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INFLUENCE OF COMPLEMENTED SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

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Abstract. P. Hall proved that a finite group G is supersoluble with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G. He called such groups complemented. A. Ballester-Bolinches and X. Guo established the structure of minimal non-complemented groups. We give the classification of finite non-soluble groups all of whose second maximal subgroups are complemented groups. We also prove that every finite group with less than 21 non-complemented non-minimal $\{2,3,5\}$-subgroups is soluble.

1. Introduction and notation

All groups considered in this paper are finite. We use conventional notions and notations, as in [9, 16]. Let π be a set of prime numbers. Denote by π' the complement to π in the set of all prime numbers. Throughout this article G stands for a finite group. A subgroup H of G is said to be *complemented* in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Such a subgroup K of G is called a *complement* to H in G. A number of authors have examined the structure of G under the assumption that certain subgroups are complemented in G. For example P. Hall proved that a group G is soluble if and only if every Sylow subgroup is complemented in G. In [8] the same author also proved that a group G is supersoluble with elementary abelian Sylow subgroups if

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and only if every subgroup of G is complemented in G. He called such groups *complemented*. The class of all complemented groups is closed under taking subgroups and under taking epimorphic images.

In [4] A. Ballester-Bolinches and X. Guo proved that the class of all complemented groups is just the class of all groups G for which every minimal subgroup is complemented in G. In [7] the same authors and K. P. Shum denoted the family of complemented groups by $\mathscr{BNS}$ and they established the structure of minimal non-$\mathscr{BNS}$-groups (minimal non-complemented groups). In this paper we give the classification of non-soluble groups all of whose second maximal subgroups are complemented groups.

By [4, Lemma 3] if all maximal subgroups of any Sylow 2-subgroup of a group G of even order are complemented in G, then G is 2-nilpotent and by Feit-Thompson's Odd Order Theorem G is soluble. By [13, Theorem] if all non-minimal subgroups of G are complemented in G, then G is soluble. We will show that a group with less than 21 non-complemented non-minimal $\{2, 3, 5\}$-subgroups is soluble.

2. Preliminaries

Here are some elementary properties of complemented subgroups.

Lemma 2.1. [4, Lemma 1] *Let $N \lhd G$. Then the following statements hold:*

(1) *If $H \leqslant K \leqslant G$ and H is complemented in G, then H is complemented in K.*

(2) *If $N \leqslant H$ and H is complemented in G, then H/N is complemented in G/N.*

(3) *Let π be a set of primes. If N is a π'-subgroup and A is a π-subgroup of G, then A is complemented in G if and only if AN/N is complemented in G/N.*

The next lemmas will be needed in Section 3.

Lemma 2.2. [7, Lemma 2.3] *and* [4, Corollary 2] *Every minimal subgroup of G is complemented in G if and only if G is a supersoluble group with elementary abelian Sylow subgroups.*

Lemma 2.3. *Let p be the smallest prime divisor of the order of a group G. Every subgroup of G of order p is complemented in G if and only if G is a p-nilpotent group with elementary abelian Sylow p-subgroups.*

Proof. This follows from [11, Theorem 1.2] and Lemmas 2.1–2.2. □

Lemma 2.4. [12, Theorem B] *Let G be a non-soluble group. Asssume that soluble subgroups of G are either 2-nilpotent or minimal non-nilpotent. Then G is one of the following groups:*

(1) *$PSL(2, 2^f)$, where f is a positive integer such that $2^f - 1$ is a prime;*

(2) *$PSL(2, q)$, where q is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod 8$;*

(3) *$SL(2, q)$, where q is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod 8$.*

The last lemma will be needed in Sections 3–4.

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Lemma 2.5. [3, Lemma 2.2] *Let G be a non-soluble group whose maximal subgroups are soluble. Then $G/\Phi(G)$ is a minimal simple group.*

3. Non-soluble groups all of whose second maximal subgroups are complemented groups

We write $\mathscr{BNS}_2$ for the family of groups G in which each subgroup of order 2 (if there exists) is complemented in G. By Lemma 2.3 every $\mathscr{BNS}_2$-group of even order is 2-nilpotent with elementary abelian Sylow 2-subgroups. The class of all $\mathscr{BNS}_2$-groups is closed under taking subgroups and under taking epimorphic images.

Theorem 3.1. *If all maximal subgroups of a group G are $\mathscr{BNS}_2$-groups, then G is either 2-nilpotent or minimal non-nilpotent. In particular, G is soluble.*

Proof. By Lemma 2.3 every maximal subgroup of G is 2-nilpotent. Then G is either 2-nilpotent or minimal non-2-nilpotent (in fact, non-nilpotent by [9, IV.5.4]). In particular by [9, IV.5.4], the definition of 2-nilpotent groups and Feit-Thompson Theorem on solubility of groups of odd order it follows that G is soluble. □

Theorem 3.2. *A group G is a minimal non-$\mathscr{BNS}_2$-group if and only if G is one of the following groups:*

(1) $G = \langle a \mid a^4 = 1\rangle$;

(2) *$G = P \ltimes Q$ is a minimal non-abelian group of order $2^\alpha p^\beta$, where p is an odd prime, $\alpha > 1$, $\beta \geqslant 1$ and Q is a normal elementary abelian Sylow 2-subgroup of G and P is a cyclic Sylow p-subgroup.*

Proof. Clearly, the group of type (1) is a minimal non-$\mathscr{BNS}_2$-group. Assume that G is a group of type (2), then G is a minimal non-abelian group with a normal elementary abelian Sylow 2-subgroup. Hence G is a minimal non-$\mathscr{BNS}_2$-group.

Now assume that every maximal subgroup of G belongs to $\mathscr{BNS}_2$, but G does not. Suppose that G is a 2-group. Then clearly G is cyclic of order 4 (see [7, Theorem 3.2]).

Now assume that G is not a 2-group. By Theorem 3.1 G is either 2-nilpotent or minimal non-nilpotent. If G is of odd order, then G is a $\mathscr{BNS}_2$-group, a contradiction. If G is 2-nilpotent of even order, then every Sylow 2-subgroup of G is elementary abelian. Hence G is a $\mathscr{BNS}_2$-group by Lemma 2.3, a contradiction. It follows that G is a minimal non-nilpotent group and by [2, Theorem 3] and Lemma 2.3, G is a group of type (2). □

In the next results we will use Dickson's Theorem [16, 3.6.25—3.6.26] and some other observations on groups $PSL(2, q)$ and $SL(2, q)$ [16, §1.9 and §3.6].

Lemma 3.3. *Assume that G is one of the following groups:*

(1) *$PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*

(2) *$PSL(2, 2^p)$, where p is a prime such that $2^p - 1$ is a prime;*

(3) *$PSL(2,3^p)$, where p is an odd prime.*

Then every second maximal subgroup of G is a $\mathscr{BNS}_2$-group.

Proof. By Dickson's Theorem a maximal subgroup of G is one of the following groups:

(I) a dihedral group of order $2(r \pm 1)/d$, where $d = (r-1, 2)$ and $r = p$ or 2^p or 3^p,

(II) a Frobenius group N with elementary abelian kernel of order r and a cyclic complement D of order $(r-1)/d$, where $d = (r-1, 2)$ and $r = p$ or 2^p or 3^p, for the structure of N see [16, p.393] (in fact, in this notation $N \cong H/Z(L)$).

(III) A_4.

If M is a group of type (I), then either $M \cong \langle a, b \mid a^2 = b^n = 1,\ b^a = b^{-1}\rangle$ or $M \cong \langle a, b \mid a^2 = b^{2n} = 1,\ b^a = b^{-1}\rangle$, where n is an odd number. By Lemma 2.3 M is a $\mathscr{BNS}_2$-group. If r is a prime, then N of type (II) is a $\mathscr{BNS}_2$-group. If $r = 3^p$, then the order of N is odd and consequently N is a $\mathscr{BNS}_2$-group. Assume that $r = 2^p$. From the structure of N of type (II) it is a minimal non-abelian group with a normal elementary abelian Sylow 2-subgroup. Clearly A_4 is also a minimal non-abelian group with a normal elementary abelian Sylow 2-subgroup. □

Lemma 3.4. *Assume that G is one of the following groups:*

(1) *$PSL(2,p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$, $p \equiv 3 \pmod 8$ and $(p-1)/2$, $(p+1)/4$ are square-free numbers;*

(2) *$PSL(2,p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$, $p \equiv 5 \pmod 8$ and $(p-1)/4$, $(p+1)/2$ are square-free numbers;*

(3) *$PSL(2,2^p)$, where p is a prime such that $2^p - 1$ is a prime and $2^p + 1$ is a square-free number;*

(4) *$PSL(2,3^p)$, where p is an odd prime such that $(3^p - 1)/2$ is a prime and $(3^p + 1)/4$ is a square-free number.*

Then every second maximal subgroup of G is a complemented group.

Proof. As in the proof of Lemma 3.3 a maximal subgroup M of G is one of the groups (I)–(III).

If M is a group of type (I), then either $M \cong \langle a, b \mid a^2 = b^n = 1, b^a = b^{-1}\rangle$ or $M \cong \langle a, b \mid a^2 = b^{2n} = 1, b^a = b^{-1}\rangle$, where n is an odd square-free number. By Lemma 2.2 M is a complemented group. If r is a prime, then N of type (II) is also a complemented group. Assume that $r = 2^p$ or 3^p. From the structure of N of type (II) it is a minimal non-abelian group with elementary abelian Sylow subgroups. Clearly, A_4 is also a minimal non-abelian group with elementary abelian Sylow subgroups. □

Theorem 3.5. *Let G be a group all of whose second maximal subgroups are $\mathscr{BNS}_2$-groups. Then G is either a soluble group or one of the following groups:*

(1) *$PSL(2,p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*

(2) *$PSL(2,2^p)$, where p is a prime such that $2^p - 1$ is a prime;*

(3) *$PSL(2,3^p)$, where p is an odd prime.*

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Proof. By Lemma 3.3 every group of type (1)–(3) is a group all of whose second maximal subgroups are $\mathscr{BNS}_2$-groups. We will show that there are no other non-soluble groups satisfying these conditions.

Let G be a non-soluble group all of whose second maximal subgroups are $\mathscr{BNS}_2$-groups. By Theorem 3.1 if M is a maximal subgroup of G, then M is either 2-nilpotent or minimal non-nilpotent. Therefore G is one of the groups from Lemma 2.4.

We make the following claims:

(i) $G \ncong PSL(2, p^f)$, where $p > 3$, $f > 1$ and $p^f \equiv 3$ or $5 \pmod 8$.

If not, by Dickson's Theorem, $PSL(2, p^f)$ contains a non-soluble proper subgroup $PSL(2, p)$, a contradiction.

(ii) $G \ncong PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.

If not, by Dickson's Theorem G contains a non-soluble proper subgroup isomorphic to A_5, a contradiction.

(iii) $G \ncong PSL(2, 3^f)$, where f is even or composite and $3^f \equiv 3$ or $5 \pmod 8$.

If not, since $3^2 \equiv 1 \pmod 8$, f is odd and $3^f \equiv 3 \pmod 8$. Assume that f is composite and let k be a prime dividing f. By Dickson's Theorem $PSL(2, 3^f)$ contains a non-soluble proper subgroup $PSL(2, 3^k)$, a contradiction.

(iv) If $G \cong PSL(2, 3^p)$, where p is an odd prime, then $3^p \equiv 3 \pmod 8$.

(v) $G \ncong SL(2, q)$, where q is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod 8$.

If not, it follows that G contains a unique element of order 2, $|Z(G)| = 2$ and $G/Z(G) \cong PSL(2, q)$ (see [16, 3.6.2–3.6.3]). Hence by [16, 3.6.17, 3.6.25, 3.6.26] G contains a subgroup $Q_8 \rtimes \langle x \rangle$, where x is an element of order 3, which acts on Q_8 permuting the three maximal subgroups of Q_8 and Q_8 is not a $\mathscr{BNS}_2$-group, a contradiction.

[] By (i)–(v) G is one of the groups (1)–(3). □

The claims (i)—(iii) of the proof above could be proved by using Lemma 2.5 and Thompson's Theorem [17] (see also [9, II.Bemerkung 7.5]).

Theorem 3.6. *Let G be a group all of whose second maximal subgroups are complemented groups. Then G is either a soluble group or one of the following groups:*

(1) $PSL(2, p)$, *where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$, $p \equiv 3 \pmod 8$ and $(p-1)/2$, $(p+1)/4$ are square-free numbers;*

(2) $PSL(2, p)$, *where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$, $p \equiv 5 \pmod 8$ and $(p-1)/4$, $(p+1)/2$ are square-free numbers;*

(3) $PSL(2, 2^p)$, *where p is a prime such that $2^p - 1$ is a prime and $2^p + 1$ is a square-free number;*

(4) $PSL(2, 3^p)$, *where p is an odd prime such that $(3^p - 1)/2$ is a prime and $(3^p + 1)/4$ is a square-free number.*

Proof. By Lemma 3.4 every group of type (1)–(4) is a group all of whose second maximal subgroups are complemented groups. We will show that there are no other non-soluble groups satisfying these conditions.

Let G be a non-soluble group all of whose second maximal subgroups are complemented groups. Hence G is a non-soluble group all of whose second maximal subgroups are $\mathscr{BNS}_2$-groups. It follows that G is one of the groups from Theorem 3.5. Therefore Sylow subgroups of G are elementary abelian and by Dickson's Theorem a maximal subgroup M of G is one of the groups (I)–(III) from the proof of Lemma 3.3. We should only show that $G \not\cong PSL(2, 3^f)$, where f is an odd prime and $(3^f - 1)/2$ is a composite square-free number. If not, then G possesses a Frobenius group N with kernel P of order 3^f and a cyclic complement D of order $(3^f - 1)/2$. For the structure of N see [16, p. 393] (in fact, in this notation $N \cong H/Z(L)$). Moreover P is an elementary abelian 3-group. Since $(3^f - 1)/2$ is composite, it follows that N possesses a proper non-abelian subgroup $\langle x\rangle P$, where $\langle x\rangle$ is a proper subgroup of D of prime order. By hypothesis $\langle x\rangle P$ is a complete group and, in consequence $\langle x\rangle P$ is supersoluble by Lemma 2.2. Thus $|\langle x\rangle| = 2$, which contradicts the fact that $(3^f - 1)/2$ is odd. Hence G is one of the groups (1)–(4). This completes the proof. □

4. Groups with at most 21 non-complemented non-minimal subgroups

We say that a group G is an $\mathscr{NC}_n$-group if it has exactly n non-complemented subgroups. The groups belonging to $\mathscr{NC}_0$ are the complemented groups [8]. We say that a group G is an $\mathscr{NCM}_n$-group if it has exactly n non-complemented non-minimal subgroups. By [13, Theorem] the groups belonging to $\mathscr{NCM}_0$-groups are soluble. Note that $\mathscr{NCM}_0$-groups do not have to be supersoluble. For example, $A_4 \in \mathscr{NCM}_0$.

Let p be a fixed prime. We write $\mathbb{P}_p$ to denote the set of primes less than or equal to p. We say that a group G is an $\mathscr{NCM}_0(\mathbb{P}_p)$-group if every non-minimal $\mathbb{P}_p$-subgroup of G is complemented in G. We will show that every $\mathscr{NCM}_0(\mathbb{P}_2)$-group is soluble. Assume that G is an $\mathscr{NCM}_0(\mathbb{P}_2)$-group and P is a Sylow 2-subgroup of G. If $|P| > 4$, then by [4, Lemma 3] G is soluble. If $|P| = 2$, then by [14, Corollary 10.24] G is soluble. Assume that $|P| = 4$. Then P is complemented in G. By Lemma 2.1 (3) $\bar{P} = PO_{2'}(G)/O_{2'}(G)$ is complemented in $\bar{G} = G/O_{2'}(G)$. Let $\bar{N}$ be a minimal normal subgroup of $\bar{G}$. Then $\bar{N}$ is of even order. If $\bar{N}$ is not soluble, then $\bar{P} < \bar{N}$ and by Lemma 2.1 (1) $\bar{P}$ is complemented in $\bar{N}$. By [1, Corollary 5.6(I)] $\bar{N} \cong PSL(2, r)$ with r a Mersenne prime. Hence $|P| = |\bar{P}| > 4$, a contradiction.

Clearly, a subgroup H of G is a non-complemented subgroup if H is not complemented in G. By Lemma 2.1(1) if $H \leqslant K \leqslant G$ and H is not complemented in K, then H is not complemented in G.

Let π be a set of primes. We say that a group G is an $\mathscr{NCM}_n(\pi)$-group if it has exactly n non-complemented non-minimal π-subgroups. Let $\bar{H}$ be a π-subgroup of $G/\Phi(G)$, then there exists a subgroup H such that $\Phi(G) \subseteq H$ and $H/\Phi(G) = \bar{H}$. By [9, Satz III.3.6 and Satz III.3.8] $\Phi(G)$ is a nilpotent group such that $\pi\big(\Phi(G)\big) \subseteq \pi\big(G/\Phi(G)\big)$. Therefore a Hall π'-subgroup of $\Phi(G)$, say F, is normal in G. Hence by [9, Hauptsatz I.18,1] there exists a π-subgroup K of G such that $H = KF$ and $\bar{H} = H/\Phi(G) = K\Phi(G)/\Phi(G)$. Assume that $H/\Phi(G)$ is not complemented in $G/\Phi(G)$. By Lemma 2.1 (2) H is not complemented in G and by Lemma 2.1 (3) $H/F = KF/F$ is not complemented in G/F and K is not complemented in G. Lemma 2.1 now gives the following easy observations.

Lemma 4.1. *Let G be a group and let π be a set of primes. Suppose that $G \in \mathscr{NCM}_n(\pi)$ and $K < G$. Then*

(1) *$K \in \mathscr{NCM}_m(\pi)$ for some $m \leqslant n$;*

(2) *$G/\Phi(G) \in \mathscr{NCM}_k(\pi)$ for some $k \leqslant n$.*

If the group $G = AB$ is the product of two subgroups A and B with $A \cap B = 1$, we say that G has an *exact factorization*. Clearly A and B are the complemented subgroups of G. Let A_1 and B_1 be any conjugate subgroups of A and B respectively. One can get A_1 by transforming A with a suitable element of B and similarly with B_1. Hence G has also the exact factorization $G = A_1B_1$. We say that this factorization of G is equivalent to the original one. In [10] N. Ito referred to the number of non-equivalent factorizations of G simply as the number of factorizations of G. He also found all factorizations of the projective linear group $PSL(2, p^n)$ by using [6, XII].

In the next results we will use some results on groups $PSL(2, q)$ and $Sz(2^p)$ due to L. E. Dickson [6], M. Suzuki [15] and N. Ito [10]. But first we consider three groups $PSL(2, q)$ of small order.

Example 4.1. Let $G \cong A_5 \cong PSL(2, 4) \cong PSL(2, 5)$. By [10] G has the one exact factorization $G = AB$, where $A \cong A_4$ and B is a Sylow 5-subgroup of G. Hence G has exactly 46 non-complemented subgroups: a single conjugacy class of 5 Sylow 2-subgroups, a single conjugacy class of 10 Sylow 3-subgroups, a single conjugacy class of 15 subgroups of order 2, a single conjugacy class of 10 subgroups isomorphic to S_3 and a single conjugacy class of 6 subgroups isomorphic to D_{10}. It follows that $G \in \mathscr{NC}_{46} \cap \mathscr{NCM}_{21}(\mathbb{P}_5)$.

Example 4.2. Let $G \cong PSL(2, 7)$. Since G has two conjugacy classes of subgroups isomorphic to S_4, by [10] G has just three exact factorizations: two $G = AB$, where A is a Sylow 7-subgroup of G and $B \cong S_4$, and $G = HK$, where H is a normalizer of a Sylow 7-subgroup of G and K is a subgroup isomorphic to dihedral group of order 8. Hence by [5] or [6, XII] G has 126 non-complemented subgroups:

- two conjugacy classes each of 7 subgroups isomorphic to A_4;
- a single conjugacy class of 28 subgroups isomorphic to S_3;
- a single conjugacy class of 21 cyclic subgroups of order 4;
- two conjugacy classes each of 7 non-cyclic subgroups of order 4;
- a single conjugacy class of 28 subgroups of order 3;
- a single conjugacy class of 21 subgroups of order 2.

Therefore $G \in \mathscr{NCM}_{77}(\mathbb{P}_5)$.

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Example 4.3. Let $G \cong PSL(2,8)$. By [10] G has the one exact factorization $G = AB$, where A is the normalizer of a Sylow 2-subgroup and B is a cyclic subgroup of order 9. Hence by [6, XII] or [5] G has exactly 347 non-complemented subgroups:

- a single conjugacy class of 28 subgroups isomorphic to D_{18};
- a single conjugacy class of 36 subgroups isomorphic to D_{14};
- a single conjugacy class of 9 Sylow 2-subgroups;
- a single conjugacy class of 36 Sylow 7-subgroups;
- a single conjugacy class of 84 subgroups isomorphic to S_3;
- a single conjugacy class of 63 non-cyclic subgroups of order 4;
- a single conjugacy class of 28 subgroups of order 3;
- a single conjugacy class of 63 subgroups of order 2;

It follows that $G \in \mathscr{N}\mathscr{C}\mathscr{M}_{148}(\mathbb{P}_5)$.

Lemma 4.2. *Let G be a minimal simple group. Then $G \in \mathscr{N}\mathscr{C}\mathscr{M}_n(\mathbb{P}_5)$, where $n \geqslant 21$. The only minimal simple $\mathscr{N}\mathscr{C}\mathscr{M}_{21}(\mathbb{P}_5)$-group is A_5.*

Proof. By Thompson's Theorem [17] (see also [9, II.Bemerkung 7.5]) G is isomorphic to one of the following groups:

(a) $PSL(3,3)$;
(b) a Suzuki group $Sz(2^p)$, where p is an odd prime;
(c) $PSL(2,3^p)$, where p is an odd prime;
(d) $PSL(2,2^p)$, where p is a prime;
(e) $PSL(2,p)$, where $p > 3$ is a prime and $p^2 - 1 \not\equiv 0 \pmod 5$

Assume that $G \cong PSL(3,3)$. There are 351 Sylow 2-subgroups of $PSL(3,3)$, $|PSL(3,3)| = 5616 = 16 \cdot 351$ and G does not have a subgroup of order 351 (see [5]). Since 2-subgroups of $PSL(3,3)$ are not complemented in G it follows that $PSL(3,3) \in \mathscr{N}\mathscr{C}\mathscr{M}_n(\mathbb{P}_5)$, where $n > 351$.

Assume that $G \cong Sz(2^p)$, where p is an odd prime. We have $|G| = 2^{2p}(2^p-1)(2^{2p}+1)$. By [1, Corollary 5.6] and [11, Lemma 2.4] every 2-subgroup of G is not complemented in G and G has $2^{2p}+1$ Sylow 2-subgroups. Hence $G \in \mathscr{N}\mathscr{C}\mathscr{M}_n(\mathbb{P}_5)$ for $n > 2^{2p}+1 \geqslant 2^6+1 = 65$.

Assume that $G \cong PSL(2,3^p)$, where p is an odd prime. We have $3^p \equiv 3 \pmod 8$ and $|G| = 3^p(3^{2p}-1)/2$. By [10] G has the only one factorization $G = ND$, where N is the normalizer of a Sylow 3-subgroup, $|N| = 3^p(3^p-1)/2$ and D is a dihedral group of order 3^p+1. By [6, XII.260] G has a single conjugacy class of $3^p(3^{2p}-1)/24$ Sylow 2-subgroups of order 4 and for $p \in \{2,3\}$ every p-subgroup is non-complemented in G. Hence $G \in \mathscr{N}\mathscr{C}\mathscr{M}_n(\mathbb{P}_5)$, where $n > 3^p(3^{2p}-1)/24 \geqslant 3^3(3^6-1)/24 = 819$.

By Examples 4.1 and 4.3 we now consider $G \cong PSL(2,2^p)$, where p is a prime and $p > 3$. By [10] G has the only one exact factorization $G = AB$, where A is the normalizer of a Sylow 2-subgroup of order $2^p(2^p-1)$ and B is a cyclic subgroup of order 2^p+1. By [6, XII.260] G has a single conjugacy class of 2^p+1 Sylow 2-subgroups. Since every 2-subgroup of G is not complemented in G, it follows that $G \in \mathscr{NCM}_n(\mathbb{P}_5)$, where $n > 2^p+1 \geqslant 2^5+1 = 33$.

Finally by Examples 4.1 and 4.2 we consider $G \cong PSL(2,p)$, where p is a prime, $p > 7$ and $p^2-1 \not\equiv 0 \pmod 5$. We have $|G| = p(p^2-1)/2$. If $p \equiv 1 \pmod 4$, then by [10] G admits no factorization. If $p \equiv 3 \pmod 4$, then by [10] G has the only one factorization $G = ND$, where N is the normalizer of a Sylow p-subgroup, $|N| = p(p-1)/2$ and D is a dihedral group of order $p+1$. Since by [6, XII.260] G has $p(p^2-1)/24$ elementary abelian subgroups of order 4 and G has a subgroup isomorphic to A_4, it follows that $G \in \mathscr{NCM}_n(\mathbb{P}_5)$, where $n > p(p^2-1)/24 \geqslant 13(13^2-1)/24 = 91$.

This completes the proof. □

Theorem 4.3. *Every $\mathscr{NCM}_n(\mathbb{P}_5)$-group with $n < 21$ is soluble.*

Proof. Assume that G is a non-soluble $\mathscr{NCM}_n(\mathbb{P}_5)$-group of minimal order with $n < 21$. Then by Lemma 4.1 (1) every maximal subgroup of G is soluble. Hence by Lemmas 2.5 and 4.1 (2) $G/\Phi(G)$ is a minimal simple $\mathscr{NCM}_n(\mathbb{P}_5)$-group with $n < 21$, contrary to Lemma 4.2. Therefore G is soluble. □

Remark 4.4. *It can be proved that*

(1) *Every $\mathscr{NC}_n$-group with $n < 46$ is soluble;*

(2) *G is a non-soluble $\mathscr{NC}_{46}$-group if and only if it is isomorphic to A_5;*

(3) *G is a non-soluble $\mathscr{NCM}_{21}$-group if and only if it is isomorphic to A_5.*

Remark 4.5. *Let $nc(G)$ be the number of isomorphic classes of non-complemented subgroups in G. It can be proved that*

(1) a group G is soluble if $nc(G) < 5$

(2) $nc(G) = 5$ if and only if $G \cong A_5$.

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