



www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 9 No. 1 (2020), pp. 7-24.
© 2020 University of Isfahan



www.ui.ac.ir

ON SOME GENERALIZATION OF THE MALNORMAL SUBGROUPS

LEONID A. KURDACHENKO, NIKOLAI N. SEMKO AND IGOR YA. SUBBOTIN*

Communicated by Patrizia Longobardi

ABSTRACT. A subgroup H of a group G is called malnormal in G if $H \cap H^x = \langle 1 \rangle$ for every element $x \notin N_G(H)$. These subgroups are generalizations of malnormal subgroups. Every malnormal subgroup is malnormal, and every selfnormalizing malnormal subgroup is malnormal. Furthermore, every normal subgroup is malnormal. In this paper we obtain a description of finite and certain infinite groups, whose subgroups are malnormal.

1. Introduction

Let G be a group. A subgroup H of a group G is called *malnormal* in G if $H \cap H^x = \langle 1 \rangle$ for every element $x \notin H$.

The term has been introduced by B. Baumslag [1].

Malnormal subgroups arise in finite groups as Frobenius complements in Frobenius groups. The Frobenius complements of Frobenius groups are described quite well (see, for example, [8, Chapter 10 Theorem 3.1]). In this connection, we note the following result:

If G is a finite group and H, K are malnormal subgroups of G , then there exists an element g such that $H^g \leq K$ or $K^g \leq H$ ([6, P. Flavell]).

The situation with malnormal subgroups in infinite groups much more complicated. The following result justifies this statement:

MSC(2010): Primary 20D15, 20D25, 20E07, 20E34, 20F19, 20F50.

Keywords: Malnormal Subgroups, Malnormal Subgroups, Frobenius Group, Locally Graded groups, Generalized Radical Groups.

Received: 17 July 2018, Accepted: 09 December 2018.

*Corresponding author.

<http://dx.doi.org/10.22108/ijgt.2018.112124.1487>

If K and L are non-trivial groups, then K is malnormal in the free product $H * K$ [12, Proposition 2].

In [12] in various cases the role of malnormal subgroups in different classes of infinite groups was shown. As can be seen from the definition, malnormal subgroups are antagonists to normal subgroups: *A normal subgroup H of a group G is malnormal in G if and only if $H = \langle 1 \rangle$ or $H = G$.*

A similar situation we have with another antagonist to normal subgroups, namely with abnormal subgroups. Recall that a subgroup H of a group G is called **abnormal** if $x \in \langle H, H^x \rangle$ for each element $x \in G$. *A normal subgroup H of a group G is abnormal in G if and only if $H = G$.*

Nevertheless, there are subgroups that are simultaneously a generalization of both abnormal and normal subgroups. Such are the pronormal subgroups. Recall that a subgroup H of a group G is called **pronormal**, if the subgroups H and H^x are conjugate in $\langle H, H^x \rangle$ for each element $x \in G$.

These types of subgroups and other types of subgroups associated with them are subjects of a vast array of articles. The results of these articles are reflected in the survey [14].

We want to introduce a similar generalization of malnormal subgroups, namely, we want to introduce here the following class of subgroups containing both malnormal subgroups and normal subgroups.

A subgroup H of a group G is called **malonormal** in G , if $H \cap H^x = \langle 1 \rangle$ for every element $x \notin N_G(H)$.

Thus every malnormal subgroup is malonormal, and every selfnormalizing malonormal subgroup is malnormal. Furthermore, every normal subgroup is malonormal.

In this paper we considered the groups, whose all subgroups are malonormal. These class of groups includes the groups, whose subgroups are normal. Recall that groups, whose subgroups are normal, are called Dedekind groups. If G is a **Dedekind group**, then either G is abelian or $G = Q \times B \times S$ where Q is a quaternion group, B is an elementary abelian 2-subgroup, S is a periodic abelian p -subgroup [2].

A special consideration here was given to finite groups, whose subgroups are malonormal. Our first main result is the description of such groups.

Theorem 1.1. *Let G be a finite group, whose subgroups are malonormal. Then G is a group of one of the following types:*

- (i) G is a Dedekind group.
- (ii) $G = \langle \nu \rangle \lambda \langle u \rangle$ where $|\nu| = p^k$, $k \geq 3$, $|u| = p$ and $v^u = v^s$ where $s = 1 + p^{k-1}$, p is a prime.
- (iii) $G = (\langle c \rangle \times \langle \nu \rangle) \lambda \langle u \rangle$ where $|c| = |\nu| = |u| = p > 2$ and $[\nu, u] = c$, $[c, u] = 1$, p is a prime.
- (iv) $G = (\langle z \rangle \times \langle a \rangle) \lambda \langle u \rangle$ where $|a| = |u| = p$, $|z| > p^k$, $k > 1$, $[z, u] = 1$, $[a, u] = c$ where $\langle c \rangle = \Omega_1(Z)$, p is a prime.
- (v) $G = QD$, $[Q, D] = \langle 1 \rangle$ where Q is a quaternion group of order 8, D is a dihedral group of order 8 and $Q \cap D = \zeta(Q) = \zeta(D)$.

- (vi) $G = Q\lambda K$ where Q is an elementary abelian q -subgroup of order q^2 , q is a prime, K is a cyclic q' -subgroup, $2 \notin \Pi(K)$, Q does not include non-trivial cyclic $\langle y \rangle$ -invariant subgroup for each $y \in K \setminus \{1\}$.
- (vii) $G = Q\lambda K$ where $|Q| = q$ is prime, $Q = C_G(Q)$, K is a cyclic subgroup of order dividing $q - 1$.

Conversely, in any listed above group each subgroup is malonormal.

The natural next stage is the consideration of infinite groups with this property. The situation with periodic groups is not very visible. A. Yu. Olshanskii [18] has constructed an example of infinite finitely generated p -group O , p is a quite big prime, all of whose proper subgroups have order p . It is clear that every proper subgroup of this group is malnormal. Therefore, some additional restrictions were required.

A group G is said to be **locally graded** if every non-trivial finitely generated subgroup of G includes a proper subgroup of finite index.

This definition belongs to S. N. Chernikov [4].

We were able to obtain the following results.

Theorem 1.2. *Let G be an infinite periodic locally graded group, whose subgroups are malonormal. Then G is a group of one of following types:*

- (i) G is a Dedekind group.
- (ii) $G = (K \times \langle c \rangle)\lambda \langle b \rangle$ where $K = \langle a_n \mid a_1^p = 1, a_{n+1}^p = a_n, n \in \mathbb{N} \rangle$ is a quasicyclic p -subgroup, $c^p = b^p = 1, [K, b] = \langle 1 \rangle, [b, c] = a_1$, where p is a prime.

Conversely, in every of these groups each subgroup is malonormal.

For non-periodic locally graded groups the situation is simpler.

Theorem 1.3. *Let G be a locally graded group, whose subgroups are malonormal. If G non-periodic, then G is abelian.*

Another restriction, which we used is connected to generalized solvability.

A group G is called **generalized radical** if G has an ascending series, whose factors are locally nilpotent or locally finite.

It is not hard to see that a generalized radical group has an ascending series of normal, indeed characteristic, subgroups with locally nilpotent or locally finite factors.

Theorem 1.4. *Let G be a locally generalized radical group, whose subgroups are malonormal. If G non-periodic, then G is abelian.*

2. Finite groups, whose subgroups are malonormal

Note some properties of malonormal subgroups.

Lemma 2.1. *Let G be a group.*

- (i) If H is a malonormal subgroup of G and K is a subgroup, then $H \cap K$ is malonormal in K . In particular, if H is a malonormal subgroup of G and K is a subgroup including H , then H is malonormal in K .
- (ii) If H is a malonormal subgroup of G , then H^x is a malonormal subgroup of G for every element $x \in G$.
- (iii) Let H and K be subgroups of G such that K is a proper nontrivial subgroup of H . If $N_G(K) \neq N_G(H)$, then H is not malonormal in G . In particular, if H is malonormal in G and $\mathbf{Core}_G(H) \neq \langle 1 \rangle$, then H is normal in G .

Proof. (i) Suppose that $x \in K \setminus N_K(H \cap K)$. If we assume that $x \in N_G(H)$, then $H^x = H$. Since $x \in K$, $K^x = K$. Then $H \cap K = H^x \cap K^x = (H \cap K)^x$, and we obtain a contradiction. Hence $x \notin N_G(H)$. It follows that $H \cap H^x = \langle 1 \rangle$. Then

$$\langle 1 \rangle = (H \cap H^x) \cap K = (H \cap K) \cap (H^x \cap K) = (H \cap K) \cap (H^x \cap K^x) = (H \cap K) \cap (H \cap K)^x.$$

It shows that $H \cap K$ is malonormal in K .

(ii) is obvious.

(iii) Let $x \in N_G(K) \setminus N_G(H)$, then $H^x \neq H$ and $K = K^x \leq H^x$, so that $K \leq H \cap H^x$ and $H \cap H^x \neq \langle 1 \rangle$. □

Corollary 2.2. Let G be a group and H be a non-trivial normal subgroup of G . If every subgroup of G is malonormal in G , then G/H is a Dedekind group.

Proof. Indeed, choose an arbitrary element $x \notin H$ and put $X = \langle x, H \rangle$. The inclusion $H \leq \mathbf{Core}_G(X)$ shows that $\mathbf{Core}_G(X) \neq \langle 1 \rangle$. Lemma 2.1 implies that X must be normal in G . It follows that every cyclic subgroup of G/H is normal. In this case, every subgroup of G/H is normal. □

Corollary 2.3. Let G be a group and P be a p -subgroup of G , p is a prime. Suppose that every subgroup of order p of P is normal in G . If every subgroup of P is malonormal in G , then every subgroup of P is normal in G . In particular, P is a Dedekind group.

Proof. Indeed, if H is an arbitrary subgroup of P . Then H contains an element x having prime order p . Since the subgroup $\langle x \rangle$ is normal in G , $\langle x \rangle \leq \mathbf{Core}_G(H)$, in particular $\mathbf{Core}_G(H) \neq \langle 1 \rangle$. Then Lemma 2.1 implies that H is normal in G . □

Lemma 2.4. Let G be a group and H be a normal finite subgroup of G , having order p^n where p is a prime. If L is a malonormal subgroup of G such that $L \leq H$, $|L| = p^k$ and $2k > n$, then L is normal in G .

Proof. Suppose the contrary, let L is not normal on G . Then $N_G(L) \neq L$. Let $x \notin N_G(L)$, then $L^x \neq L$. Since L is malonormal, $L^x \cap L = \langle 1 \rangle$. On the other hand, $L^x \leq H$, so that H includes the subgroup $\langle L^x, L \rangle$. The last subgroup includes the product $L^x L$. The equality $L^x \cap L = \langle 1 \rangle$ shows that

$|L^x L| = |L^x||L| = |L|^2 = p^{2k}$. It follows that $|\langle L^x, L \rangle| \geq p^{2k} > p^n$, so we obtain a contradiction with the inclusion $\langle L^x, L \rangle \leq H$. This contradiction proves that L is a normal subgroup of G . \square

Corollary 2.5. *Let G be a group and P be a proper finite normal p -subgroup of G , p is a prime. Suppose that every subgroup of P is malonormal in G . If $|P| > p^2$, then every maximal subgroup of P is normal in G .*

Corollary 2.6. *Let G be a group and P be a finite abelian normal p -subgroup of G . Suppose that $P = \langle a_1 \rangle \times \dots \times \langle a_n \rangle$ where $|a_j| = p^k$ for all $j, 1 \leq j \leq n$. If $n > 2$ and every subgroup of P is malonormal in G , then every subgroup of P is normal in G .*

Proof. We have $|P| = p^{kn}$. Put $L_j = \langle a_1 \rangle \times \mathbf{Dr}_{2 \leq m \leq n, m \neq j} \langle a_m \rangle$. Then $|L_j| = p^{k(n-1)}$. Since $n > 2, 2k(n-1) > 2k$ and Lemma 2.4 implies that every subgroup L_j is normal in $G, 2 \leq j \leq n$. The equality $\langle a_1 \rangle = \cap_{2 \leq j \leq n} L_j$ shows that the subgroup $\langle a_1 \rangle$ is normal in G . \square

Let d be an arbitrary element of P , then $d = a_1^{t_1} \dots a_n^{t_n}$ where $0 \leq t_j \leq k-1, 1 \leq j \leq n$. Let m be a number such that $|a_{m^{t_m}}| \geq |a_{j^{t_j}}|$ for all $j, 1 \leq j \leq n$. Let $|a_{m^{t_m}}| = p^r$. If $r = k$, then $|d| = p^k$. In this case, there exists a subgroup K of P such that $P = \langle d \rangle \times K$. Repeating the above arguments, we obtain that the subgroup $\langle d \rangle$ is normal in G .

Suppose now that $r < k$. Then the equation $x^s = d$ where $s = p^{k-r}$ has a solution in P . Let b be a solution of this equation. Then b has order p^k . Using the above arguments we obtain that the subgroup $\langle b \rangle$ is normal in G . Since every subgroup of a cyclic group is characteristic, the subgroup $\langle d \rangle$ is also normal in G . Thus every cyclic subgroup of P is normal in G . It follows that every subgroup of P is normal in G .

Let p be a prime and A be an abelian p -group. We say that a group A is **homogeneous** if $A = \mathbf{Dr}_{\lambda \in \Lambda} \langle a_\lambda \rangle$ where $|a_\lambda| = p^k$ for all $\lambda \in \Lambda$.

Corollary 2.7. *Let G be a group and P be a finite abelian normal p -subgroup of G . Suppose that $P = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_1| = |a_2| = p^k$ and $k \geq 2$. If every subgroup of P is malonormal in G , then every subgroup of P is normal in G .*

Proof. We have $|P| = p^{2k} \geq p^4$. Let a be an arbitrary element of P , and let $\langle b \rangle = \langle a \rangle \cap \Omega_1(P)$. Then there exists an element d of P such that $\langle b \rangle = \langle d \rangle \cap \Omega_1(P)$ and $P = \langle d \rangle \times \langle v \rangle$ for some subgroup $\langle v \rangle$ (see, for example, [7, Corollary 27.2],). Since a direct decomposition of finite abelian p -group is unique up to isomorphism, $|d| = |v| = p^k$. By Corollary 2.5 a subgroup $\langle d \rangle \times \langle v^p \rangle$ is normal in G . Put $v_1 = v^p$. Suppose first that $|v_1| = p$. Suppose that the subgroup $\langle d \rangle$ is not normal in G . Then we can find an element $x \notin N_G(\langle d \rangle)$. Then $d^x \notin \langle d \rangle$. Since $|d^x| = |d| = p^k, d^x = d^t v_1^s$ where $\mathbf{GCD}(p, t) = 1 = \mathbf{GCD}(p, s)$. In this case, $(d^x)^p = d^{tp} \in \langle d^p \rangle$ and $\langle d \rangle^x \cap \langle d \rangle = \langle d^x \rangle \cap \langle d \rangle = \langle d^p \rangle \neq \langle 1 \rangle$, and we obtain a contradiction. This contradiction shows that $\langle d \rangle$ is normal in G . \square

Suppose now that $|v_1| > p$. Then $|\langle d \rangle \times \langle v_1 \rangle| > p^2$. Using again Corollary 2.5, we obtain that the subgroup $\langle d \rangle \times \langle v_1^p \rangle$ is normal in G . Put $v_2 = v_1^p$. If $|v_2| = p$, then repeating the above arguments, we

obtain that the subgroup $\langle d \rangle$ is normal in G . If $|v_2| > p$, then we consider the subgroup $\langle d \rangle \times \langle v_2 \rangle$, and so on. After finitely many steps we find an element v_m such that $|v_m| = p$ and the subgroup $\langle d \rangle \times \langle v_m \rangle$ is normal in G . Repeating again the above arguments, we obtain that the subgroup $\langle d \rangle$ is normal in G .

Since $\Omega_1(\langle d \rangle)$ is a characteristic subgroup of $\langle d \rangle$, it is normal in G . The choice of d yields that $\Omega_1(\langle d \rangle) = \Omega_1(\langle a \rangle)$, so that $\Omega_1(\langle a \rangle)$ is normal in G . It follows that $\mathbf{Core}_G(\langle a \rangle) \geq \Omega_1(P)$, in particular, $\mathbf{Core}_G(\langle a \rangle) \neq \langle 1 \rangle$ and Lemma 2.1 implies that $\langle a \rangle$ is normal in G . The fact, that every cyclic subgroup of P is normal in G , implies that every subgroup of P is normal in G .

Corollary 2.8. *Let G be a group and P be a finite abelian normal p -subgroup of G . Suppose that $P = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_1|, |a_2| \geq p^2$. If every subgroup of P is malnormal in G , then every subgroup of P is normal in G .*

Proof. We will apply induction on $|P|$. If $|P| = p^4$, then the result follows from Corollary 2.7. Suppose now that $|P| > p^4$, and we have already proved this assertion for all normal abelian p -subgroup B such that $|B| < |P|$. If $|a_1| = |a_2|$, then we can use again Corollary 2.7. Therefore suppose now that $|a_1| > |a_2|$ and consider the subgroup $A = \langle a_1^p \rangle \times \langle a_2 \rangle$. By Corollary 2.5, the subgroup A is normal in G . Since $|a_2| \geq p^2$, $|A| \geq p^4$. By the induction hypothesis, every subgroup of A is normal in G . The choice of A shows that $\Omega_1(P) \leq A$, so we obtain that every subgroup of $\Omega_1(P)$ is normal in G . If H is a subgroup of P , then $\mathbf{Core}_G(H) \geq \Omega_1(H)$, and in particular, $\mathbf{Core}_G(H) \neq \langle 1 \rangle$. Lemma 2.1 implies that H is normal in G . \square

Let p be a prime. We say that a group G has **finite section p -rank** $\mathbf{sr}_p(G) = r$ if every elementary abelian p -section of G is finite of order at most p^r and there is an elementary abelian p -section A/B of G such that $|A/B| = p^r$.

Let A be an abelian p -group. Then A has finite section p -rank r if and only if $|\Omega_1(A)| = p^r$ (see, for example, [5, Lemma 3.1.3]).

Here for every positive integer n we put

$$\Omega_n(A) = \{a \in A \mid |a| \text{ divides } p^n\}.$$

Corollary 2.9. *Let G be a group and P be a finite abelian normal p -subgroup of G . Suppose that $\mathbf{sr}_p(P) \geq 3$. If every subgroup of P is malnormal in G , then every subgroup of P is normal in G .*

Proof. We will apply induction on $|P|$. If $|P| = p^3$, then the condition $\mathbf{sr}_p(P) \geq 3$ implies that $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle$ where $|a_j| = p, j \in \{1, 2, 3\}$. In this case, using Corollary 2.6 we obtain that every subgroup of P is normal in G . \square

Suppose now that $|P| > p^3$, and we have already proved this assertion for every normal abelian p -subgroup B such that $|B| < |P|$. Let e be a positive integer such that $|a| \leq p^e$ for all elements $a \in P$ and there exists an element $d \in P$ such that $|d| = p^e$. If we suppose that $e = 1$, then P is elementary

abelian and $\text{sr}_p(P) \geq 4$. Using again Corollary 2.6 we obtain that every subgroup of P is normal in G . Therefore we can suppose now that $e > 1$.

Put $D = \langle d \rangle$. It is not hard to prove that $P = D \times B$ for some subgroup B . In this case, $\Omega_1(P) \leq D^p \times B$. The last subgroup has index p in P , and using Corollary 2.5 we obtain that the subgroup $D^p \times B$ is normal in G . By its choice, $\text{sr}_p(D^p \times B) = \text{sr}_p(P)$, $|D^p \times B| \geq p^3$ and $|D^p \times B| < |P|$. By the induction hypothesis every subgroup of $D^p \times B$ is normal in G . In particular, every subgroup of $\Omega_1(P)$ is normal in G . If H is a subgroup of P , then $\text{Core}_G(H) \geq \Omega_1(H)$; in particular, $\text{Core}_G(H) \neq \langle 1 \rangle$. Lemma 2.1 implies that H is normal in G .

Lemma 2.10. *Let G be a group and P be a normal finite p -subgroup of G , p is a prime. Suppose that P includes a normal abelian subgroup A such that $\text{sr}_p(A) \geq 3$. If every subgroup of P is malonormal in G , then every subgroup of P is normal in G ; in particular, P is a Dedekind group.*

Proof. Using Corollary 2.9 we obtain that every subgroup of A is normal in P . In particular, every subgroup of $\Omega_1(A)$ is normal in P . We note that $\text{sr}_p(\Omega_1(A)) \geq 3$. Let $1 \neq d$ be an element of $\Omega_1(A)$. By Corollary 2.2 the factor-group $P/\langle d \rangle$ is Dedekind. Let B be a maximal normal in P elementary abelian p -subgroup of P , including $\Omega_1(A)$. Then $\text{sr}_p(B) \geq 3$. Using again Corollary 2.9 we obtain that every subgroup of B is normal in P . In particular, every cyclic subgroup of B is normal in P . Recall that the center of a finite p -group includes every normal cyclic subgroup of prime order p . It follows that $B \leq \zeta(P)$. If we assume that P contains an element c of order p such that $c \notin B$, then $\langle B, c \rangle$ is an elementary abelian p -subgroup. The inclusion $\langle d \rangle \leq \langle B, c \rangle$ implies that $\langle B, c \rangle$ is normal in P . Thus we obtain a contradiction with a choice of B . This contradiction shows that B contains every element of P , having order p . In particular, B is a characteristic subgroup of P , so that B is normal in G . An application of Corollary 2.9 shows that every subgroup of B is normal in G . Since $B \cap H \neq \langle 1 \rangle$ for every non-trivial subgroup H of P , Lemma 2.1 implies that every subgroup of P is normal in G . In particular, P is a Dedekind group. \square

Lemma 2.11. *Let G be a group and P be a normal finite p -subgroup of G , where p is a prime. Suppose that P includes a normal abelian subgroup A such that $A = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_1|, |a_2| \geq p^2$. If every subgroup of P is malonormal in G , then every subgroup of P is normal in G , in particular, P is a Dedekind group.*

Proof. Using Corollary 2.8 we obtain that every subgroup of A is normal in P . In particular, every subgroup of $\Omega_1(A)$ is normal in P . Let $1 \neq d$ be an element of $\Omega_1(A)$. By Corollary 2.2, the factor-group $P/\langle d \rangle$ is Dedekind. Let B be a maximal normal in P elementary abelian p -subgroup of P including $\Omega_1(A)$. If $\text{sr}_p(B) \geq 3$, then using Corollary 2.9 we obtain that every subgroup of B is normal in P . In particular, every cyclic subgroup of B is normal in P . Recall that the center of a finite p -group includes every normal cyclic subgroup of prime order p . It follows that $B \leq \zeta(P)$. If we assume that P contains an element c of order p such that $c \notin B$, then $\langle B, c \rangle$ is an elementary abelian p -subgroup. The inclusion $\langle d \rangle \leq \langle B, c \rangle$ implies that $\langle B, c \rangle$ is normal in P . Thus we obtain a

contradiction with a choice of B . This contradiction shows that B contains every element of P , having order p . In particular, B is a characteristic subgroup of P , so that B is normal in G . An application of Corollary 2.9 shows that every subgroup of B is normal in G .

Suppose that $\text{sr}_p(B) = 2$. In this case, $B = \Omega_1(A)$, and again $B \leq \zeta(P)$. If we assume that P contains an element c of order p such that $c \notin B$, then $\langle B, c \rangle$ is an elementary abelian p -subgroup. The inclusion $\langle d \rangle \leq \langle B, c \rangle$ implies that $\langle B, c \rangle$ is normal in P . Thus we obtain a contradiction with a choice of B . This contradiction shows that B contains every element of P having order p .

Since $B \cap H \neq \langle 1 \rangle$ for every non-trivial subgroup H of P , Lemma 2.1 implies that every subgroup of P is normal in G . In particular, P is a Dedekind group. \square

Lemma 2.12. *Let P be a finite p -group, where p is a prime. If every subgroup of P is malonormal in G , then P is a group of one of following types:*

- (a) P is a Dedekind group.
- (b) $P = \langle \nu \rangle \lambda \langle u \rangle$ where $|\nu| = p^k, k \geq 3, |u| = p$ and $\nu^u = \nu^s$ where $s = 1 + p^{k-1}$.
- (c) $P = (\langle c \rangle \times \langle \nu \rangle) \lambda \langle u \rangle$ where $|c| = |\nu| = |u| = p > 2$ and $[\nu, u] = c, [c, u] = 1$.
- (d) $P = (\langle z \rangle \times \langle a \rangle) \lambda \langle u \rangle$ where $|a| = |u| = p, |z| > p, [z, u] = 1, [a, u] = c$ where $\langle c \rangle = \Omega_1(Z)$.
- (e) $P = QD, [Q, D] = \langle 1 \rangle$ where Q is a quaternion group of order 8, D is a dihedral group of order 8 and $Q \cap D = \zeta(Q) = \zeta(D)$.

Conversely, in every of these above listed groups each subgroup is malonormal.

Proof. Let A be an arbitrary maximal normal abelian subgroup of P . If $\text{sr}_p(A) \geq 3$, then Lemma 2.10 implies that every subgroup of P is normal. Therefore suppose that $\text{sr}_p(A) \leq 2$ for every maximal normal abelian subgroup A of P . It follows that $A = \langle a_1 \rangle \times \langle a_2 \rangle$. If $|a_1|, |a_2| \geq p^2$, then using Lemma 2.11 we obtain again that every subgroup of P is normal in G .

Suppose now that $|a_1| = p^n \geq p^2, |a_2| = p$.

Let a be an element of A and $|a| \geq p^2$. Then $a = a_1^t a_2^s$ where $t = p^k t_1, k \leq n - 2, 0 \leq s < p$. Then $a^p = (a_1^t a_2^s)^p = a_1^{tp}$. It follows that $\langle a \rangle \geq \Omega_1(\langle a_1 \rangle)$. Assume that the subgroup $\langle a \rangle$ is not normal in G and choose an element $x \in G$ such that $\langle a \rangle^x = \langle a^x \rangle \neq \langle a \rangle$. Then $|a^x| = |a|$ and therefore $\langle a \rangle^x \geq \Omega_1(\langle a_1 \rangle)$. But in this case $\langle 1 \rangle \neq \langle a \rangle^x \cap \langle a \rangle$. This contradiction shows that the subgroup $\langle a \rangle$ must be normal in P . In particular, the subgroup $\langle a \rangle$ is normal in P . It follows that $\langle c_1 \rangle = \Omega_1(\langle a_1 \rangle)$ also is normal in P . Corollary 2.2 shows that the factor-group $P/\langle c_1 \rangle$ is Dedekind.

Suppose that $|a_1| = |a_2| = p$. Since A is a normal subgroup of a p -group P , then $A \cap \zeta(P) \neq \langle 1 \rangle$. In this case, A includes a P -invariant subgroup $\langle c_1 \rangle$, having order p . Using again Corollary 2.2 we obtain that the factor-group $P/\langle c_1 \rangle$ is Dedekind.

Finally, if A is cyclic, then $\Omega_1(A) = \langle c_1 \rangle$ is a P -invariant subgroup of order p , and again the factor-group $P/\langle c_1 \rangle$ is Dedekind.

If every proper subgroup of P is cyclic, then P is a quaternion group of order 8 (see, for example, [3, §1, Exercise 2]). In particular, P is a Dedekind group.

Suppose now that P includes a non-cyclic subgroups and let H be an arbitrary non-cyclic subgroup of P . If $c_1 \in H$, then the fact that $P/\langle c_1 \rangle$ is a Dedekind group implies that H is normal in P . Suppose that $c_1 \notin H$. Then $\langle H, c_1 \rangle = H \times \langle c_1 \rangle$, in particular, H is maximal in $H\langle c_1 \rangle$. Since H is not cyclic, $|H| \geq p^2$. It follows that $|H\langle c_1 \rangle| \geq p^3$. Using Corollary 2.5, we obtain that the subgroup H is normal in P . Thus every non-cyclic subgroup of P is normal in P . Finite p -groups with this property have been described by F.N. Liman [16], [17]. One can find the description of these groups in the book [3, Theorem 16.2]). We use this description.

Suppose that P is a group of type (i) of Theorem 16.2 of the book [3]. Then either $P = \langle v \rangle \lambda \langle u \rangle$ where $|v| = p^k, |u| = p^t, k \geq 2, t \geq 1$, and $v^u = v^s$ where $s = 1 + p^{k-1}$; or $P = (\langle c \rangle \times \langle v \rangle) \lambda \langle u \rangle$ where $|v| = p^k, |u| = p^t, |c| = p, [v, u] = c, [u, c] = 1$, and if $p = 2$, then $k + t > 2$ (see, for example, [3, §1, Exercise 8a]). In the first group the subgroup $\langle u \rangle$ is not normal. Then Lemma 2.1 shows that $\langle u \rangle$ does not include proper P -invariant subgroups. It follows that $u^p = 1$. In particular, if $p = 2, k = 2$, then P is a dihedral group of order 8. Thus we obtain a group of type (b).

Consider the second case. If $t \geq 2$, then $[v, u^p] = 1$, so that P has an abelian subgroup, having section p -rank 3. Thus $|u| = p$. If we suppose that $k \geq 2$, then $[v^p, u] = 1$, and again P has an abelian subgroup, having section p -rank 3. Thus $|u| = |v| = p$. In particular, it follows that $p > 2$, and we obtain a group of type (c).

Let P be a group of type (ii) of Theorem 16.2 of the book [3]. Then $P = RZ$, where R is a non-abelian subgroup of order $p^3, Z = \langle z \rangle$ is cyclic, $[R, Z] = \langle 1 \rangle$, and $R \cap Z = \zeta(R)$. Moreover, if $p = 2$, then $|Z| > 2$. Thus if $p > 2$ and $|Z| = p$, then $P = R$ is a non-abelian group of order p^3 . In this case, every proper subgroup of P is abelian. It was considered above. Therefore we assume that $|Z| > p$. Since R is a non-abelian group of order p^3 , then either $R = \langle v \rangle \lambda \langle u \rangle$ where $|v| = p^2, |u| = p$ and $v^u = v^{1+p}$, or $R = (\langle c \rangle \times \langle v \rangle) \lambda \langle u \rangle$ where $|v| = |u| = |c| = p, [v, u] = c, [u, c] = 1$ and $p > 2$.

In the first case, we consider the subgroup $\langle v, z \rangle$. It is abelian and the equality $\langle v \rangle \cap Z = \langle v^p \rangle$ implies that $\langle v, z \rangle / \langle v^p \rangle$ is a direct product of two cyclic subgroups. In particular, $\langle v, z \rangle$ is not cyclic. We have $|z| \geq p^2 = |v|$. It follows that $\langle z \rangle$ is a cyclic subgroup of $\langle v, z \rangle$ of maximal possible order. In this case, $\langle v, z \rangle = \langle z \rangle \times \langle a \rangle$ for some element $a \in \langle v, z \rangle$. We note that $|a| = p$. The equalities $R \cap Z = \langle v^p \rangle$ and $\langle v \rangle \cap \langle u \rangle = \langle 1 \rangle$ imply that $\langle v, z \rangle \cap \langle u \rangle = \langle 1 \rangle$, so we obtain that $P = \langle v, z \rangle \lambda \langle u \rangle = (\langle a \rangle \times \langle z \rangle) \lambda \langle u \rangle$. Furthermore, $a = v^k z^t$ where $\text{GCD}(k, p) = 1$, because $[v, u] \neq 1$, but $[v^p, u] = 1$. It follows that $[a, u] = [v^k, u] = c \neq 1$ and $\langle c \rangle = \langle v^p \rangle = \Omega_1(Z)$. Thus we obtain a group of type (d).

Consider the second case. In this case, $Z \cap R = \langle c \rangle$, so that $\langle z, v \rangle = \langle z \rangle \times \langle v \rangle$, and again $P = (\langle z \rangle \times \langle v \rangle) \lambda \langle u \rangle, [v, u] = c$ where $\langle c \rangle = \Omega_1(Z)$. Thus we come again to a group of type (d).

Let P be a group of type (iii) of Theorem 16.2 of the book [3]. Then $P = Q \times Z$, where Q is a quaternion group of order 8 and Z is a cyclic 2-subgroup of order 2^k where $k \geq 2$. In this case, P includes the subgroup $\langle v \rangle \times \langle z \rangle$ where $|v| = 2^2, |z| = 2^k$ and $k \geq 2$. As we have seen above, in this situation every subgroup of P is normal in G .

Let P be a group of type (iv) of Theorem 16.2 of the book [3]. Then P is a group of order 3^4 and of maximal class (i.e. a nilpotency class of P is 3) with $\Omega_1(P) = [P, P]$ and the latter is an elementary

abelian subgroup of order 3^2 . However, we have already proved above that P includes a normal cyclic subgroup $\langle c_1 \rangle$ of prime order such that the factor-group $P/\langle c_1 \rangle$ is Dedekind. Since G is a 3-group, we obtain that $P/\langle c_1 \rangle$ is abelian. On the other hand, the center of P includes $\langle c_1 \rangle$, so that nilpotency class of P is 2. It shows that a group of this type includes a subgroup which is not malonormal.

Let P be a group of type (v) of Theorem 16.2 of the book [3]. Then $P = \langle u, v | u^8 = v^8 = 1, v^u = v^{-1}, v^4 = u^4 \rangle$ where $|P| = 2^5$, $[P, P]$ and $\zeta(P)$ are cyclic subgroups of order 4, $[P, P] \cap \zeta(P)$ has order 2, $\Omega_2(P) = \langle x | x^4 = 1 \rangle$ is a direct product of cyclic subgroups of order 4 and a subgroup of order 2. In this case, we have $[v, u] = v^{-1}u^{-1}vu = v^{-1}v^{-1} = v^{-2}$. It follows that $v^{-1}u^{-1}v = v^{-2}u^{-1}$, hence $v^{-1}uv = uv^2$. Then the subgroup $\langle u \rangle^v$ contains an element uv^2 (in particular, it shows that $\langle u \rangle^v \neq \langle u \rangle$), and therefore $\langle u \rangle^v$ contains an element $uv^2uv^2 = uu^{-2}v^2 = u^2$. It follows that $\langle u \rangle^v \cap \langle u \rangle$ contains an element $u^2 \neq 1$, which shows that the subgroup $\langle u \rangle$ cannot be malonormal.

Let P be a group of type (vi) of Theorem 16.2 of a book [3]. Then P is a generalized quaternion group of order 2^4 , that is $P = \langle u, v | v^8 = u^4 = 1, v^4 = u^2 = c \neq 1, c^2 = 1, v^u = v^{-1} \rangle$. The subgroup $\langle u \rangle$ is not normal, but $\langle u \rangle$ includes a normal subgroup $\langle c \rangle$. Thus a subgroup $\langle u \rangle$ cannot be malonormal.

Let P be a group of type (vii) of Theorem 16.2 of the book [3]. Then $P = QD$, $[Q, D] = \langle 1 \rangle$ where Q is a quaternion group of order 8, D is a dihedral group of order 8, and $Q \cap D = \zeta(Q) = \zeta(D)$. We have $Q = \langle u, v | u^4 = v^4 = 1, v^u = v^{-1}, v^2 = u^2 = c \neq 1 \rangle$ and $D = \langle d, b | d^4 = b^2 = 1, d^b = d^{-1} \rangle$. In this situation $\langle c \rangle = \zeta(P) = [P, P]$ and $P/\langle c \rangle$ is an elementary abelian group of order 2^4 . If x is an element of order 4, then $1 \neq x^2 \in \langle c \rangle$. Since $\langle c \rangle = \{1, c\}$, $x^2 = c$. It follows that $c \in \langle x \rangle$, so that $\langle x \rangle$ is a normal subgroup of P . Suppose now that $|x| = 2$ and $x \neq c$. The subgroup $\langle x \rangle$ is not normal in P , otherwise $\langle x \rangle \leq \zeta(P)$ and $\zeta(P) > \langle c \rangle$, so we obtain a contradiction. Let $y \notin N_P(\langle x \rangle)$. The fact that $P/\langle c \rangle$ is abelian implies that $x \neq x^y \in x\langle c \rangle$. This means that $x^y = xc$. In this case, $\langle x^y \rangle \cap \langle x \rangle = \langle xc \rangle \cap \langle x \rangle = \langle 1 \rangle$. Thus every cyclic subgroup of P is malonormal in P . On the other hand, by Theorem 16.2 of the book [3] every non-cyclic subgroup of P is normal. Consequently every subgroup of P is malonormal, so we come to a group of type (e).

Let P be a group of type (viii) of Theorem 16.2 of the book [3]. Then $P = (\langle u \rangle \times \langle v \rangle)\langle c \rangle$, where $u^4 = v^4 = 1, v^2 = c^2, v^c = vu^2, u^c = uv^2$. In this case, P includes a direct product of two subgroups of order 2^2 . As we have proved above, in this case, every subgroup of P is normal in G .

Let P be a group of type (ix) of Theorem 16.2 of the book [3]. Then this group includes a maximal subgroup isomorphic to a group of type (viii) of Theorem 16.2 of the book [3]. As we have seen, in this case, every subgroup of P is normal in G . \square

Lemma 2.13. *Let G be a finite group and let P be a Sylow p -subgroup of G , where p is a prime. Suppose that P is not normal in G . If every subgroup of G is malonormal, then $G = Q\lambda K$ where K is a Frobenius complement. Moreover, K is a Dedekind group and $K = N_G(P)$.*

Proof. Since P is not normal in G , $K = N_G(P) \neq G$. Since P is a Sylow p -subgroup of G , $N_G(K) = K$. In this case, K is a malnormal subgroup of G . Then G is a Frobenius group with K as a Frobenius

complement, that is $G = Q\lambda K$ (see, for example, [8, Chapter 2 Theorem 7.6]). The fact that $K \neq G$ implies that Q is non-trivial. Then Corollary 2.2 implies that $G/R \cong K$ is a Dedekind group. \square

Corollary 2.14. *Let G be a finite group, whose subgroups are malonormal. If G is non-nilpotent, then $G = Q\lambda K$, and the following conditions hold:*

- (i) Q is an abelian Sylow q -subgroup of G where q is a prime.
- (ii) K is a Dedekind group, moreover, either K is cyclic or $K = D \times S$ where D is a quaternion group of order 8 and S is a cyclic Sylow 2-subgroup of K .
- (iii) $C_Q(x) = \langle 1 \rangle$ for each $1 \neq x \in K$.

Proof. Since G is not nilpotent, there exists a prime p such that the Sylow p -subgroup P of G is not normal in G . By Lemma 2.13, $G = Q\lambda K$ is a Frobenius group with the Frobenius complement $K = N_G(P)$. We note that a Sylow r -subgroup of the Frobenius complement is cyclic whenever $r \neq 2$, and it is cyclic or generalized quaternion whenever $r = 2$ (see, for example, [8, Chapter 10 Theorem 3.1]). The fact that $K \neq G$ implies that Q is non-trivial. Then Corollary 2.2 implies that $G/Q \cong K$ is a Dedekind group. In particular, if a Sylow 2-subgroup of K is not cyclic, then being generalized quaternion and Dedekind, it is a quaternion group of order 8. Thus we obtain that either K is cyclic or $K = D \times S$ where D is a quaternion group of order 8 and S is a cyclic 2-subgroup.

Note that a Frobenius kernel Q is nilpotent (see, for example, [8, Chapter 10 Theorem 3.1]). Suppose that $\Pi(Q)$ contains two different primes q_1 and q_2 . Let S_j be a Sylow q_j -subgroup of Q , $j \in \{1, 2\}$. Clearly S_j is normal in G . Since S_j is non-trivial, Corollary 2.2 shows that G/S_j is a Dedekind group. In particular, it is nilpotent. The equality $S_1 \cap S_2 = \langle 1 \rangle$ together with Remak's theorem imply an imbedding G in $G/S_1 \times G/S_2$, which shows that G is nilpotent, and we obtain a contradiction. This contradiction proves that Q is a q -subgroup for some prime q .

For each $1 \neq x \in K$ we have $C_Q(x) = \langle 1 \rangle$ (see, for example, [8, Chapter 10 Theorem 3.1]). In particular, it follows that $q \notin \Pi(K)$, so that Q is a Sylow q -subgroup of G . Suppose that Q is non-abelian. Then $\zeta(Q)$ is a non-trivial proper G -invariant subgroup of Q . By Corollary 2.2, the factor-group $G/\zeta(Q)$ is Dedekind. It follows that $[Q, K] \leq \zeta(Q)$, in particular, $[Q, K] \neq Q$. On the other hand, $q \notin \Pi(K)$, so that $Q = [Q, K]C_Q(K)$ (see, for example, [8, Chapter 5 Theorem 3.5]). From last equality it follows that $C_Q(K) \neq \langle 1 \rangle$, and we obtain a contradiction. This contradiction proves that Q is abelian. \square

Lemma 2.15. *Let G be a finite group, whose all subgroups are malonormal. Suppose that G is not nilpotent. Then G is a group of one of following types:*

- (i) $G = Q\lambda K$ where Q is an elementary abelian q -subgroup of order q^2 , q is a prime, K is a cyclic q' -subgroup, $2 \notin \Pi(K)$, Q does not include a non-trivial cyclic $\langle y \rangle$ -invariant subgroup for each $y \in K \setminus \{1\}$.
- (ii) $G = Q\lambda K$ where $|Q| = q$ is prime, $Q = C_G(Q)$, K is a cyclic subgroup of order dividing $q - 1$.

Proof. Corollary 2.14 shows that $G = Q\lambda K$ is a Frobenius group with a Frobenius complement K and a Frobenius kernel Q . By this Corollary, Q is an abelian Sylow q -subgroup of G . Suppose that $\text{sr}_q(Q) \geq 3$. Then Lemma 2.10 implies that every subgroup of Q is normal in G . In this case, we have $Q = \mathbf{Dr}_{1 \leq j \leq n} \langle a_j \rangle$ where $n \geq 3$. By above noted, every subgroup $\langle a_j \rangle$ is normal in G , and Corollary 2.2 implies that $G/\langle a_j \rangle$ is a Dedekind group, $1 \leq j \leq n$. In particular, it is nilpotent. From $\langle 1 \rangle = \langle a_1 \rangle \cap \langle a_2 \rangle$ and Remak's theorem we obtain that G is isomorphic to some subgroup of $G/\langle a_1 \rangle \times G/\langle a_2 \rangle$, so that G is nilpotent, and we obtain a contradiction. This contradiction shows that $\text{sr}_q(Q) \leq 2$.

Suppose now that $Q = \langle a_1 \rangle \times \langle a_2 \rangle$, where $|a_1|, |a_2| \geq q^2$. In this case, using Lemma 2.11 we obtain that every subgroup of Q is normal in G , and repeating the above arguments again we obtain a contradiction.

Suppose now that $Q = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_1| > q, |a_2| = q$. Suppose that the subgroup $\langle a_1 \rangle$ is not normal in G and choose an element $x \notin N_G(\langle a_1 \rangle)$. Since Q is normal in $G, a_1^x \in Q$. We have $|a_1^x| = |a_1| > q$, and therefore $a_1^x = a_1^k a_2^s$ where $\mathbf{GCD}(k, q) = 1, 0 \leq s < q$. It follows that $1 \neq (a_1^x)^q = (a_1^k a_2^s)^q = a_1^{kq} a_2^{sq}$. This means that $\langle a_1^q \rangle \leq \langle a_1^x \rangle = \langle a_1 \rangle^x$ and hence $\langle a_1 \rangle \cap \langle a_1 \rangle^x \neq \langle 1 \rangle$. This contradiction shows that $\langle a_1 \rangle$ is normal in G . Taking into account that $q \notin \Pi(K)$, we obtain that $Q = \langle a_1 \rangle \times \langle b_1 \rangle$ where the subgroup $\langle b_1 \rangle$ is G -invariant (see, for example, [15, Corollary 5.13]). Using the above arguments, we again obtain a contradiction. This contradiction shows that $Q = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_1| = |a_2| = q$.

Assume that K has an element $y \neq 1$ such that Q includes a non-trivial $\langle y \rangle$ -invariant cyclic subgroups $\langle c_1 \rangle$. Suppose that $\langle c_1, y \rangle$ is non-normal in G and choose an element $z \notin N_G(\langle c_1, y \rangle)$. The fact that G/Q is a Dedekind group implies that a subgroup $\langle y \rangle Q$ is normal in G . It follows that $\langle c_1, y \rangle^z \leq Q \langle y \rangle$. We have $q = |Q \langle y \rangle : \langle c_1, y \rangle|$. Thus

$$q = |\langle Q, y \rangle^z : \langle c_1, y \rangle^z| = |\langle Q, y \rangle : \langle c_1, y \rangle^x|,$$

which shows that $\langle c_1, y \rangle \cap \langle c_1, y \rangle^z \neq \langle 1 \rangle$, and we obtain a contradiction. This contradiction proves that a subgroup $\langle c_1, y \rangle$ must be normal in G .

Since Q is elementary abelian, $\langle c_1 \rangle$ has a complement in Q . The fact that $\mathbf{GCD}(q, |y|) = 1$ implies that Q includes an $\langle y \rangle$ -invariant subgroup $\langle c_2 \rangle$ such that $Q = \langle c_1 \rangle \times \langle c_2 \rangle$ (see, for example, [15, Corollary 5.13]). Using the above arguments, we obtain that the subgroup $\langle c_2, y \rangle$ is normal in G . It follows that $\langle y \rangle = \langle c_1, y \rangle \cap \langle c_2, y \rangle$ is normal in G . By Corollary 2.2, the factor-group $G/\langle y \rangle$ is Dedekind. In particular, it is nilpotent. From $\langle 1 \rangle = Q \cap \langle y \rangle$ and Remak's theorem it follows that G is isomorphic to some subgroup of $G/Q \times G/\langle y \rangle$, so that G is nilpotent, and we obtain a contradiction. This contradiction shows that Q does not include non-trivial $\langle y \rangle$ -invariant cyclic subgroups for each element $1 \neq y \in K$.

Suppose now that $2 \in \Pi(K)$ and let d be an element of K of order 2. For an element $1 \neq a \in Q$ we have only the following options: $a^d \in \langle a \rangle$ or $\langle a^d \rangle \cap \langle a \rangle = \langle 1 \rangle$. By above proved first case is impossible. In the second case a $a^d \neq 1$ and $d^{-1}(a d^{-1} a d) d = (d^{-1} a d)(d^{-2} a d^2) = a d^{-1} a d$, and we again obtain a

contradiction. This contradiction shows that $2 \notin \Pi(K)$. Corollary 2.14 implies that K is cyclic. Thus we come to a group of the type (i).

Finally, suppose that Q is cyclic, $Q = \langle a \rangle$. Also suppose that $|a| > p$. Then $a^p \neq 1$. By Corollary 2.2, the factor-group $G/\langle a^p \rangle$ is Dedekind. In particular, it is nilpotent. It follows that $[Q, K] \leq \langle a^p \rangle$. In particular, $[Q, K] \neq Q$. On the other hand, $q \notin \Pi(K)$, so that $Q = [Q, K] \times C_Q(K)$ (see, for example, [8, Chapter 5 Theorem 2.3]). From the last equality it follows that $C_Q(K) \neq \langle 1 \rangle$, and we obtain a contradiction. This contradiction proves that Q has order q . Thus we come to a group of the type (ii). □

Corollary 2.16. *Let G be a finite group, whose subgroups are malonormal. If G includes an elementary abelian p -subgroup of order p^3 for some prime p , then G is nilpotent.*

Lemma 2.17. *Let G be a periodic locally nilpotent group, whose subgroups are malonormal. If $\Pi(G)$ contains two different primes, then G is a Dedekind group.*

Proof. Let p be the least prime from the set $\Pi(G)$. Then $G = P \times Q$ where P is a Sylow p -subgroup of G and Q is a Sylow p' -subgroup of G . By Corollary 2.2, $G/P \cong Q$ is a Dedekind group. The choice of p shows that $2 \notin \Pi(G/P)$, hence Q is abelian. Corollary 2.2 shows that $P = G/Q$ is a Dedekind p -group. It follows that G is a Dedekind group. □

Corollary 2.18. *Let G be a group, whose subgroups are malonormal. Suppose that G includes a normal periodic locally nilpotent subgroup K . If $\Pi(K)$ contains two different primes, then G is a Dedekind group.*

Proof. Let p be the least prime from a set $\Pi(G)$. Then $G = P \times Q$ where P is a Sylow p -subgroup of G and Q is a Sylow p' -subgroup of G . Clearly P and Q are G -invariant. By Corollary 2.2, G/P and G/Q are Dedekind groups. If G is not periodic, then the both these factor-groups are abelian. The equality $P \cap Q = \langle 1 \rangle$ together with Remak’s theorem imply that G is abelian. If G is periodic, then G/P and G/Q are nilpotent of nilpotency class at most 2. The equality $P \cap Q = \langle 1 \rangle$ together with Remak’s theorem imply that G is nilpotent, and we can apply Lemma 2.17. □

Proof of Theorem 1.1. If G is a p -group for some prime p , then Lemma 2.12 shows that G is a group of one of the types (i)-(v). Suppose that G is nilpotent and $|\Pi(G)| \geq 2$. Then Lemma 2.17 implies that G is a Dedekind group.

If G is not nilpotent, then Lemma 2.15 shows that G is a group of the types (vi) or (vii).

3. Infinite groups, whose subgroups are malonormal

Lemma 3.1. *Let P be an infinite locally finite p -group, whose subgroups are malonormal, p is a prime. Then either P is a Dedekind group or $P = (K \times \langle c \rangle)\lambda\langle b \rangle$ where $K = \langle a_n | a_1^p = 1, a_{n+1}^p = a_n, n \in \mathbf{N} \rangle$ is a quasicyclic p -subgroup, $c^p = b^p = 1, [K, b] = \langle 1 \rangle, [b, c] = a_1$. Conversely, in every of these groups each subgroup is malonormal.*

Proof. Let K be an arbitrary finite subgroup of P . Choose in the center of K an element c of order p . By Corollary 2.2, the factor-group $K/\langle c \rangle$ is Dedekind. In particular, it is nilpotent of nilpotency class at most 2. Thus K is nilpotent of nilpotency class at most 3. Since it is true for every finitely generated subgroup of P , P is also nilpotent of nilpotency class at most 3. In particular, $\zeta(P) \neq \langle 1 \rangle$. Choose in $\zeta(P)$ an element c_1 having order p . Suppose that P includes a finite abelian subgroup A such that $\text{sr}_p(A) \geq 3$. If $c_1 \notin A$, then instead of A we consider a subgroup $\langle A, c_1 \rangle$. Therefore without loss of generality we can suppose that $c_1 \in A$. By Corollary 2.2, the factor-group $P/\langle c_1 \rangle$ is Dedekind. It follows that the subgroup A is normal in P . Let g and x be arbitrary elements of P . By Lemma 2.10, every subgroup of $\langle A, g, x \rangle$ is normal, in particular, $x^{-1}\langle g \rangle x = \langle g \rangle$. Since it is true for each $x \in P$, the subgroup $\langle g \rangle$ is normal in P . Thus every cyclic subgroup of P is normal in P . It follows that P is a Dedekind group.

Assume that $P/\langle c_1 \rangle$ is non-abelian. Then $P/\langle c_1 \rangle = Q/\langle c_1 \rangle \times E/\langle c_1 \rangle$ where $Q/\langle c_1 \rangle$ is a quaternion group of order 8, and $E/\langle c_1 \rangle$ is an elementary abelian 2-group. Since P is infinite, $E/\langle c_1 \rangle$ is infinite. But in this case, P includes an infinite elementary abelian subgroup. By above proved P is a Dedekind group.

Suppose that P includes an abelian subgroup $A = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_1|, |a_2| > p$. As above without loss of generality we can suppose that $c_1 \in A$. By Corollary 2.2, the factor-group $P/\langle c_1 \rangle$ is Dedekind. It follows that the subgroup A is normal in P . Let g and x be arbitrary elements of P . By Lemma 2.11, every subgroup of $\langle A, g, x \rangle$ is normal; in particular, $x^{-1}\langle g \rangle x = \langle g \rangle$. Since it is true for each $x \in P$, the subgroup $\langle g \rangle$ is normal in P . Thus every cyclic subgroup of P is normal in P . It follows that P is a Dedekind group.

Suppose now that every finite abelian subgroup A of P has a form $A = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_2| = p$. Furthermore, we assume also that $P/\langle c_1 \rangle$ is abelian. If we suppose that every proper subgroup of P is cyclic, then, being locally finite, P must be quasicyclic [13], [11]. In particular, it is abelian. Therefore we can assume that P includes non-cyclic subgroups. As in the proof of Lemma 2.12, we can show that every non-cyclic subgroup of P is normal in P . Locally finite p -groups with this property have been described by F.N. Liman [16], [17].

Suppose that $p = 2$. We note that only groups of type (4) and (10) of the main theorem of the paper [16] are infinite. Let P be a group of type (4), then $P = (K \times \langle c \rangle)\lambda\langle b \rangle$ where $K = \langle a_n | a_1^2 = 1, a_{n+1}^2 = a_n, n \in \mathbf{N} \rangle$ is a quasicyclic 2-subgroup, $c^2 = b^2 = 1$, $[K, b] = \langle 1 \rangle$, $[b, c] = a_1$. If x is an element of order 2^k , $k \geq 2$, then $x = a_t c^m b^s$ where $t \geq 2, m, s \in \{0, 1\}$. If $m = s = 1$, then $1 \neq x^2 = a_t c b a_t c b = a_{t+2} a_1 c^2 b^2 = a_{t-1} a_1 \in K$. If $m = 0, s = 1$, then $1 \neq x^2 = a_t b a_t b = a_t^2 b^2 = a_{t-1} \in K$; if $m = 1, s = 0$, then $1 \neq x^2 = a_t c : a_t c = a_t^2 c^2 = a_{t-1} \in K$; if $m = s = 0$, then $1 \neq x^2 = a_t a_t = a_{t-1} \in K$. Thus in every case $a_1 \in \langle x \rangle$. Since $P/\langle a_1 \rangle$ is abelian, $\langle x \rangle$ is normal in P . Suppose now that $|x| = 2$. Then $x = a_{1r} c^m b^s$ where $r, m, s \in \{0, 1\}$. If $m = s = 0$, then $\langle x \rangle = \langle a_1 \rangle$ is normal in P . Otherwise the subgroup $\langle x \rangle$ is not normal in P . Let $y \notin N_P(\langle x \rangle)$. The fact that $P/\langle a_1 \rangle$ is abelian implies that $x \neq x^y \in x\langle a_1 \rangle$. This means that $x^y = x a_1$. In this case, $\langle x^y \rangle \cap \langle x \rangle = \langle x a_1 \rangle \cap \langle x \rangle = \langle 1 \rangle$. Thus every

cyclic subgroup of P is malonormal in P . On the other hand, by the main theorem of the paper [16], each non-cyclic subgroup of P is normal, and therefore malonormal.

Let P be a group of type (10), then $P = K \times Q$ where K is a quasicyclic 2-subgroup, Q is a quaternion group of order 8. This group includes an abelian subgroup $A = \langle a_1 \rangle \times \langle a_2 \rangle$ where $|a_1| = |a_2| = 4$. As we have seen above, in this case, P must be a Dedekind group.

Suppose first that $p > 2$. Then Theorem 2.1 of the paper [17] shows that $P = (K \times \langle c \rangle) \lambda \langle b \rangle$ where $K = \langle a_n | a_{1^p} = 1, a_{n+1^p} = a_n, n \in \mathbf{N} \rangle$ is a quasicyclic p -subgroup, $c^p = b^p = 1, [K, b] = \langle 1 \rangle, [b, c] = a_1$. Repeating the above arguments, we prove that in this group every subgroup is malonormal. \square

Lemma 3.2. *Let G be an infinite locally finite group, whose subgroups are malonormal. If $\Pi(G)$ contains two different primes, then G is a Dedekind group.*

Proof. If G is locally nilpotent, then Lemma 2.17 shows that G is a Dedekind group. Suppose that G is not locally nilpotent. Then G includes a finite subgroup F which is not nilpotent. Let \mathfrak{L} be the family of all finite subgroup of G , including F . If F is a group of type (i) of Lemma 2.15, then every finite subgroup K , including F , cannot be a group of type (ii) of Lemma 2.15. Thus K has a normal Sylow q -subgroup Q , which is elementary abelian of order q^2 . Moreover, $Q = C_K(Q)$. It follows that $|K/Q| \leq q(q - 1)(q^2 - 1)$ (see, for example, [8, Chapter 2 Theorem 8.1]). It follows that $|K| \leq q^3(q - 1)(q^2 - 1)$. Since it is true for each $K \in \mathfrak{L}$, G cannot be infinite, and we obtain a contradiction. If F is a group of type (ii) of Lemma 2.15, then an application of similar arguments also bring us to a contradiction. These contradictions show that G must be locally nilpotent.

If G is a group, then denote by $\mathbf{Tor}(G)$ a maximal normal periodic subgroup of G . Note that if G is locally nilpotent, then $\mathbf{Tor}(G)$ contains all elements, having finite order. \square

Lemma 3.3. *Let G be an infinite residually finite group, whose subgroups are malonormal. Then G is nilpotent of nilpotency class at most 2.*

Proof. Being residually finite, G has a family \mathfrak{R} of normal subgroups, having finite index, such that $\cap \mathfrak{R} = \langle 1 \rangle$. If H is a normal subgroup, having finite index, then H is non-trivial and Corollary 2.2 implies that G/H is a Dedekind group. In particular, G/H is nilpotent of nilpotency class at most 2. The equality $\cap \mathfrak{R} = \langle 1 \rangle$ together with Remak’s theorem imply that G is nilpotent of nilpotency class at most 2. \square

Proof of Theorem 1.2. Suppose first that G is a locally finite group. If the set $\Pi(G)$ contains two different primes, then Lemma 3.2 implies that G is a Dedekind group. If G is a p -group for some prime p , then we can apply Lemma 3.1.

Suppose now that G is locally graded group and let K be a finitely generated subgroup of G . Suppose that K is infinite. Denote by \mathfrak{R} the family of normal subgroups of K having in K finite index. If $H \in \mathfrak{R}$, then, by our assumption, H is non-trivial and Corollary 2.2 implies that K/H is a Dedekind group. Let $R = \cap_{H \in \mathfrak{R}} H$. If $R = \langle 1 \rangle$, then Remak’s theorem imply that K is nilpotent of nilpotency class at most 2. Being periodic and finitely generated, K is finite, and we obtain a contradiction

with our assumption. Thus $R \neq \langle 1 \rangle$. Finiteness of K/R implies that R is finitely generated (see, for example, [5, Proposition 1.2.13]). Being locally graded, R includes a proper subgroup S having finite index in R . Then $|K : S|$ is also finite. It follows that S includes R , and again we obtain a contradiction. This contradiction proves that K is finite. Then G is locally finite, and this finish the proof.

Lemma 3.4. *Let G be a finitely generated nilpotent group, whose subgroups are malonormal. If G is infinite, then G is abelian.*

Proof. Suppose the contrary, let G be non-abelian. Being infinite, G is non-periodic. Let $T = \mathbf{Tor}(G)$, then T is finite and there is a normal subgroup K of G such that $K \cap T = \langle 1 \rangle$ and G/K is finite. By Corollary 2.2, G/T is a Dedekind group. Being not periodic, G/T is abelian. It follows that K is abelian. Finiteness of G/K implies that K is finitely generated (see, for example, [5, Proposition 1.2.13]). Then $K \neq K^8$. The subgroup K^8 is G -invariant and non-trivial. Corollary 2.2 implies that G/K^8 is a Dedekind group. Since this factor-group contains some elements of order 8, it is abelian. The equality $T \cap K^8 = \langle 1 \rangle$ together with Remak's theorem imply that G is abelian. \square

Corollary 3.5. *Let G be a locally nilpotent group, whose subgroups are malonormal. If G is not periodic, then G is abelian.*

Proof. Being not periodic, G has a finitely generated subgroup K , which is not periodic. Let x, y be the arbitrary elements of G . Then the subgroup $\langle x, y, K \rangle$ is not periodic, and Lemma 3.4 shows that it is abelian. In particular, $xy = yx$. Thus G is abelian. \square

Corollary 3.6. *Let G be an infinite finitely generated group, whose subgroups are malonormal. If G is residually finite, then G is abelian.*

Proof. By Lemma 3.3 G is nilpotent, and we can apply Lemma 3.4. \square

Corollary 3.7. *Let G be a residually finite group, whose subgroups are malonormal. If G is non-periodic, then G is abelian.*

Corollary 3.8. *Let G be an infinite polycyclic-by-finite group, whose subgroups are malonormal. Then G is abelian.*

Proof. Indeed, a polycyclic-by-finite group is finitely generated and residually finite, and therefore we may apply Corollary 3.6. \square

Recall that the group G has **finite special rank** $\mathbf{r}(G) = \mathbf{r}$, if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property.

Proof of Theorem 1.3. Since G is not periodic, G contains an element g , having infinite order. Let K be an arbitrary finitely generated subgroup of G containing g . Then K is infinite. Let $K = \langle g, g_1, \dots, g_n \rangle$. Denote by \mathfrak{A} the family of normal subgroups of K , having in K finite index. If $H \in \mathfrak{A}$,

then, by our assumption, H is non-trivial and Corollary 2.2 implies that K/H is a Dedekind group. If K/H is abelian, then $\mathbf{r}(K/H) \leq n + 1$. If K/H is non-abelian, then K/H is a direct product of a quaternion group and abelian group, so that $\mathbf{r}(K/H) \leq n + 2$. Thus the special ranks of finite factor-groups of K are bounded. Let L be a subgroup of K , and x be an element of K such that $L^x \leq L$. If we suppose that $x \notin N_K(L)$, then $L \cap L^x = \langle 1 \rangle$. It shows that $x \in N_K(L)$, so that $L^x = L$. Using now Theorem B of the paper [19], we obtain that K is polycyclic-by-finite. Corollary 3.8 implies that K is abelian. It follows that G is also abelian.

Lemma 3.9. *Let G be a finitely generated generalized radical group, whose subgroups are malnormal. If G is not periodic, then G is abelian.*

Proof. Let $T = \mathbf{Tor}(G)$ and suppose that T is non-trivial. Using Corollary 2.2 we obtain that G/T is a Dedekind group. Being non-periodic, G/T is abelian. If T is finite, then G is residually finite, and Corollary 3.6 shows that G is abelian.

Suppose that T is infinite. If $\Pi(T)$ contains two different primes, then Lemma 3.2 implies that T is a Dedekind group. Using now Corollary 2.18, we obtain that G is also Dedekind. Being non-periodic, G is abelian. Consider now the case, when T is a p -group for some prime p . If we suppose that T is non-abelian, then Lemma 3.1 shows that $[T, T]$ has order p and $T/[T, T]$ includes a quasicyclic subgroup. On the other hand, $G/[T, T]$ is metabelian and finitely generated, so that it satisfies the maximal condition on normal subgroups [9, Theorem 3], and we obtain a contradiction. This contradiction shows that T is abelian. In this case, G is residually finite [10, Theorem 1], and Corollary 3.6 proves that G is abelian. \square

Proof of Theorem 1.4. Since G is not periodic, G contains an element g , having infinite order. Let K be an arbitrary finitely generated subgroup of G , containing g . Then K is finitely generated, non-periodic and generalized radical. Using Lemma 3.9, we obtain that K is abelian. It follows that G is abelian.

REFERENCES

- [1] B. Baumslag, Generalized free products whose two-generator subgroups are free, *J. London Math. Soc.*, **43** (1968) 601–606.
- [2] R. Baer, Situation der Untergruppen und Struktur der Gruppe, *Sitz. Ber. Heidelberg Akad.*, **2** (1933) 12–17.
- [3] Ya. Berkovich, *Groups of prime power order*, **1**, Walter de Gruyter: Berlin-2008.
- [4] S. N. Chernikov, Infinite nonabelian groups with an invariance condition for infinite nonabelian subgroups, *Dokl. Akad. Nauk SSSR*, **194** (1970) 1280–1283.
- [5] M. R. Dixon, L. A. Kurdachenko and I. Ya. Subbotin, *Ranks of groups. The tools, characteristics and restrictions*, Wiley, New York, 2017.
- [6] P. Flavell, A note on Frobenius groups, *J. Algebra*, **228** (2000) 367–376.
- [7] L. Fuchs, *Infinite abelian groups*, **1**, Academic Press, New York, 1970.
- [8] D. Gorenstein, *Finite groups*, Chelsea Publ. Co., New York, 1980.

- [9] P. Hall *Finiteness conditions for soluble groups*, *Proc. London Math. Soc.*, **4** (1954) 419–436.
- [10] P. Hall, On the finiteness of certain soluble groups, *Proc. London Math. Soc.*, **9** (1959) 595–622.
- [11] P. Hall and C. R. Kulatilaka, A property of locally finite groups, *J. London Math. Soc.*, **39** (1964) 235–239.
- [12] P. De la Harpe and C. Weber, Malnormal subgroups and Frobenius groups: basics and examples, *Confluentes Math.*, **6** (2014) 65–76.
- [13] M. I. Kargapolov, On a problem of O. Ju. midt, *Sibirsk. Mat. Z.*, **4** (1963) 232–235.
- [14] V. V. Kirichenko, L. A. Kurdachenko and I. Ya. Subbotin, Some related to pronormality subgroup families and the properties of a group, *Algebra Discrete Math.*, **11** (2011) 75–108.
- [15] L. A. Kurdachenko, *Artinian modules over group rings*, Frontiers in Mathematics, Birkhuser Verlag, Basel, 2007.
- [16] F. N. Liman, 2-groups with normal non-cyclic subgroups, *Math. Notes*, **4** (1968) 535–539.
- [17] F. N. Liman, Periodic groups, whose abelian normal non-cyclic subgroups are invariant, In book “Groups with restrictions on subgroups”, Naukova Dumka, 1971 65–95.
- [18] A. Yu. Olshanskii, Infinite group with subgroups of prime orders, *Izv. Akad. Nauk SSSR Ser. Mat.*, **44** (1980) 309–321.
- [19] D. J. S. Robinson, a. Russo and G. Vincenzi, On groups which contain no HNN-extension, *Internat. J. Algebra Comput.*, **17** (2007) 1377–1387.

Leonid A. Kurdachenko

Department of Algebra, Dnipro National University, Gagarin Prospect 70, Dnipro 10 (Ukraine)

Email: lkurdachenko@i.ua

Nikolai N. Semko

Department of Mathematics, University of State Fiscal Service of Ukraine Irpin, 08200 (Ukraine)

Email: dr.mykola.semko@gmail.com

Igor Ya. Subbotin

Department of Mathematics, National University, 5245 Pacific Concourse Drive, Los Angeles, CA 90045 (USA)

Email: isubboti@nu.edu