \sigma'$ ; we claim that the  $p$ -component of  $R/M$  is radicable and abelian. If this false, there exists  $S \triangleleft G$  such that  $M \leq S < R$  and  $R/S$  is a finite abelian  $p$ -group. Set  $C = C_G(R/S)$ ; then  $G/C$  is finite, so  $C$  satisfies  $\mathcal{F}(\pi)$ . Next  $R/S \leq Z(C/S)$  and  $C/R$  is a  $\pi$ -group, while  $R/S$  is a  $\pi'$ -group. It follows via a well known theorem of Schur that the  $\pi$ -elements of  $C/S$  form a subgroup

$D/S$  where  $D \triangleleft G$  and  $C/D$  is a finite  $p$ -group. As  $p \notin \pi$  and  $C$  has  $\mathcal{F}(\pi)$ , we obtain that  $D = C$  and  $C/S$  is a  $\pi$ -group. This leads to the contradiction  $R = S$ . Therefore the  $p$ -component of  $R/M$  is a radicable abelian group for all  $p \in \pi'$ .

Next let  $L/M$  denote the  $\pi'$ -component of  $R/M$ . Then  $R/L$ , and hence  $G/L$ , is a  $\pi$ -group, while  $L/M$  is a radicable abelian  $\pi'$ -group.  $\square$

A case of particular interest is when  $\sigma(G)$  is empty, so the minimal condition holds for all primes. In this situation the hypothesis of local solubility is unnecessary.

**Corollary 6.2.** *Let  $G$  be a locally finite group satisfying min- $p$  for all primes  $p$  and let  $\pi$  be any set of primes. Then  $G$  has the property  $\mathcal{F}(\pi)$  if and only if it is an extension of a radicable abelian  $\pi'$ -group by a  $\pi$ -group.*

*Proof.* Assume that  $G$  has  $\mathcal{F}(\pi)$ . By a well known theorem of Belyaev [2], (see also [6, 3.5.15]), the group  $G$  has a locally soluble normal subgroup of finite index. Clearly we can assume  $G$  to be locally soluble. Now apply Theorem 6.1 with  $\sigma(G)$  the empty set and the result follows.  $\square$

**Corollary 6.3.** *If in addition  $\pi$  is finite, then  $G$  satisfies  $\mathcal{F}(\pi)$  if and only if it is an extension of a radicable abelian  $\pi'$ -group by a finite  $\pi$ -group.*

*Proof.* With the notation of the proof of Theorem 6.1,  $\bar{G} = G/R$  is a  $\pi$ -group and it has finite Sylow subgroups since it is residually finite. By [6, 2.5.13] the group  $\bar{G}/O_{p'}(\bar{G})$  is finite for each  $p \in \pi$ . Since  $\pi$  is finite, it follows that  $\bar{G}$  is finite.  $\square$

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