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SPLICES, LINKS, AND THEIR EDGE-DEGREE DISTANCES

MAHDIEH AZARI* AND HOJJATOLLAH DIVANPOUR

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Abstract. The edge-degree distance of a simple connected graph G is defined as the sum of the terms $\big(d(e\,|G) + d(f\,|G)\big)d(e,f\,|G)$ over all unordered pairs $\{e,f\}$ of edges of G, where $d(e\,|G)$ and $d(e,f\,|G)$ denote the degree of the edge e in G and the distance between the edges e and f in G, respectively. In this paper, we study the behavior of two versions of the edge-degree distance under two graph products called splice and link.

1. Introduction

In this paper, we consider connected finite graphs without any loops or multiple edges. In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of *molecular–graph–based structure–descriptors*, which are also referred to as *topological indices* [11, 16]. The oldest molecular structure descriptor is the one put forward in 1947 by Harold Wiener [30], nowadays referred to as the *Wiener index*. Wiener used his index for the calculation of the boiling points of alkanes. The Wiener index of a graph G is defined as the sum of distances between all pairs of vertices of G:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v\,|G),$$

where $d(u,v\,|G)$ denotes the distance between the vertices u and v of G which is defined as the length of any shortest path in G connecting them.

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∗Corresponding author.

The *degree distance* was introduced by Dobrynin and Kochetova [12] and at the same time by Gutman [15] as a weighted version of the Wiener index. The degree distance of a graph G is defined as:

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} \big[d(u\,|G) + d(v\,|G)\big] d(u,v\,|G),$$

where $d(u\,|G)$ denotes the degree of the vertex u in G. It can also be expressed as:

$$DD(G) = \sum_{u\in V(G)} d(u\,|G) D(u\,|G),$$

where $D(u\,|G) = \sum_{v\in V(G)} d(u,v\,|G)$. In fact, if T is a tree on n vertices, the Wiener index and degree distance are closely related by $DD(T) = 4W(T) - n(n-1)$ (see [15]). The interested readers may consult [1, 8, 9, 10, 14, 20, 21, 22, 27, 28, 29] for mathematical properties of the degree distance.

Edge versions of the degree distance were introduced by Iranmanesh *et al.* [24] in 2011. Two possible distances between the edges $e = uv$ and $f = zt$ of a graph G can be considered [23]. The first distance is denoted by $d_0(e,f\,|G)$ and defined as:

$$d_0(e,f\,|G) = \begin{cases} d_1(e,f\,|G) + 1 & e \neq f, \\ 0 & e = f, \end{cases}$$

where $d_1(e,f\,|G) = \min\{d(u,z\,|G), d(u,t\,|G), d(v,z\,|G), d(v,t\,|G)\}$. It is easy to see that, $d_0(e,f\,|G) = d(e,f\,|L(G))$, where $L(G)$ is the line graph of G. The second distance is denoted by $d_4(e,f\,|G)$ and defined as:

$$d_4(e,f\,|G) = \begin{cases} d_2(e,f\,|G) & e \neq f, \\ 0 & e = f, \end{cases}$$

where $d_2(e,f\,|G) = \max\{d(u,z\,|G), d(u,t\,|G), d(v,z\,|G), d(v,t\,|G)\}$. Related to these two distances, two edge versions of the degree distance can be defined. The first and second *edge-degree distances* of G are denoted by $DD_{e_0}(G)$ and $DD_{e_4}(G)$, respectively and defined as:

$$DD_{e_i}(G) = \sum_{\{e,f\}\subseteq E(G)} \big[d(e\,|G) + d(f\,|G)\big] d_i(e,f\,|G), \qquad i \in \{0,4\},$$

where $d(e\,|G)$ denotes the degree of the edge e in G which is the degree of the vertex e in the line graph $L(G)$.

In this paper, we present explicit formulas for the first and second edge-degree distances of two graph products called splice and link. Readers interested in more information on computing topological indices of splice and link of graphs can be referred to [2, 3, 4, 5, 6, 7, 13, 18, 19, 25, 26].

2. Definitions and preliminaries

In this section, we state some definitions and introduce some useful notations. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $V(e)$ the set of two end-vertices of the edge

e of G. For $u \in V(G)$, we define:

$$D_i(u\,|G) = \sum_{e \in E(G)} D_i(u, e\,|G), \qquad i \in \{1, 2\} \ ,$$

where for $e = ab \in E(G)$,

$$D_1(u, e\,|G) = \min\{d(u, a\,|G), d(u, b\,|G)\},$$

$$D_2(u, e\,|G) = \max\{d(u, a\,|G), d(u, b\,|G)\}.$$

$D_1(u, e\,|G)$ and $D_2(u, e\,|G)$ can be considered as two possible distances between the vertex u and the edge e in G. Note that, $D_1(u, e\,|G)$ is a nonnegative integer and $D_1(u, e\,|G) = 0$ if and only if $u \in V(e)$. Also, $D_2(u, e\,|G)$ is a positive integer and $D_2(u, e\,|G) = 1$ if and only if $u \in V(e)$ or u and the end vertices of e form a 3-cycle in G. For $u \in V(G)$, $D_1(u\,|G)$ and $D_2(u\,|G)$ can also be expressed by:

$$D_1(u\,|G) = \sum_{e \in E(G); u \notin V(e)} D_1(u, e\,|G),$$

$$D_2(u\,|G) = \sum_{e \in E(G); u \notin V(e)} D_2(u, e\,|G) + d(u\,|G).$$

For $u \in V(G)$, let $N(u\,|G)$ denote the set of all first neighbors of u in G. We denote by $\delta(u\,|G)$, the sum of degrees of all neighbors of u in G, i.e.,

$$\delta(u\,|G) = \sum_{v \in N(u|G)} d(v\,|G).$$

We denote by $M_1(G)$, the *first Zagreb index* [17] of G which is defined as:

$$M_1(G) = \sum_{u \in V(G)} d(u\,|G)^2.$$

It can also be expressed as a sum over edges of G:

$$M_1(G) = \sum_{uv \in E(G)} \big[d(u\,|G) + d(v\,|G)\big].$$

Let e be an edge of G with $V(e) = \{a, b\}$. It is easy to see that, $d(e\,|G) = d(a\,|G) + d(b\,|G) - 2$. Therefore,

$$\sum_{e \in E(G)} d(e\,|G) = M_1(G) - 2\,|E(G)|\,,$$

and for $u \in V(G)$,

$$\sum_{e \in E(G); u \in V(e)} d(e\,|G) = d(u\,|G)\big(d(u\,|G) - 2\big) + \delta(u\,|G).$$

3. Results and discussion

In this section, we compute the first and second edge-degree distances for splice and link of graphs.

3.1. **Splice.** Let G_1 and G_2 be two graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively. Following Došlić [13], for given vertices $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$, a *splice* of G_1 and G_2 by vertices a_1 and a_2 is denoted by $(G_1.G_2)(a_1, a_2)$ and defined by identifying the vertices a_1 and a_2 in the union of G_1 and G_2. We denote by n_i, m_i and α_i, the order and size of the graph G_i, and the degree of a_i in G_i, respectively, where $i \in \{1, 2\}$. It is easy to see that, $\left|V\big((G_1.G_2)(a_1, a_2)\big)\right| = n_1 + n_2 - 1$ and $\left|E\big((G_1.G_2)(a_1, a_2)\big)\right| = m_1 + m_2$.

In the following lemma, the distance between vertices of $(G_1.G_2)(a_1, a_2)$ is computed. The result follows easily from the definition of splice of graphs, so the proof is omitted.

Lemma 3.1. *Let $G = (G_1.G_2)(a_1, a_2)$. For $u, v \in V(G)$,*

$$d(u, v\,|G) = \begin{cases} d(u, v\,|G_1) & u, v \in V(G_1), \\ d(u, v\,|G_2) & u, v \in V(G_2), \\ d(u, a_1\,|G_1) + d(a_2, v\,|G_2) & u \in V(G_1), v \in V(G_2). \end{cases}$$

In the following lemma, the distances d_0 and d_4 between edges of $(G_1.G_2)(a_1, a_2)$ are computed. The results follow easily from Lemma 3.1, so the proofs are omitted.

Lemma 3.2. *Let $G = (G_1.G_2)(a_1, a_2)$. For $e, f \in E(G)$,*

(i) $$d_0(e, f\,|G) = \begin{cases} d_0(e, f\,|G_1) & e, f \in E(G_1), \\ d_0(e, f\,|G_2) & e, f \in E(G_2), \\ D_1(a_1, e\,|G_1) + D_1(a_2, f\,|G_2) + 1 & e \in E(G_1), f \in E(G_2), \end{cases}$$

(ii) $$d_4(e, f\,|G) = \begin{cases} d_4(e, f\,|G_1) & e, f \in E(G_1), \\ d_4(e, f\,|G_2) & e, f \in E(G_2), \\ D_2(a_1, e\,|G_1) + D_2(a_2, f\,|G_2) & e \in E(G_1), f \in E(G_2). \end{cases}$$

In the following lemma, the degree of an arbitrary edge of $(G_1.G_2)(a_1, a_2)$ is computed. The result follows easily from the definition of splice of graphs, so the proof is omitted.

Lemma 3.3. *Let $G = (G_1.G_2)(a_1, a_2)$. For $e \in E(G)$,*

$$d(e\,|G) = \begin{cases} d(e\,|G_1) & e \in E(G_1), a_1 \notin V(e), \\ d(e\,|G_1) + \alpha_2 & e \in E(G_1), a_1 \in V(e), \\ d(e\,|G_2) & e \in E(G_2), a_2 \notin V(e), \\ d(e\,|G_2) + \alpha_1 & e \in E(G_2), a_2 \in V(e). \end{cases}$$

Theorem 3.4. *Let $G = (G_1.G_2)(a_1, a_2)$. The first and second edge-degree distances of G are given by:*

(i) $$DD_{e_0}(G) = DD_{e_0}(G_1) + DD_{e_0}(G_2) + \alpha_2 \sum_{e,f \in E(G_1); a_1 \in V(e) \setminus V(f)} d_0(e, f \,|G_1\,) + \alpha_1 \sum_{e,f \in E(G_2); a_2 \in V(e) \setminus V(f)} d_0(e, f \,|G_2\,) + m_2 \sum_{e \in E(G_1)} d(e \,|G_1\,) D_1(a_1, e \,|G_1\,) + m_1 \sum_{f \in E(G_2)} d(f \,|G_2\,) D_1(a_2, f \,|G_2\,) + \big[M_1(G_2) - 2m_2 + \alpha_1\alpha_2\big]\big[D_1(a_1 \,|G_1\,) + m_1\big] + \big[M_1(G_1) - 2m_1 + \alpha_1\alpha_2\big]\big[D_1(a_2 \,|G_2\,) + m_2\big] + \alpha_1\alpha_2(\alpha_1 + \alpha_2 - 2),$$

(ii) $$DD_{e_4}(G) = DD_{e_4}(G_1) + DD_{e_4}(G_2) + 2\alpha_2 \sum_{\{u,v\} \subseteq N(a_1|G_1\,)} d(u, v \,|G_1\,) + 2\alpha_1 \sum_{\{u,v\} \subseteq N(a_2|G_2\,)} d(u, v \,|G_2\,) + \alpha_2 \sum_{e,f \in E(G_1); a_1 \in V(e) \setminus V(f)} d_4(e, f \,|G_1\,) + \alpha_1 \sum_{e,f \in E(G_2); a_2 \in V(e) \setminus V(f)} d_4(e, f \,|G_2\,) + m_2 \sum_{e \in E(G_1)} d(e \,|G_1\,) D_2(a_1, e \,|G_1\,) + m_1 \sum_{f \in E(G_2)} d(f \,|G_2\,) D_2(a_2, f \,|G_2\,) + \big[M_1(G_2) - 2m_2 + \alpha_1\alpha_2\big] D_2(a_1 \,|G_1\,) + \big[M_1(G_1) - 2m_1 + \alpha_1\alpha_2\big] D_2(a_2 \,|G_2\,) + (m_1 + m_2)\alpha_1\alpha_2.$$

Proof. *(i)* By definition of the first edge-degree distance,

$$DD_{e_0}(G) = \sum_{\{e,f\} \subseteq E(G)} \big[d(e \,|G\,) + d(f \,|G\,)\big] d_0(e, f \,|G\,).$$

Now, we partition the above sum into three sums as follows:

The first sum S_1 consists of contributions to $DD_{e_0}(G)$ of pairs of edges from G_1:

$$S_1 = \sum_{\{e,f\} \subseteq E(G_1)} \big[d(e \,|G\,) + d(f \,|G\,)\big] d_0(e, f \,|G\,).$$

In order to compute the sum S_1, we partition it into three sums S_{11}, S_{12} and S_{13} as follows:

The sum S_{11} is equal to:

$$S_{11} = \sum_{\{e,f\} \subseteq E(G_1); a_1 \notin V(e) \cup V(f)} \big[d(e \,|G\,) + d(f \,|G\,)\big] d_0(e, f \,|G\,).$$

Using Lemmas 3.2 and 3.3, we obtain:

$$S_{11} = \sum_{\{e,f\} \subseteq E(G_1); a_1 \notin V(e) \cup V(f)} \big[d(e \,|G_1\,) + d(f \,|G_1\,)\big] d_0(e, f \,|G_1\,).$$

The sum S_{12} is equal to:

$$S_{12} = \sum_{\{e,f\} \subseteq E(G_1); a_1 \in V(e) \cap V(f)} \big[d(e \,|G\,) + d(f \,|G\,)\big] d_0(e, f \,|G\,).$$

Using Lemmas 3.2 and 3.3, we obtain:

$$\begin{aligned} S_{12} &= \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} \Big[\big(d(e\,|G_1\,)+\alpha_2\big)+\big(d(f\,|G_1\,)+\alpha_2\big)\Big]d_0(e,f\,|G_1\,) \\ &= \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} \big[d(e\,|G_1\,)+d(f\,|G_1\,)\big]d_0(e,f\,|G_1\,)+2\alpha_2\binom{\alpha_1}{2}. \end{aligned}$$

The sum S_{13} is equal to:

$$S_{13} = \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

Using Lemmas 3.2 and 3.3, we obtain:

$$\begin{aligned} S_{13} &= \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \Big[\big(d(e\,|G_1\,)+\alpha_2\big)+d(f\,|G_1\,)\Big]d_0(e,f\,|G_1\,) \\ &= \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \big[d(e\,|G_1\,)+d(f\,|G_1\,)\big]d_0(e,f\,|G_1\,) \\ &\quad +\alpha_2 \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d_0(e,f\,|G_1\,). \end{aligned}$$

By adding the quantities S_{11}, S_{12} and S_{13}, we obtain:

$$S_1 = DD_{e_0}(G_1)+2\alpha_2\binom{\alpha_1}{2}+\alpha_2 \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d_0(e,f\,|G_1\,).$$

The second sum S_2 consists of contributions to $DD_{e_0}(G)$ of pairs of edges from G_2:

$$S_2 = \sum_{\{e,f\}\subseteq E(G_2)} \big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

Using the same argument as in the computation of S_1, we obtain:

$$S_2 = DD_{e_0}(G_2)+2\alpha_1\binom{\alpha_2}{2}+\alpha_1 \sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d_0(e,f\,|G_2\,).$$

The third sum S_3 is taken over all pairs of edges $e,f\in E(G)$ such that $e\in E(G_1)$ and $f\in E(G_2)$:

$$S_3 = \sum_{e\in E(G_1)}\sum_{f\in E(G_2)} \big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

In order to compute the sum S_3, we partition it into four sums S_{31}, S_{32}, S_{33} and S_{34} as follows:

The sum S_{31} is equal to:

$$S_{31} = \sum_{e\in E(G_1);a_1\notin V(e)}\ \sum_{f\in E(G_2);a_2\notin V(f)} \big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

Using Lemmas 3.2 and 3.3, we obtain:

$$\begin{aligned}S_{31} =& \sum_{e\in E(G_1);a_1\notin V(e)} \sum_{f\in E(G_2);a_2\notin V(f)} \big[d(e\,|G_1\,) + d(f\,|G_2\,)\big]\big[D_1(a_1,e\,|G_1\,) + D_1(a_2,f\,|G_2\,)+1\big]\\ =&(m_2-\alpha_2)\sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1,e\,|G_1\,) + (m_1-\alpha_1)\sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2,f\,|G_2\,)\\ &+\big[M_1(G_1)-2m_1-\alpha_1(\alpha_1-2)-\delta(a_1\,|G_1\,)\big]\big[D_1(a_2\,|G_2\,)+m_2-\alpha_2\big]\\ &+\big[M_1(G_2)-2m_2-\alpha_2(\alpha_2-2)-\delta(a_2\,|G_2\,)\big]\big[D_1(a_1\,|G_1\,)+m_1-\alpha_1\big].\end{aligned}$$

The sum S_{32} is equal to:

$$S_{32} = \sum_{e\in E(G_1);a_1\notin V(e)} \sum_{f\in E(G_2);a_2\in V(f)} \big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

Using Lemmas 3.2 and 3.3, we obtain:

$$\begin{aligned}S_{32} =& \sum_{e\in E(G_1);a_1\notin V(e)} \sum_{f\in E(G_2);a_2\in V(f)} \big[d(e\,|G_1\,) + \big(d(f\,|G_2\,)+\alpha_1\big)\big]\big[D_1(a_1,e\,|G_1\,)+1\big]\\ =&\alpha_2\big[\sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1,e\,|G_1\,) + M_1(G_1)-2m_1-\alpha_1(\alpha_1-2)-\delta(a_1\,|G_1\,)\big]\\ &+\big[\alpha_2(\alpha_2-2)+\delta(a_2\,|G_2\,)+\alpha_1\alpha_2\big]\big[D_1(a_1\,|G_1\,)+m_1-\alpha_1\big].\end{aligned}$$

The sum S_{33} is equal to:

$$S_{33} = \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\notin V(f)} \big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

Using the same argument as in the computation of S_{32}, we obtain:

$$\begin{aligned}S_{33} =& \alpha_1\big[\sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2,f\,|G_2\,) + M_1(G_2)-2m_2-\alpha_2(\alpha_2-2)-\delta(a_2\,|G_2\,)\big]\\ &+\big[\alpha_1(\alpha_1-2)+\delta(a_1\,|G_1\,)+\alpha_1\alpha_2\big]\big[D_1(a_2\,|G_2\,)+m_2-\alpha_2\big].\end{aligned}$$

The sum S_{34} is equal to:

$$S_{34} = \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\in V(f)} \big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

Using Lemmas 3.2 and 3.3, we obtain:

$$\begin{aligned}S_{34} =& \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\in V(f)} \big[\big(d(e\,|G_1\,)+\alpha_2\big)+\big(d(f\,|G_2\,)+\alpha_1\big)\big]\\ =&\alpha_2\big[\alpha_1(\alpha_1-2)+\delta(a_1\,|G_1\,)\big]+\alpha_1\big[\alpha_2(\alpha_2-2)+\delta(a_2\,|G_2\,)\big]+(\alpha_1+\alpha_2)\alpha_1\alpha_2.\end{aligned}$$

By adding the quantities S_{31}, S_{32}, S_{33} and S_{34}, we obtain:

$$\begin{aligned}S_3 =&m_2\sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1,e\,|G_1\,)+m_1\sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2,f\,|G_2\,)\\ &+\big[M_1(G_2)-2m_2+\alpha_1\alpha_2\big]\big[D_1(a_1\,|G_1\,)+m_1\big]+\big[M_1(G_1)-2m_1+\alpha_1\alpha_2\big]\big[D_1(a_2\,|G_2\,)+m_2\big].\end{aligned}$$

The formula of $DD_{e_0}(G)$ is obtained by adding the quantities S_1, S_2 and S_3 and simplifying the resulting expression.

(ii) Using the same argument as in the proof of part *(i)*, we can get the desired result. □

3.2. **Link.** Let G_1 and G_2 be two connected graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively. Following Došlić [13], for given vertices $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$, a *link* of G_1 and G_2 by vertices a_1 and a_2 is denoted by $(G_1 \sim G_2)(a_1, a_2)$ and obtained by joining a_1 and a_2 by an edge in the union of these graphs. We denote by n_i, m_i and α_i, the order and size of the graph G_i and the degree of a_i in G_i, respectively, where $i \in \{1, 2\}$. It is easy to see that, $\left|V\big((G_1 \sim G_2)(a_1, a_2)\big)\right| = n_1 + n_2$ and $\left|E\big((G_1 \sim G_2)(a_1, a_2)\big)\right| = m_1 + m_2 + 1$.

In the following lemma, the distance between vertices of $(G_1 \sim G_2)(a_1, a_2)$ is computed. The result follows easily from the definition of link of graphs, so the proof is omitted.

Lemma 3.5. *Let $G = (G_1 \sim G_2)(a_1, a_2)$. For $u, v \in V(G)$,*

$$d(u, v\,|G) = \begin{cases} d(u, v\,|G_1) & u, v \in V(G_1), \\ d(u, v\,|G_2) & u, v \in V(G_2), \\ d(u, a_1\,|G_1) + d(a_2, v\,|G_2) + 1 & u \in V(G_1), v \in V(G_2). \end{cases}$$

In the following lemma, the distances d_0 and d_4 between edges of $(G_1 \sim G_2)(a_1, a_2)$ are computed. The results follow easily from Lemma 3.5, so the proofs are omitted.

Lemma 3.6. *Let $G = (G_1 \sim G_2)(a_1, a_2)$. For $e, f \in E(G)$,*

(i) $$d_0(e, f\,|G) = \begin{cases} d_0(e, f\,|G_1) & e, f \in E(G_1), \\ d_0(e, f\,|G_2) & e, f \in E(G_2), \\ D_1(a_1, e\,|G_1) + 1 & e \in E(G_1), f = a_1a_2, \\ D_1(a_2, e\,|G_2) + 1 & e \in E(G_2), f = a_1a_2, \\ D_1(a_1, e\,|G_1) + D_1(a_2, f\,|G_2) + 2 & e \in E(G_1), f \in E(G_2), \end{cases}$$

(ii) $$d_4(e, f\,|G) = \begin{cases} d_4(e, f\,|G_1) & e, f \in E(G_1), \\ d_4(e, f\,|G_2) & e, f \in E(G_2), \\ D_2(a_1, e\,|G_1) + 1 & e \in E(G_1), f = a_1a_2, \\ D_2(a_2, e\,|G_2) + 1 & e \in E(G_2), f = a_1a_2, \\ D_2(a_1, e\,|G_1) + D_2(a_2, f\,|G_2) + 1 & e \in E(G_1), f \in E(G_2). \end{cases}$$

In the following lemma, the degree of an arbitrary edge of $(G_1 \sim G_2)(a_1, a_2)$ is computed. The result follows easily from the definition of link of graphs, so the proof is omitted.

Lemma 3.7. *Let $G=(G_1\sim G_2)(a_1,a_2)$. For $e\in E(G)$,*

$$d(e\,|G\,)=\begin{cases} d(e\,|G_1\,) & e\in E(G_1), a_1\notin V(e),\\ d(e\,|G_1\,)+1 & e\in E(G_1), a_1\in V(e),\\ d(e\,|G_2\,) & e\in E(G_2), a_2\notin V(e),\\ d(e\,|G_2\,)+1 & e\in E(G_2), a_2\in V(e),\\ \alpha_1+\alpha_2 & e=a_1a_2.\end{cases}$$

Theorem 3.8. *Let $G=(G_1\sim G_2)(a_1,a_2)$. The first and second edge-degree distances of G are given by:*

(i) $$\begin{aligned} DD_{e_0}(G) =& DD_{e_0}(G_1)+DD_{e_0}(G_2)+\sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d_0(e,f\,|G_1\,)\\ &+\sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d_0(e,f\,|G_2\,)+(m_2+1)\sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1,e\,|G_1\,)\\ &+(m_1+1)\sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2,f\,|G_2\,)+\big[M_1(G_2)-2m_2+2\alpha_2+\alpha_1\big]D_1(a_1\,|G_1\,)\\ &+\big[M_1(G_1)-2m_1+2\alpha_1+\alpha_2\big]D_1(a_2\,|G_2\,)+(2m_2+1)\big(M_1(G_1)-2m_1\big)\\ &+(2m_1+1)\big(M_1(G_2)-2m_2\big)+\alpha_1(\alpha_1+2m_2)+\alpha_2(\alpha_2+2m_1)+(m_1+m_2)(\alpha_1+\alpha_2), \end{aligned}$$

(ii) $$\begin{aligned} DD_{e_4}(G) =& DD_{e_4}(G_1)+DD_{e_4}(G_2)+\sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d_4(e,f\,|G_1\,)\\ &+\sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d_4(e,f\,|G_2\,)+2\sum_{\{u,v\}\subseteq N(a_1|G_1\,)} d(u,v\,|G_1\,)\\ &+2\sum_{\{u,v\}\subseteq N(a_2|G_2\,)} d(u,v\,|G_2\,)+(m_2+1)\sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,)\\ &+(m_1+1)\sum_{f\in E(G_2)} d(f\,|G_2\,)D_2(a_2,f\,|G_2\,)+\big[M_1(G_2)-2m_2+2\alpha_2+\alpha_1\big]D_2(a_1\,|G_1\,)\\ &+\big[M_1(G_1)-2m_1+2\alpha_1+\alpha_2\big]D_2(a_2\,|G_2\,)+(m_2+1)\big[M_1(G_1)-2m_1+2\alpha_1\big]\\ &+(m_1+1)\big[M_1(G_2)-2m_2+2\alpha_2\big]+(m_1+m_2)(\alpha_1+\alpha_2). \end{aligned}$$

Proof. *(i)* By definition of the first edge-degree distance,

$$DD_{e_0}(G)=\sum_{\{e,f\}\subseteq E(G)}\big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

Now, we partition the above sum into five sums as follows:

The first sum S_1 consists of contributions to $DD_{e_0}(G)$ of pairs of edges from G_1:

$$S_1=\sum_{\{e,f\}\subseteq E(G_1)}\big[d(e\,|G\,)+d(f\,|G\,)\big]d_0(e,f\,|G\,).$$

In order to compute the sum S_1, we partition it into three sums S_{11}, S_{12} and S_{13} as follows:

The sum S_{11} is equal to:

$$S_{11} = \sum_{\{e,f\}\subseteq E(G_1);a_1\notin V(e)\cup V(f)} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$S_{11} = \sum_{\{e,f\}\subseteq E(G_1);a_1\notin V(e)\cup V(f)} \big[d(e\,|G_1) + d(f\,|G_1)\big]d_0(e,f\,|G_1).$$

The sum S_{12} is equal to:

$$S_{12} = \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$\begin{aligned} S_{12} &= \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} \Big[\big(d(e\,|G_1)+1\big) + \big(d(f\,|G_1)+1\big)\Big]d_0(e,f\,|G_1) \\ &= \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} \big[d(e\,|G_1) + d(f\,|G_1)\big]d_0(e,f\,|G_1) + 2\binom{\alpha_1}{2}. \end{aligned}$$

The sum S_{13} is equal to:

$$S_{13} = \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$\begin{aligned} S_{13} &= \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \Big[\big(d(e\,|G_1)+1\big) + d(f\,|G_1)\Big]d_0(e,f\,|G_1). \\ &= \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \big[d(e\,|G_1) + d(f\,|G_1)\big]d_0(e,f\,|G_1) + \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d_0(e,f\,|G_1). \end{aligned}$$

By adding the quantities S_{11}, S_{12} and S_{13}, we obtain:

$$S_1 = DD_{e_0}(G_1) + 2\binom{\alpha_1}{2} + \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d_0(e,f\,|G_1).$$

The second sum S_2 consists of contributions to $DD_{e_0}(G)$ of pairs of edges from G_2:

$$S_2 = \sum_{\{e,f\}\subseteq E(G_2)} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

Using the same argument as in the computation of S_1, we obtain:

$$S_2 = DD_{e_0}(G_2) + 2\binom{\alpha_2}{2} + \sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d_0(e,f\,|G_2).$$

The third sum S_3 is taken over all pairs of edges $e, f \in E(G)$ such that $e \in E(G_1)$ and $f = a_1a_2$:

$$S_3 = \sum_{e\in E(G_1),f=a_1a_2} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

In order to compute the sum S_3, we partition it into two sums S_{31} and S_{32} as follows:

The sum S_{31} is equal to:

$$S_{31} = \sum_{e\in E(G_1);a_1\notin V(e),f=a_1a_2} \big[d(e\,|G\,) + d(f\,|G\,)\big] d_0(e, f\,|G\,).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$\begin{aligned} S_{31} =& \sum_{e\in E(G_1);a_1\notin V(e)} \big[d(e\,|G_1\,) + (\alpha_1 + \alpha_2)\big] \big[D_1(a_1, e\,|G_1\,) + 1\big] \\ =& \sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1, e\,|G_1\,) + M_1(G_1) - 2m_1 - \alpha_1(\alpha_1 - 2) - \delta(a_1\,|G_1\,) \\ &+ (\alpha_1 + \alpha_2)\big[D_1(a_1\,|G_1\,) + m_1 - \alpha_1\big]. \end{aligned}$$

The sum S_{32} is equal to:

$$S_{32} = \sum_{e\in E(G_1);a_1\in V(e),f=a_1a_2} \big[d(e\,|G\,) + d(f\,|G\,)\big] d_0(e, f\,|G\,).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$\begin{aligned} S_{32} =& \sum_{e\in E(G_1);a_1\in V(e)} \big[\big(d(e\,|G_1\,) + 1\big) + (\alpha_1 + \alpha_2)\big] \\ =&\alpha_1(\alpha_1 - 2) + \delta(a_1\,|G_1\,) + \alpha_1(\alpha_1 + \alpha_2 + 1). \end{aligned}$$

By adding the quantities S_{31} and S_{32}, we obtain:

$$S_3 = \sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1, e\,|G_1\,) + M_1(G_1) - 2m_1 + \alpha_1 + (\alpha_1 + \alpha_2)\big[D_1(a_1\,|G_1\,) + m_1\big].$$

The fourth sum S_4 is taken over all pairs of edges $e, f \in E(G)$ such that $e \in E(G_2)$ and $f = a_1a_2$:

$$S_4 = \sum_{e\in E(G_2),f=a_1a_2} \big[d(e\,|G\,) + d(f\,|G\,)\big] d_0(e, f\,|G\,).$$

Using the same argument as in the computation of S_3, we obtain:

$$S_4 = \sum_{e\in E(G_2)} d(e\,|G_2\,)D_1(a_2, e\,|G_2\,) + M_1(G_2) - 2m_2 + \alpha_2 + (\alpha_1 + \alpha_2)\big[D_1(a_2\,|G_2\,) + m_2\big].$$

The fifth sum S_5 is taken over all pairs of edges $e, f \in E(G)$ such that $e \in E(G_1)$ and $f \in E(G_2)$:

$$S_5 = \sum_{e\in E(G_1)} \sum_{f\in E(G_2)} \big[d(e\,|G\,) + d(f\,|G\,)\big] d_0(e, f\,|G\,).$$

In order to compute the sum S_5, we partition it into four sums S_{51}, S_{52}, S_{53} and S_{54} as follows:

The sum S_{51} is equal to:

$$S_{51} = \sum_{e\in E(G_1);a_1\notin V(e)} \sum_{f\in E(G_2);a_2\notin V(f)} \big[d(e\,|G\,) + d(f\,|G\,)\big] d_0(e, f\,|G\,).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$\begin{aligned}S_{51} =& \sum_{e\in E(G_1);a_1\notin V(e)}\ \sum_{f\in E(G_2);a_2\notin V(f)} \big[d(e\,|G_1) + d(f\,|G_2)\big]\big[D_1(a_1,e\,|G_1) + D_1(a_2,f\,|G_2) + 2\big]\\ =&(m_2-\alpha_2)\sum_{e\in E(G_1)} d(e\,|G_1)D_1(a_1,e\,|G_1) + (m_1-\alpha_1)\sum_{f\in E(G_2)} d(f\,|G_2)D_1(a_2,f\,|G_2)\\ &+\big[M_1(G_1) - 2m_1 - \alpha_1(\alpha_1-2) - \delta(a_1\,|G_1)\big]\big[D_1(a_2\,|G_2) + 2(m_2-\alpha_2)\big]\\ &+\big[M_1(G_2) - 2m_2 - \alpha_2(\alpha_2-2) - \delta(a_2\,|G_2)\big]\big[D_1(a_1\,|G_1) + 2(m_1-\alpha_1)\big].\end{aligned}$$

The sum S_{52} is equal to:

$$S_{52} = \sum_{e\in E(G_1);a_1\notin V(e)}\ \sum_{f\in E(G_2);a_2\in V(f)} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$\begin{aligned}S_{52} =& \sum_{e\in E(G_1);a_1\notin V(e)}\ \sum_{f\in E(G_2);a_2\in V(f)} \big[d(e\,|G_1) + \big(d(f\,|G_2)+1\big)\big]\big[D_1(a_1,e\,|G_1) + 2\big]\\ =&\alpha_2\Big[\sum_{e\in E(G_1)} d(e\,|G_1)D_1(a_1,e\,|G_1) + 2\big(M_1(G_1) - 2m_1 - \alpha_1(\alpha_1-2) - \delta(a_1\,|G_1)\big)\Big]\\ &+\big[\alpha_2(\alpha_2-1) + \delta(a_2\,|G_2)\big]\big[D_1(a_1\,|G_1) + 2(m_1-\alpha_1)\big].\end{aligned}$$

The sum S_{53} is equal to:

$$S_{53} = \sum_{e\in E(G_1);a_1\in V(e)}\ \sum_{f\in E(G_2);a_2\notin V(f)} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

Using the same argument as in the computation of S_{52}, we obtain:

$$\begin{aligned}S_{53} =&\alpha_1\Big[\sum_{f\in E(G_2)} d(f\,|G_2)D_1(a_2,f\,|G_2) + 2\big(M_1(G_2) - 2m_2 - \alpha_2(\alpha_2-2) - \delta(a_2\,|G_2)\big)\Big]\\ &+\big[\alpha_1(\alpha_1-1) + \delta(a_1\,|G_1)\big]\big[D_1(a_2\,|G_2) + 2(m_2-\alpha_2)\big].\end{aligned}$$

The sum S_{54} is equal to:

$$S_{54} = \sum_{e\in E(G_1);a_1\in V(e)}\ \sum_{f\in E(G_2);a_2\in V(f)} \big[d(e\,|G) + d(f\,|G)\big]d_0(e,f\,|G).$$

Using Lemmas 3.6 and 3.7, we obtain:

$$\begin{aligned}S_{54} =&2\sum_{e\in E(G_1);a_1\in V(e)}\ \sum_{f\in E(G_2);a_2\in V(f)} \big[\big(d(e\,|G_1)+1\big) + \big(d(f\,|G_2)+1\big)\big]\\ =&2\alpha_2\big[\alpha_1(\alpha_1-2) + \delta(a_1\,|G_1)\big] + 2\alpha_1\big[\alpha_2(\alpha_2-2) + \delta(a_2\,|G_2)\big] + 4\alpha_1\alpha_2.\end{aligned}$$

By adding the quantities S_{51}, S_{52}, S_{53} and S_{54}, we obtain:

$$\begin{aligned} S_5 =& m_2 \sum_{e\in E(G_1)} d(e\,|G_1)D_1(a_1, e\,|G_1) + m_1 \sum_{f\in E(G_2)} d(f\,|G_2)D_1(a_2, f\,|G_2) \\ &+ \big[M_1(G_1) - 2m_1 + \alpha_1\big]D_1(a_2\,|G_2) + \big[M_1(G_2) - 2m_2 + \alpha_2\big]D_1(a_1\,|G_1) \\ &+ 2m_2\big[M_1(G_1) - 2m_1 + \alpha_1\big] + 2m_1\big[M_1(G_2) - 2m_2 + \alpha_2\big]. \end{aligned}$$

The formula of $DD_{e_0}(G)$ is obtained by adding the quantities S_1, S_2, S_3, S_4 and S_5 and simplifying the resulting expression.

(ii) Using the same argument as in the proof of part *(i)*, we can get the desired result. □

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Mahdieh Azari

Department of Mathematics, Kazerun Branch, Islamic Azad University, P. O. Box: 73135-168, Kazerun, Iran

Email: mahdie.azari@gmail.com, azari@kau.ac.ir

Hojjatollah Divanpour

Department of Science, Shiraz Technical College, Technical and Vocational University, Shiraz, Iran

Email: h.divanpour@yahoo.com