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REDUCED ZERO-DIVISOR GRAPHS OF POSETS

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ABSTRACT. This paper investigates properties of the reduced zero-divisor graph of a poset. We show that a vertex is an annihilator prime ideal if and only if it is adjacent to all other annihilator prime ideals and there are always two annihilator prime ideals which are not adjacent to a non-annihilator prime ideal. We also classify all posets whose reduced zero-divisor graph is planar or toroidal and the number of distinct annihilator prime ideals is four or seven.

1. Introduction

Let Q be a non-empty subset of a poset P . If there exists $y \in Q$ such that $y \leq x$ for every $x \in Q$, then y is called the *least element* of Q . The least element of P , if exists, is usually denoted by 0 .

Let P be a poset with least element 0 . An element $x \in P$ is called a *zero-divisor* of P if there exists $y \in P^\times := P \setminus \{0\}$ such that the set $L(x, y) := \{z \in P \mid z \leq x \text{ and } z \leq y\} = \{0\}$. We denote the set of zero-divisors of P by $Z(P)$ and write $Z(P)^\times := Z(P) \setminus \{0\}$. Given $x \in P$, the *annihilator* of x in P is defined to be the set $\text{ann}(x) := \{y \in P \mid L(x, y) = \{0\}\}$. Given $x, y \in P$, set $x \sim y$ if $\text{ann}(x) = \text{ann}(y)$. Clearly, \sim is an equivalence relation in P . Let $[x]$ denote the equivalence class of $x \in P$. Observe that $[x] \subseteq Z(P)^\times$, $[y] = P \setminus Z(P)$ and $[0] = \{0\}$, for all $x \in Z(P)^\times$ and $y \in P \setminus Z(P)$. The *reduced zero-divisor graph* of P denoted by $\Gamma_E(P)$ is the graph with all equivalence classes of the elements of $Z(P)^\times$ as vertex set and two vertices $[x]$ and $[y]$ are adjacent if and only if $L(x, y) = \{0\}$. This graph has been studied extensively in [3].

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In this paper, we demonstrate how this graph helps in identifying annihilator prime ideals that satisfy the ascending chain condition for its proper annihilator ideals. We also classify all posets with planar or toroidal reduced zero-divisor graph and number of annihilator prime ideals is four or seven.

2. Prerequisites

In this section, we put together some well-known concepts, most of which can be found in [2, 4, 5, 6].

We begin by recalling some of the basic terminologies from the theory of graphs. Needless to mention that all graphs considered here are simple graphs, that is, without loops or multiple edges. Let G be a graph and $x, y \in V(G)$, the vertex set of G . Then, x and y are said to be *adjacent* if $x \neq y$ and there is an edge $x - y$ between x and y . A *walk* between x and y is a sequence of adjacent vertices, often written as $x - x_1 - x_2 - \cdots - x_n - y$. A walk between x and y is called *path* if the vertices in it are all distinct. If in a path $x - x_1 - x_2 - \cdots - x_n - y$, x and y are adjacent in G , then the walk $x - x_1 - x_2 - \cdots - x_n - y - x$ is called a *cycle*. The number of edges in a walk (counting repeats), path or a cycle, is called its *length*. The distance between x and y , denoted by $d(x, y)$, is the number of edges in a shortest path between x and y . The length of the shortest cycle in a graph G is called *girth* of G and denoted by $\text{girth}(G)$. The largest distance among all distances between pairs of the vertices of a graph G is called the *diameter* of G and is denoted by $\text{diam}(G)$. A graph G is called *connected* if for any vertices x and y of G there is a path between x and y . Otherwise, G is called *disconnected*. The *neighborhood* of a vertex x in a graph G , denoted by $\text{nbr}(x)$, is defined to be the set of all vertices adjacent to x while the *degree* of x in G , denoted by $\text{deg}(x)$, is defined to be the number of vertices adjacent to x . The minimum degree of G will be denoted by $\delta(G)$. If $\text{deg}(x) = 1$, then x is said to be an *end vertex* in G . A graph G is said to be *complete* if there is an edge between every pair of distinct vertices in G . We denote the complete graph with n vertices by K_n . Given a graph G , let U be a nonempty subset of $V(G)$. Then the *induced subgraph* of G on U is defined to be the graph $G[U]$ in which the vertex set is U and the edge set consists precisely of those edges in G whose endpoints lie in U .

A *chord* of a cycle in a graph is an edge of the graph which does not lie in the edge set of the cycle but whose endpoints lie in the vertex set of the cycle. A *chordless cycle* of a graph is a cycle without any chord. A cycle of a graph, embedded on a surface, is called *contractible with respect to the embedding* if it can be contracted continuously on the surface to a point. A cycle of a toroidal graph is said to be *flat* if it is contractible in every torus embedding of the graph. Given a cycle C of a graph G , we write $G - C$ to denote the graph obtained from G by deleting the vertices of C and the edges of the graph incident to the vertices of C .

The *genus* of a graph G , denoted by $\gamma(G)$, is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. Graphs having genus zero are called *planar graphs*, while those having genus one are called *toroidal graphs*.

Proposition 2.1. [9, Corollary 6-14] *If G is a connected graph with p number of vertices and q number of edges, $p \geq 3$, then, $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, equality holds if and only if a triangular embedding can be found for G .*

Theorem 2.2. [9, Theorem 6-38] $\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil, n \geq 3.$

A subdivision of an edge $x - y$ in a graph is a path $x - x_1 - x_2 - \dots - x_n - y$ obtained by inserting some new vertices x_1, x_2, \dots, x_n into the edge $x - y$. A subdivision of a graph G is the result of some subdivisions of the edges of G . Furthermore, every graph can be considered as a subdivision of itself. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski’s Theorem [2, Page 153] says that a graph is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5 . As a consequence of Kuratowski’s Theorem, one has the following result.

Theorem 2.3. [8, Theorem 2.1] *If a cycle C of a toroidal graph G is such that $G - C$ is nonplanar, then C is flat in G . Further, if flat C is chordless and $G - C$ is connected, then C is a flat face in any torus embedding of G .*

Next we turn to partially ordered sets. A non-empty set is said to be a *partially ordered set* (in short, a *poset*) if it is equipped with a *partial order*, that is, a reflexive, anti-symmetric and transitive binary relation. It is customary to denote a partial order by ‘ \leq ’.

By an *ideal* of P we mean a non-empty subset I of P such that $y \in I$ whenever $y \leq x$ for some $x \in I$. We say that the ideal I is *proper* if $I \neq P$. For $x \in P$, $\text{ann}(x)$ is an ideal of P . Note that $x \notin \text{ann}(x)$ for all $x \in P^\times$. A proper ideal \mathfrak{p} of P is called a *prime ideal* of P if for every $x, y \in P, L(x, y) \subseteq \mathfrak{p}$ implies that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. A prime ideal \mathfrak{p} of P is said to be an *annihilator prime ideal* if there exists $x \in P$ such that $\mathfrak{p} = \text{ann}(x)$. Two annihilator prime ideals $\text{ann}(x)$ and $\text{ann}(y)$ are distinct if and only if $L(x, y) = \{0\}$ (see [7, Lemma 2.3]). We write $\text{Ann}(P)$ to denote the set of all annihilator prime ideals of P . There is a natural injective map from $\text{Ann}(P)$ to the vertex set of $\Gamma_E(P)$ given by $\mathfrak{p} \mapsto [y]$ where $\mathfrak{p} = \text{ann}(y)$. As a result, we will slightly abuse terminology and refer to $[y]$ as an annihilator prime ideal. We denote the set of all annihilator prime ideals which are adjacent to a vertex $[x]$ in $\Gamma_E(P)$ by $\text{anbd}([x])$ (the *annihilator neighborhood* of $[x]$ in $\Gamma_E(P)$) and the set of all the vertices of $\Gamma_E(P)$ which are adjacent to m number of annihilator prime ideals by V_m .

Let P be a poset. We say that P is a *poset with ACC for annihilators* if the ascending chain condition holds for its annihilator ideals, that is, if there is no infinite strictly ascending chain in the set $\mathfrak{A} := \{\text{ann}(x) \mid x \in P^\times\}$ under set inclusion. Equivalently, P is a poset with ACC for annihilators if and only if every non-empty subset of \mathfrak{A} has a maximal element. Thus, if P is a poset with ACC for annihilators, then \mathfrak{A} has a maximal element and every element of \mathfrak{A} is contained in a maximal element of \mathfrak{A} . We denote the set of all maximal elements of \mathfrak{A} by $\text{Max}(\mathfrak{A})$.

Throughout the paper by a poset P we mean a non-trivial poset with least element $0, Z(P)^\times \neq \emptyset$ and with ACC for annihilators.

3. properties of the reduced zero-divisor graph

In this section, we study some properties of the reduced graph of the zero-divisor graph of a poset. We begin with the following lemma.

Lemma 3.1. *Let P be a poset. If $x, y \in P$ with $x \leq y$, then for every $z \in P$, we have $L(x, z) \subseteq L(y, z)$ and so $\text{ann}(y) \subseteq \text{ann}(x)$.*

Proof. This is straightforward (see also [1, Lemma 3.2, part (d)]). \square

Theorem 3.2. *Let P be a poset with $|\text{Ann}(P)| = n$, where $n \in \mathbb{N}$. Let $[p], [q] \in V(\Gamma_E(P))$. Then the following assertions hold:*

- (1) $[p] \in V_{n-1}$, if and only if $\text{ann}(p) \in \text{Ann}(P)$.
- (2) $[p] \in V_i, 1 \leq i \leq n - 2$ if and only if $\text{ann}(p)$ is not an annihilator prime ideal.
- (3) $|\text{anbd}([p]) \cap \text{anbd}([q])| \leq n - 4$ and $|\text{anbd}([p]) \cup \text{anbd}([q])| = n$ if and only if $[p]$ and $[q]$ are adjacent and both $\text{ann}(p)$ and $\text{ann}(q)$ are not annihilator prime ideals.

Proof. Let $\text{Ann}(P) = \{[v_1], [v_2], \dots, [v_n]\}$.

(a) Suppose $\text{ann}(p) \in \text{Ann}(P)$. Then by [3, Proposition 11, part(a)], we have, $[p] \in V_{n-1}$. Conversely, suppose $[p] \in V_{n-1}$. Suppose $[p]$ is not an annihilator prime ideal. Without any loss of generality, we can assume that $\text{anbd}([p]) = \{[v_1], [v_2], \dots, [v_{n-1}]\}$. Now, $L(p, v_n) \neq \{0\}$, implies that there exists $y \in P^\times$ such that $y \leq p$ and $y \leq v_n$. Therefore, $\text{ann}(p) \subseteq \text{ann}(y)$ and $\text{ann}(v_n) \subseteq \text{ann}(y)$. By [3, Proposition 9], we have $\text{ann}(v_n) \in \text{Max}(\mathfrak{A})$ and so $\text{ann}(y) = \text{ann}(v_n)$. Thus, since $[p]$ is not an annihilator prime ideal, we have $\text{ann}(p) \subsetneq \text{ann}(v_n)$. This further implies that there exists $z \in P^\times$, such that $z \in \text{ann}(v_n)$, but $z \notin \text{ann}(p)$. Hence, $L(z, p) \neq \{0\}$ and therefore there exists $w \in P^\times$, such that $w \leq p$ and $w \leq z$. Thus, $v_n \in \text{ann}(z) \subseteq \text{ann}(w)$ and $v_i \in \text{ann}(p) \subseteq \text{ann}(w)$, for all $i \in \{1, 2, \dots, n - 1\}$. Now, by [3, Proposition 9], we have, $\text{ann}(w) \subseteq \text{ann}(v_j)$, for some j . Since, v_i is an element of $\text{ann}(w)$ for every $i \in \{1, 2, \dots, n\}$, we have, $L(v_j, v_j) = \{0\}$ and thus $v_j = 0$, which is a contradiction to the fact that $v_j \in Z(P)^\times$.

(b) Follows from part (a), [3, Proposition 12] and [3, Proposition 11, part(c)].

(c) Suppose $|\text{anbd}([p]) \cap \text{anbd}([q])| \leq n - 4$ and $|\text{anbd}([p]) \cup \text{anbd}([q])| = n$. By part (a), we have both $[p]$ and $[q]$ are not annihilator prime ideals. Suppose $[p]$ and $[q]$ are not adjacent. Then, there exists $z \in P^\times$ such that $z \leq p$ and $z \leq q$. Thus, $\text{ann}(p) \subseteq \text{ann}(z)$, $\text{ann}(q) \subseteq \text{ann}(z)$ and $z \in Z(P)^\times$. Hence, $[z] \in V_n$. Which is a contradiction to part (a) and part (b).

Conversely, suppose $[p]$ and $[q]$ are adjacent and $\text{ann}(p), \text{ann}(q)$ are not annihilator prime ideals. Suppose $[p]$ is not adjacent to $[v_i]$. Let $w \in L(p, v_i)$ and $w \neq 0$. Then $w \leq p$ and $w \leq v_i$. Thus $\text{ann}(p) \subseteq \text{ann}(w)$ and $\text{ann}(v_i) \subseteq \text{ann}(w)$. Now, by [3, Proposition 9], we have $\text{ann}(w) = \text{ann}(v_i)$ and so $q \in \text{ann}(p) \subset \text{ann}(v_i)$. Therefore p or $q \in \text{ann}(v_i)$ for all $i \in \{1, 2, \dots, n\}$. Thus $|\text{anbd}([p]) \cup \text{anbd}([q])| = n$. Now, by part (b) we have, $[p], [q] \in V_i, 1 \leq i \leq n - 2$, that is $|\text{anbd}([p])|, |\text{anbd}([q])| \leq n - 2$. Since, $n = |\text{anbd}([p]) \cup \text{anbd}([q])| = |\text{anbd}([p])| + |\text{anbd}([q])| - |\text{anbd}([p]) \cap \text{anbd}([q])|$, implies that $|\text{anbd}([p]) \cap \text{anbd}([q])| \leq n - 4$. \square

As an immediate consequence we have the following result.

Corollary 3.3. *Let P be a poset with $|\text{Ann}(P)| = n$, where $n \in \mathbb{N}$. Let $[p], [q] \in V(\Gamma_E(P))$. Then the following assertions hold:*

- (1) If $\text{anbd}([p]) = \text{anbd}([q])$, then $[p] = [q]$.

- (2) $|\text{anbd}([p])| = 1$ if and only if $[p]$ is an end vertex.
- (3) If $|\text{anbd}([p])| = 2$, then $\text{deg}([p]) = 2$ or 3 .

Proof. If $[p]$ is an annihilator prime ideal, then $[q]$ is also an annihilator prime ideal. Since any two annihilator prime ideals are adjacent, we have $[p] = [q]$. If $[p]$ is not an annihilator prime ideal, then $[q]$ is also not an annihilator prime ideal. Then, by Theorem 3.2, part (c), any vertices which is adjacent to $[p]$ is also adjacent to $[q]$ and vice versa. Hence using [3, Proposition 4], the assertion (a) follows.

Part (b) and (c) follow from Theorem 3.2. □

Proposition 3.4. *Let P be a poset with $|\text{Ann}(P)| < \infty$ and $\text{Ann}(P) \subsetneq V(\Gamma_E(P))$. Then, for any $[x] \in \text{Ann}(P)$, we have $\text{deg}([x]) > \delta(\Gamma_E(P))$.*

Proof. Suppose to the contrary that $\text{deg}([x]) = \delta(\Gamma_E(P))$, for some $[x] \in \text{Ann}(P)$. Suppose there exists $[v] \in V(\Gamma_E(P)) \setminus \text{Ann}(P)$, such that $[x]$ and $[v]$ are not adjacent. Let V be the set of all vertices which are adjacent to $[v]$ and are not annihilator prime ideal. By Theorem 3.2, we have $|\text{anbd}([x])| = n - 1$ and $|\text{anbd}([v])| \leq n - 2$. Now, for any $[z] \in V$, by Theorem 3.2, we have $|\text{anbd}([v]) \cup \text{anbd}([z])| = n$ and thus it follows that $L(x, z) = \{0\}$. Therefore $\text{deg}([x]) \geq n - 1 + |V|$ and $\text{deg}([v]) \leq n - 2 + |V|$, which is a contradiction. Thus, we can assume that $L(v, x) = \{0\}$ for any $[v] \in V(\Gamma_E(P)) \setminus \text{Ann}(P)$. Let $m = |V(\Gamma_E(P)) \setminus \text{Ann}(P)|$. Then $\text{deg}([x]) = n - 1 + m$ and $\text{deg}([v]) \leq n - 2 + m$, which is again a contradiction. This completes the proof. □

We denote the induced subgraph of $\Gamma_E(P)$ by the set V_{n-2} by $\Gamma_E(P)[V_{n-2}]$, where $n = |\text{Ann}(P)|$. From now onwards, in this paper, if $|\text{Ann}(P)| = n$, then we denote the elements of $\text{Ann}(P)$ by v_1, v_2, \dots, v_n . In view of Corollary 3.3, we denote a vertex, which is not an annihilator prime ideal, not an end vertex and adjacent to annihilator prime ideal $v_{i_1}, v_{i_2}, \dots, v_{i_m}$ by v_{i_1, i_2, \dots, i_m} .

Lemma 3.5. *Suppose $x, y \in V_{n-2}$, x is not adjacent to v_{i_1} and v_{i_2} and y is not adjacent to v_{i_3} and v_{i_4} . Then x and y are adjacent if and only if $i_1, i_2 \notin \{i_3, i_4\}$.*

Proof. Follows directly from Theorem 3.2. □

Proposition 3.6. *Let P be a poset with ACC for annihilators and $|\text{Ann}(P)| = n$. Then $\text{girth}(\Gamma_E(P)[V_{n-2}]) \in \{3, 4, 5, 6, \infty\}$.*

Proof. Suppose $\text{girth}(\Gamma_E(P)[V_{n-2}]) \notin \{3, 4, 5, \infty\}$. Let $x_1 - x_2 - \dots - x_m - x_1$ be a cycle in $\Gamma_E(P)[V_{n-2}]$ of length greater than or equal to 6. Since x_1 is adjacent to x_2 and x_m and also x_2 and x_m are not adjacent, we can assume that x_1 is not adjacent to v_1 and v_2 , x_2 is not adjacent to v_{n-1} and v_n and thus x_m is not adjacent to v_n and v_j , where $j \notin \{1, 2, n - 1\}$. Suppose x_3 is not adjacent to v_r and v_s . Now, x_3 is adjacent to x_2 and not adjacent to x_1 and x_m and so $r, s \notin \{n - 1, n\}$, $r \in \{1, 2\}$ and $s \in \{n, j\}$. It follows that $r = 1$ and $s = j$. Using a similar argument, we can show that x_4 is not adjacent to v_2 and v_n . Now, using the fact that x_5 is adjacent to x_4 and not adjacent to x_1, x_2 and x_3 , we see that x_5 is not adjacent to v_1 and v_{n-1} . It follows that any annihilator prime ideal is adjacent to at least one of x_5 or x_m . Thus, by Theorem 3.2, we have x_5 and x_m are adjacent. This completes the proof. □

Proposition 3.7. *Let P be a poset with ACC for annihilators and $|\text{Ann}(P)| = n$. If $\Gamma_E(P)[V_{n-2}]$ is connected, then $\text{diam}(\Gamma_E(P)[V_{n-2}]) \leq 4$.*

Proof. Let $x, y \in V_{n-2}$. Suppose $x = x_1 - x_2 - \cdots - x_m = y$ is a shortest path in $\Gamma_E(P)[V_{n-2}]$ of length greater than or equal to 5. Then, we can assume that x_1 is not adjacent to v_1 and v_2 , x_2 is not adjacent to v_{n-1} and v_n . Then x_3 is not adjacent to v_1 and v_j , where $j \notin \{1, 2, n-1, n\}$, x_4 is not adjacent to v_2 and v_n and x_5 is not adjacent to v_1 and v_{n-1} . Now, x_6 is adjacent to x_i , where $i \in \{1, 2, 3, 4\}$, which is a contradiction. Thus $d(x, y) \leq 4$ and hence $\text{diam}(\Gamma_E(P)[V_{n-2}]) \leq 4$. \square

Proposition 3.8. *Let P be a poset with ACC for annihilators and $|\text{Ann}(P)| = n$. If $\Gamma_E(P)[V_{n-2}]$ has a 3 or 5-cycle, then $\Gamma_E(P)[V_{n-2}]$ is connected. Moreover, if $\text{girth}(\Gamma_E(P)[V_{n-2}]) = 3$, then $\text{diam}(\Gamma_E(P)[V_{n-2}]) \leq 3$.*

Proof. Let $x_1 - x_2 - x_3 - x_1$ be a 3-cycle. Then, by a similar argument as in the proof of Proposition 3.6, we can show that, if x_1 is not adjacent to v_1 and v_2 , x_2 is not adjacent to v_{n-1} and v_n , then x_3 is not adjacent to v_i and v_j , where $i, j \notin \{1, 2, n-1, n\}$. Then any vertex $y \in V(\Gamma_E(P)[V_{n-2}]) \setminus \{x_1, x_2, x_3\}$ will be adjacent to one of the x_i . It follows that $\Gamma_E(P)[V_{n-2}]$ is connected and $\text{diam}(\Gamma_E(P)[V_{n-2}]) \leq 3$. In a similar way we can show that if $\Gamma_E(P)[V_{n-2}]$ has a 5-cycle, then any vertex which is not in the given cycle is adjacent to one of the vertex in the cycle. Thus it follows that $\Gamma_E(P)[V_{n-2}]$ is connected. \square

4. Genus of the reduced zero-divisor graphs

Consider a poset P with $|\text{Ann}(P)| \geq 4$ and an embedding of the graph G induced by the set $V(\Gamma_E(P)) \setminus V$, where $V = \{[x] \mid \deg([x]) \leq 2\}$. It is obvious that any end vertex can be put on the surface without disturbing the genus of G . Also if $\deg([v]) = 2$, then by Corollary 3.3, we have $[v] \in V_2$ and thus can be put on the surface along the edge form by the two annihilator prime ideals, in which $[v]$ is adjacent. Thus any vertex $[v]$ with $\deg([v]) = 2$ can be put on the surface without disturbing the genus of G . Thus the genera of G and $\Gamma_E(P)$ are same. Therefore, from now onward, all poset with $|\text{Ann}(P)| \geq 4$, consider in this section are without vertex of degree less than or equal to 2.

Theorem 4.1. *Let P be a poset. Then $\Gamma_E(P)$ is planar if and only if one of the following conditions holds:*

- (1) $|\text{Ann}(P)| = 2$ or 3.
- (2) $|\text{Ann}(P)| = 4$ and $V(\Gamma_E(P)) = \text{Ann}(P)$.

Proof. Suppose $|\text{Ann}(P)| = 2$. Then by [3, Proposition 18], we have $|V(\Gamma_E(P))| = 2$ and hence $\Gamma_E(P)$ is planar.

Suppose $|\text{Ann}(P)| = 3$. Then by [3, Proposition 18], we have $|V(\Gamma_E(P))| \leq 2^{|\text{Ann}(P)|} - 2 = 6$. Therefore, in view of Theorem 3.2 and Corollary 3.3, three vertices are annihilator prime ideals and are adjacent to each other and the rest are end vertices. Thus $\Gamma_E(P)$ is planar.

Suppose $|\text{Ann}(P)| = 4$ and $V(\Gamma_E(P)) = \text{Ann}(P)$. Thus $|V(\Gamma_E(P))| = 4$ and $\Gamma_E(P)$ is planar.

Conversely, Suppose $|\text{Ann}(P)| \geq 5$, then by Theorem 3.2, K_5 is a subgraph of $\Gamma_E(P)$ and hence $\Gamma_E(P)$ is not planar.

Suppose $|\text{Ann}(P)| = 4$ and there exists $[v] \in V_2$ with $\deg([v]) = 3$. Then, there exists a vertex $[w]$ which is not an annihilator prime ideal and adjacent to $[v]$. By Theorem 3.2, $[w] \in V_2$ and the induced subgraph of $\Gamma_E(P)$ by the set $\text{Ann}(P) \cup \{[v], [w]\}$ is a subdivision of K_5 . Thus by Kuratowski's theorem, $\Gamma_E(P)$ is not planar. \square

Proposition 4.2. *Let P be a poset with $|\text{Ann}(P)| = 4$. If $\text{Ann}(P) \subsetneq V(\Gamma_E(P))$, then $\Gamma_E(P)$ is toroidal.*

Proof. In view of Theorem 3.2, without any loss of generality, we can assume that

$$V(\Gamma_E(P)) = \text{Ann}(P) \cup \{v_{1,2}, v_{3,4}, v_{1,3}, v_{2,4}, v_{1,4}, v_{2,3}\}.$$

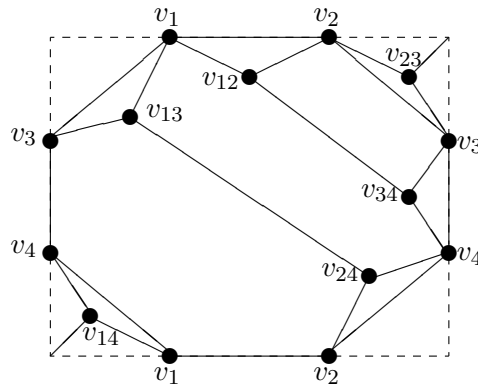


FIGURE 1. Embedding of the reduced zero-divisor graph $\Gamma_E(P)$ on a torus.

By Theorem 4.1, $\Gamma_E(P)$ is not planar. Now the graph can be embed on a torus as shown in figure 1. Therefore $\Gamma_E(P)$ is toroidal. \square

Proposition 4.3. *Let P be a poset, $|\text{Ann}(P)| = 7$ and there exists $[v] \in \Gamma_E(P)$ such that $|\text{anbd}(v)| = 4$ or 5. Then $\Gamma_E(P)$ is not toroidal.*

Proof. Let G be the induced subgraph of $\Gamma_E(P)$ by the set $\text{Ann}(P) \cup \{[v]\}$. For the graph G , $m \geq 25$ and $n = 8$, where m and n denote the number of edges and vertices, respectively. Therefore, $\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1 > 1$. Hence $\Gamma_E(P)$ is not toroidal. \square

Proposition 4.4. *Let P be a poset, $|\text{Ann}(P)| = 7$. If $|V_3| > 14$, then $\Gamma_E(P)$ is not toroidal.*

Proof. Suppose $\Gamma_E(P)$ is toroidal. Let $v_{i,j,k} \in V_3$. Then by Theorem 2.3, each of the cycle $(v_i, v_{i,j,k}, v_k)$, $(v_i, v_{i,j,k}, v_j)$ and $(v_j, v_{i,j,k}, v_k)$ are flat faces in any torus embedding of $\Gamma_E(P)$. Hence the number of faces is greater than or equal to $3k$, where $k = |V_3|$. Now, by Euler's formula, we have, $7 + k - 21 - 3k + r = 0$, which implies $r = 14 + 2k < 3k$, a contradiction. Hence $\Gamma_E(P)$ is not toroidal. \square

Theorem 4.5. *Let P be a poset with $|\text{Ann}(P)| = 7$. Suppose $V_4 = V_5 = \emptyset$ and $|V_3| \leq 14$. Then, $\Gamma_E(P)$ is toroidal if and only if distinct vertices of V_3 get attached to distinct faces of an embedding of K_7 forming triplets of vertices.*

Proof. Suppose $\Gamma_E(P)$ is toroidal. Observe that the induced subgraph of $\Gamma_E(P)$ by the set $\text{Ann}(P)$ is isomorphic to K_7 . Let $[v] \in V_3$. Let H be the induced graph on $\{v\} \cup \text{Ann}(P)$. Then H is toroidal and has 8 vertices and 24 edges. Now, by Euler's formula, H has $24 - 8 = 16$ faces. Without any loss of generality, we can assume that $[v] = v_{123}$. By Theorem 2.3, each of the three cycles $([v], v_1, v_2)$, $([v], v_2, v_3)$, $([v], v_1, v_3)$ are flat faces in any torus embedding of H . Now, K_7 has 14 faces and if (v_1, v_2, v_3) is not a face in the embedding of K_7 , then the number of faces in H becomes $14 + 3 = 17$, a contradiction. Hence (v_1, v_2, v_3) is a face. Hence all the vertices of V_3 get attached to distinct faces of an embedding of K_7 forming triplets of vertices. The converse is trivial. \square

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REFERENCES

- [1] M. Alizadeh, A. K. Das, H. R. Maimani, M. R. Pournaki and S. Yassemi, On the diameter and girth of zero-divisor graphs of posets, *Discrete Appl. Math.*, **160** (2012) 1319–1324.
- [2] G. Chartrand and O. R. Oellermann, *Applied and Algorithmic Graph Theory*, McGraw-Hill, Inc., New York, 1993.
- [3] A. K. Das and D. Nongsiang, On Reduced Zero-Divisor Graphs of Posets, *J. Discrete Math.*, **2015** (2015) pp. 7.
- [4] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, second ed., Cambridge University Press, New York, 2002.
- [5] R. Halas, Annihilators and ideals in ordered sets, *Czechoslovak Math. J.*, **45** (1995) 127–134.
- [6] R. Halas, Relative polars in ordered sets, *Czechoslovak Math. J.*, **50** (2000) 415–429.
- [7] R. Halas and M. Jukl, On Beck's coloring of posets, *Discrete Math.*, **309** (2009) 4584–4589.
- [8] E. Neufeld and W. Myrvold, *Practical Toroidality Testing*, Proceedings of the eighth annual ACM-SIAM symposium on Discrete Algorithms, New Orleans, LA, 1997, 574–580.
- [9] A. T. White, *Graphs, groups and surfaces*, North-Holland Mathematics Studies, no. 8., American Elsevier Publishing Co., Inc., New York, 1973.

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