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PRODUCTS OF GRAPHS AND NORDHAUS-GADDUM TYPE INEQUALITIES

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ABSTRACT. In this paper, we obtain α as coefficient for the $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and by which we discuss Nikiforov's conjecture for λ_1 and Aouchiche and Hansen's conjecture for q_1 in Nordhaus-Gaddum type inequalities. Furthermore, by the properties of the products of graphs we put forward a new approach to find some bounds of Nordhaus-Gaddum type inequalities.

1. Introduction

Let G be a simple connected graph with n vertices and m edges. Suppose that A is the adjacency matrix of G and $d_1, d_2, \ldots, d_n$ the vertex degrees of G such that $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. We denote the complement graph of G by $\overline{G}$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian and the signless Laplacian of G where $D(G)$ is the diagonal matrix whose diagonal entries are the vertex degrees of G. The eigenvalues of the matrices $A(G)$, $L(G)$ and $Q(G)$ are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and $q_1 \geq q_2 \geq \cdots \geq q_n$, respectively. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors corresponding to λ_1 and q_1 and by Perron-Frobenius theorem, assume that all entries of $\boldsymbol{v}$ and $\boldsymbol{w}$ are positive. We call these vectors Perron-eigenvectors. Also we consider the maximum entries of $\boldsymbol{v}$ and $\boldsymbol{w}$ are 1. In case we have several maximum entries for Perron-eigenvector, we just choose and fix one of them. For any vertex $u \in V(G)$, we denote the entry of the eigenvector $\boldsymbol{v}$ on u by $\boldsymbol{v}(u)$. The set of the neighbors of vertex v_i are denoted by $N(v_i)$ and $\overline{N(v_i)}$ is the set $V - N(v_i)$. Suppose G and H are two disjoint graphs , denote by $G \cup H$ and $G \bigtriangledown H$

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disjoint union and join of G and H, respectively. The Cartesian product of two simple graphs G and H denoted by $G\Box H$.

A classical paper of Nordhaus and Gaddum [9] established the following inequalities for the chromatic numbers $\chi(G)$ and $\chi(\overline{G})$

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1,$$
$$n \leq \chi(G)\chi(\overline{G}) \leq \frac{(n+1)^2}{4}.$$

Initially, Aouchiche and Hansen in the paper [1] said that, this type of relation did not attract much attention. Actually, the first detailed study of these relations came almost a decade after the publication of the paper of Nordhaus and Gaddum [9]. But later, there was broader attention to the problem of Nordhaus-Gaddum in obtaining the upper and lower bounds for sum and product parameters of a graph and its complement. We recall some of these results for parameters λ_1 and q_1. For λ_1 and $\overline{\lambda_1}$ of graphs G and $\overline{G}$, Nikiforov [8] conjectured that

$$\lambda_1 + \overline{\lambda_1} \leq \frac{4}{3}n + O(1). \tag{1.1}$$

This conjecture was proved by Terpai [12]. Some other results on $\lambda_1 + \overline{\lambda}_1$ are

$$\lambda_1 + \overline{\lambda_1} \leq \sqrt{2n(n-1) - 4\delta(n-1-\Delta)+1} - 1, \quad [6] \tag{1.2}$$

$$\lambda_1 + \overline{\lambda_1} \leq \sqrt{2((n-1)^2 - 2\delta n + 2\Delta\delta - \Delta + 3\delta)}, \quad [10] \tag{1.3}$$

$$\lambda_1 + \overline{\lambda_1} \leq \frac{n - \Delta + \delta - 3 + \sqrt{2((n-\Delta)^2 + 4n(\Delta-\delta)^2(\delta+1))}}{2}. \quad [10] \tag{1.4}$$

There was another conjecture for q_1 and $\overline{q_1}$,

$$q_1 + \overline{q_1} \leq 3n - 4. \quad [1] \tag{1.5}$$

This conjecture was proved by Ashraf and Tayfeh-Rezaie [2]. Other conjecture by Aouchiche and Hansen was

$$q_1\overline{q_1} \leq 2n(n-2), \quad [1] \tag{1.6}$$

that disproved by Ashraf and Tayfeh-Rezaie [2].

In this paper, we obtain α as coefficient for graph $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and discuss obtained bound by Nikiforov [8] for $\lambda_1 + \overline{\lambda}_1$, also conjectured bound by Aouchiche and Hansen in [1] for $q_1 + \overline{q}_1$. In section 2, we use the structure of graph products to provide a theory to obtain upper bounds for Nordhaus-Gaddum type inequalities for sum or product of the greatest eigenvalues of the adjacency matrix and Laplacian and singless Laplacian matrices of the graphs G and $\overline{G}$.

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2. Extremal Graphs Of Nordhaus- Gaddum Bounds

In this section we provide a value of α as the coefficient of the number of vertices of $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and then occurrence of equality of the two bounds that can be seen in the following.

Conjecture 2.1. [8] *For any graph G on n vertices,*

$$\lambda_1 + \overline{\lambda_1} \leq \frac{4}{3}n + O(1). \tag{2.1}$$

Conjecture 2.2. [1] *Let G be a simple graph on $n \geq 2$ vertices. Then,*

$$q_1 + \overline{q_1} \leq 3n - 4. \tag{2.2}$$

For $0 \leq \alpha \leq 1$, suppose that $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\overline{G} = \overline{K_{\alpha n}} \bigtriangledown K_{(1-\alpha)n}$ that are displayed in Figure 1. To obtain eigenvalues $\overline{\lambda}_1$ and $\overline{q}_1$ of $\overline{G}$ we use the following theorems.

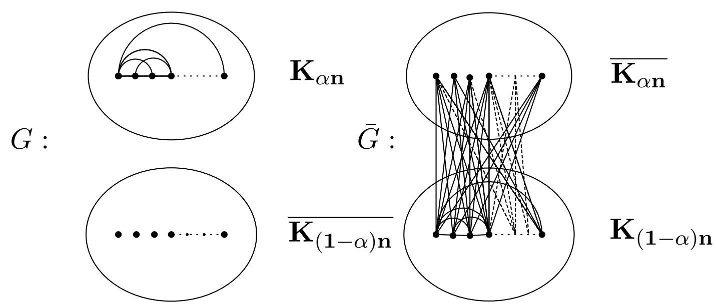

FIGURE 1. $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\overline{G} = \overline{K_{\alpha n}} \bigtriangledown K_{(1-\alpha)n}$

Theorem 2.3. [11] *Let G_i be an r_i-regular graph on n_i vertices for $i = 1,2$. If $P(A(G_i), \lambda)$ is the characteristic polynomial of G_i and $i = 1,2$, then*

$$P(A(G_1 \bigtriangledown G_2), \lambda) = \frac{P(A(G_1), \lambda)P(A(G_2), \lambda)}{(\lambda - r_1)(\lambda - r_2)}[(\lambda - r_1)(\lambda - r_2) - n_1 n_2]. \tag{2.3}$$

Theorem 2.4. [5] *Let G_i be an r_i-regular graph on n_i vertices for $i = 1,2$. If $\phi_{G_i}(q)$ is the signless Laplacian characteristic polynomial of G_i and $i = 1,2$, then*

$$\phi(Q(G_1 \bigtriangledown G_2, q) = \frac{\phi(Q(G_1), q - n_2)\phi(Q(G_2), q - n_1)}{(q - 2r_1 - n_2)(q - 2r_2 - n_1)} f(q) \tag{2.4}$$

where $f(q) = q^2 - (2(r_1 + r_2) + (n_1 + n_2))q + 2(2r_1 r_2 + r_1 n_1 + r_2 n_2)$.

First, we consider $\lambda_1 + \overline{\lambda}_1$ for the graphs $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\overline{G} = \overline{K_{\alpha n}} \bigtriangledown K_{(1-\alpha)n}$,

$$(\lambda_1 + \overline{\lambda}_1)(\alpha) = \alpha n - 1 + \frac{1}{2}(n - \alpha n - 1 + \sqrt{n^2 - 3\alpha^2 n^2 + 2\alpha n^2 + 2\alpha n - 2n}). \tag{2.5}$$

For the maximum value of $(\lambda + \overline{\lambda}_1)(\alpha)$ by using derivative we have,

$$\alpha_{max} = \frac{n + 1 + \sqrt{n^2 - n + 1}}{3n}. \tag{2.6}$$

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By substituting α_{max} in relation (2.5), we have

$$(\lambda_1 + \overline{\lambda}_1)(\alpha) \leq \frac{4n - 5 + 4\sqrt{n^2 - n + 1}}{6} < \frac{4}{3}n - 1. \tag{2.7}$$

The obtained α in relation (2.6) tends to $\frac{2}{3}$, when n tends to infinity and relation (2.7) tends to $\frac{4}{3}n$. Now, we conjecture the following:

Conjecture 2.5. *Let G be a simple graph on n vertices. Then*

$$\lambda_1 + \overline{\lambda}_1 \leq \frac{4n - 5 + 4\sqrt{n^2 - n + 1}}{6}. \tag{2.8}$$

Now, we consider $q_1 + \overline{q}_1$ for the graphs $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\overline{G} = \overline{K_{\alpha n}} \bigtriangledown K_{(1-\alpha)n}$.

$$(q_1 + \overline{q}_1)(\alpha) = 2(\alpha n - 1) + \frac{1}{2}(3n - 2\alpha n - 2 + \sqrt{n^2 - 4\alpha^2 n^2 + 4\alpha n^2 - 4n + 4}). \tag{2.9}$$

As is mentioned above, the maximum value of $(q_1 + \overline{q}_1)(\alpha)$ by using its derivative, is at

$$\alpha_{max} = \frac{1 + \sqrt{1 - \frac{2(n-1)}{n^2}}}{2}. \tag{2.10}$$

By substituting α_{max} in relation (2.9), we have

$$(q_1 + \overline{q}_1)(\alpha_{max}) = \sqrt{n^2 - 2n + 2} + 2n - 3. \tag{2.11}$$

Because $n - 1 < n\alpha_{max} < n$ and $1 \leq n\alpha \leq n - 1$, the maximum value of $q_1 + \overline{q}_1$ for simple graphs is for $n\alpha = n - 1$ and we have

$$q_1 + \overline{q}_1 \leq (q_1 + \overline{q}_1)_{(\alpha = \frac{n-1}{n})} = 3n - 4 < \sqrt{n^2 - 2n + 2} + 2n - 3. \tag{2.12}$$

3. Upper Bound For Problems Of Nordhaus-Gaddum Type By Using Product Of Graph

In the following, some definitions of the products of graphs are recalled :

Definition 3.1. [3] *Direct product of two simple graphs G and H denoted by $G \times H$ has the vertex-set $V(G) \times V(H)$. For any $u, v \in V(G)$ and $x, y \in V(H)$, (u, x) is adjacent to (v, y) if $uv \in E(G)$ and $xy \in E(H)$. The adjacency matrix of $G \times H$ is the direct product of the adjacency matrices of G and H.*
If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are eigenvectors for G and H with eigenvalues $\lambda_i\,(1 \leq i \leq m)$ and ${\lambda'}_j\,(1 \leq j \leq n)$, respectively, then the vector $\boldsymbol{w} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is an eigenvector of $G \times H$ with eigenvalues $\lambda_i \lambda'_j$.

Remark 3.2. *The direct product is also called, the tensor product categorical product, cardinal product, relational product, Kronecker product, weak direct product, or conjunction. The notation $G \times H$ is also sometimes used to represent another construction known as the Cartesian product of graphs, but more commonly refers to the direct product.*

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The tensor product of matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of orders $m \times p$ and $n \times q$, respectively, is the matrix $A \otimes B$ of order $mn \times pq$ defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1p}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \cdots & a_{mp}B \end{bmatrix}$$

Definition 3.3. [3] *The Cartesian product of two simple graphs G and H denoted by $G\Box H$ has the vertex-set $V(G) \times V(H)$. For any $u, v \in V(G)$ and $x, y \in V(H)$, (u, x) is adjacent to (v, y) if $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$. Let A and B be adjacency matrices of graphs G and H of orders m and n, respectively. Its adjacency is $A \otimes I_n + I_m \otimes B$.*
If $\boldsymbol{v}$ and $\boldsymbol{u}$ are eigenvectors for G and H with eigenvalues $\lambda_i\,(1 \leq i \leq m)$ and $\lambda'_j\,(1 \leq j \leq n)$, respectively, then the vector $\boldsymbol{w} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is an eigenvector of $G\Box H$ with eigenvalue $\lambda_i + \lambda'_j$.

Definition 3.4. [3] *The strong product of two simple graphs G and H denoted by $G \boxtimes H$ has the vertex-set $V(G) \times V(H)$. For any $u, v \in V(G)$ and $x, y \in V(H)$, (u, x) is adjacent to (v, y) if $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$ or. If A and B are the adjacency matrices of G and H of orders m and n then, $A \otimes B + A \otimes I_n + I_m \otimes B$ is the adjacency matrix of $G \boxtimes H$. It follows that the eigenvalues of $G \boxtimes H$ are the numbers $(\lambda_i + 1)(\lambda'_j + 1) - 1$, where λ_i and λ'_j the eigenvalues of G and H, respectively.*

Theorem 3.5. [5] *Let the Laplacian eigenvalues of graphs G and H are μ_i and μ_j. The Laplacian eigenvalues of the Cartesian product $G\Box H$ of graphs G and H are equal to all the possible sums of eigenvalues of the two factors:*

$$\mu_i + \mu_j \qquad 1 \leq i \leq |V_G| \quad \textit{and} \quad 1 \leq j \leq |V(H)|$$

Remark 3.6. *Suppose that G and H are two arbitrary graphs. We have for the matrices $A(G)$ and $A(H)$*

$$\begin{aligned} Q(G\Box H) &= D(G\Box H) + A(G\Box H) = (D(G) \otimes I + I \otimes D(H)) + (A(G) \otimes I + I \otimes A(H)) \\ &= (D(G) + A(G)) \otimes I + I \otimes (D(H) \otimes A(H)) = Q(G) \otimes I + I \otimes Q(H). \end{aligned}$$

So we can conclude similar to Theorem 3.5 for singless Laplacian eigenvalues.

Using the properties of Cartesian product of adjacency, Laplacian, and singless Laplacian matrices in [3] , [5], and Remark 3.6, we have

$$\lambda_1 + \overline{\lambda}_1 = \lambda_1(G\Box\overline{G}).$$
$$\mu + \overline{\mu}_1 = \mu_1(G\Box\overline{G}).$$
$$q_1 + \overline{q}_1 = q_1(G\Box\overline{G}).$$

In conclusion, by using the properties and structure of Cartisan product, we can obtain upper bounds for Nordhaus-Gaddum type inequalities of sum of eigenvalues. In fact, it is hoped that, by using the

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properties of product of graphs, we can obtain better bounds of Nordhaus-Gaddum type inequalities by easier and better methods.

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