



www.combinatorics.ir

Transactions on Combinatorics

ISSN (print): 2251-8657, ISSN (on-line): 2251-8665

Vol. 6 No. 2 (2017), pp. 37-48.

© 2017 University of Isfahan



www.ui.ac.ir

THE SITE-PERIMETER OF WORDS

AUBREY BLECHER, CHARLOTTE BRENNAN^{*†}, ARNOLD KNOPFMACHER[‡] AND TOUFIK MANSOUR

Communicated by Jamshid Moori

ABSTRACT. We define $[k] = \{1, 2, 3, \dots, k\}$ to be a (totally ordered) *alphabet* on k letters. A *word* w of length n on the alphabet $[k]$ is an element of $[k]^n$. A word can be represented by a bargraph which is a family of column-convex polyominoes whose lower edge lies on the x -axis and in which the height of the i -th column in the bargraph equals the size of the i -th part of the word. Thus these bargraphs have heights which are less than or equal to k . We consider the site-perimeter, which is the number of nearest-neighbour cells outside the boundary of the polyomino. The generating function that counts the site-perimeter of words is obtained explicitly. From a functional equation we find the average site-perimeter of words of length n over the alphabet $[k]$. We also show how these statistics may be obtained using a direct counting method and obtain the minimum and maximum values of the site-perimeters.

1. Introduction

The enumeration of polyominoes according to their area and perimeter has been well documented and researched and is a basic problem in combinatorics [3]. The *site-perimeter* is the number of nearest-neighbour cells outside the boundary of the polyomino. In essence, this notion is useful as a mathematical correlate of the physics and probability of percolation [7, 10]. Site-perimeter studies were initiated for specific combinatorial objects (staircase polyominoes) in [5]. The site-perimeter for directed animals was investigated in [3, 6, 9], and advanced for combinatorial objects (in this

MSC(2010): Primary: 05A15; Secondary: 05A05.

Keywords: words, bargraphs, site-perimeter, generating functions.

Received: 26 January 2016, Accepted: 29 January 2017.

*Corresponding author.

†This material is based upon work supported by the National Research Foundation under grant number 86329.

‡This material is based upon work supported by the National Research Foundation under grant number 81021.

case, bargraphs) by Bousquet-Mélou and Rechnitzer in [4]. The current authors have investigated the site-perimeter in permutations (see [2]).

In this paper we extend the site-perimeter idea to another fundamental type of combinatorial object, namely words over a fixed alphabet of k letters. Words are used as the underlying object in many probabilistic, mathematical, physical and chemical research studies (see [8] and references therein). We define $[k]$ to be a (totally ordered) *alphabet* on k letters. A *word* w of length n on the alphabet $[k]$ is an element of $[k]^n$ in which each letter is also called a part. A word can be represented by a bargraph which is a family of column-convex polyominoes whose lower edge lies on the x -axis.

These bargraphs are drawn on a regular planar lattice grid and are made up of square cells. Given a word over an alphabet $[k]$, the height of the i -th column of the representing bargraph matches the size of the i -th part. The length of the word is the number of columns in the representing bargraph in which each column with height at most k . The perimeter is the number of edges on the boundary of the bargraph. We illustrate, in the figure below, the site-perimeter of the word 25463 over any alphabet $[k]$ where $k \geq 6$. The site-perimeter 20 is the sum of the cells marked by “o”.

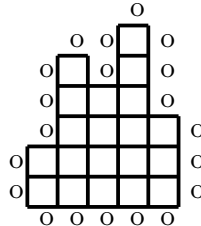


FIGURE 1. Site-perimeter of the word 25463

From now on, we will consider a word to be its associated bargraph.

In this paper, we find the explicit generating function for words over the alphabet $[k]$ according to the site-perimeter and the number of columns. We also derive the average site-perimeter for words of length n by two different methods.

2. Appending a well

We adapt the method used by Bousquet-Mélou and Rechnitzer in [4] to the case of words over an alphabet $[k]$. Let $M(x, p, s)$ be the generating function that enumerates words where s indicates the height of the last column of the word (or value of the last letter), x marks the length of the word and p marks the site-perimeter. For a particular word B whose last letter is n , we append first the letter b and then the letter c to the right of n . Here we only consider the cases where abc form a well (i.e., by definition $b < n$ and $b < c$). See Figure 2, below. The solid circles indicate new site-perimeter whereas hollow circles indicate existing site-perimeter.

In the vertical column to the right of the word, the site-perimeter increases by $c - b$. The number of columns increases by 2 (adding 3 to the site-perimeter). The well construction contributes either 0

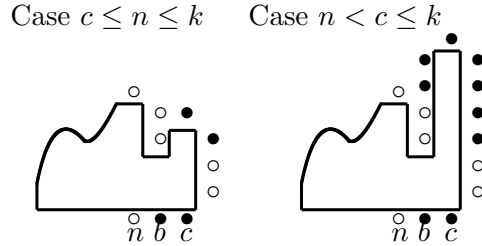


FIGURE 2. Adding a well to a word over $[k]$

(where $c \leq n$) or $c - n$ (where $n < c \leq k$) from the two columns of the well construction. Therefore the increase in the site-perimeter is

$$(c - b) + \max\{0, c - n\} + 3.$$

Since B is any word the full generating function for all such words (that ignores the last letter) is $M(x, p, 1)$. Hence the generating function $M(x, p, s)$ for all words obtained by appending a well is

$$M(x, p, s) = M(x, p, 1)x^2 \sum_{b=1}^{n-1} \left(\sum_{c=b+1}^n p^3 p^{-b} (sp)^c + \sum_{c=n+1}^k p^{3-n} p^{-b} (sp^2)^c \right).$$

After summing these geometric series, we obtain

$$(2.1) \quad M(x, p, s) = M(x, p, 1)x^2 \times \left(\frac{p^{5-2n}(p - p^n)s(p^{2n}s^n - p^{2k}s^k)}{(1 - p)(1 - p^2s)} - \frac{p^4s(s^n - ps^{1+n} - (ps)^n - s(1 - p - (sp)^n))}{(1 - p)(1 - s)(1 - ps)} \right).$$

Let $B(x, p, s)$ be the generating function that counts the number of words where s marks the size of the last part, x the number of parts and p the site-perimeter. Our first result is:

Lemma 2.1. *The generating function $B(s)$ for the site-perimeter for words satisfies the functional equation*

$$(2.2) \quad \begin{aligned} B(s) &:= B(x, p, s) \\ &= \frac{sp^4}{1 - sp^2}(1 - (sp^2)^k) + xp^2B(s) + \frac{px}{1 - s}(sB(1) - B(s)) \\ &+ \frac{xp^3s}{1 - p^2s} \left[B(s) - \frac{px}{1 - s}(sB(1) - B(s)) - s^k p^{2k} \left(B(p^{-2}) - \frac{px}{1 - p^{-2}}(p^{-2}B(1) - B(p^{-2})) \right) \right] \\ &+ x^2 p^5 s \frac{-p^{1+2k}s^k B(p^{-2}) + p^{2k}s^k B(p^{-1}) + pB(s) - B(ps)}{(1 - p)(1 - p^2s)} \\ &+ x^2 p^4 s \frac{-B(s) + psB(s) + B(ps) + s(1 - p)B(1) - sB(ps)}{(1 - p)(1 - s)(1 - ps)}. \end{aligned}$$

Proof. Here we adapt the method of Bousquet-Mélou and Rechnitzer in [4, Proposition 4]. Append columns to the right of an existing bargraph. If the rightmost column is strictly greater than the preceding one we call the word *ascending*. Likewise, we call it *steady* if the two last columns are equal

and *descending* when the last column is strictly less than its predecessor. There are five cases to consider, described as follows:

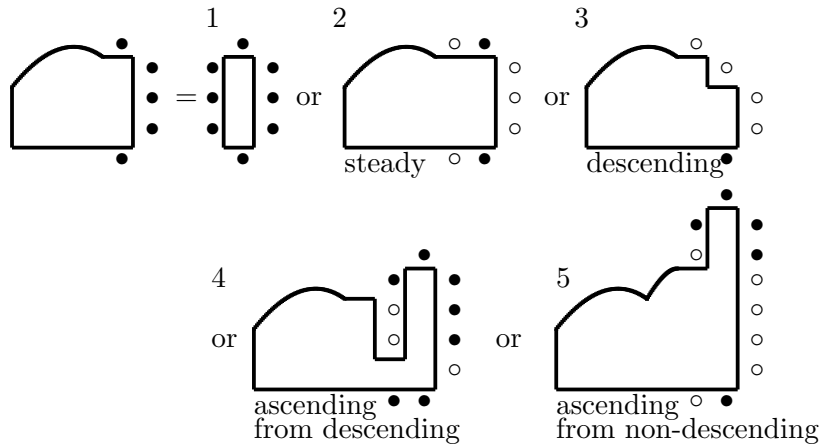


FIGURE 3. Building words over $[k]$ adapted from paper [4]

(1) The smallest word consisting of one column is the word 1, the site-perimeter is 4. Thus the generating function for words consisting of a single column is

$$sxp^4 \sum_{c=1}^k (sp^2)^c = \frac{sxp^4}{1-sp^2} (1-(sp^2)^k).$$

(2) We construct *steady* words by duplicating the last column of some word; doing this increases the site-perimeter by two and the number of columns by 1 to yield $xp^2B(s)$.

(3) *Descending* words are constructed by appending a shorter column to the right of some word; doing so increases the site-perimeter by 1, and yields $\frac{px}{1-s}(sB(1)-B(s))$. The subtraction of the term $B(s)$ is because the geometric series $\frac{p}{1-s}$ counts all last columns including the steady and ascending cases which now have to be excluded.

(4) We split the construction of *ascending* words into two subcases, the first of which is precisely the well construction from Section 2. So we consider how to construct an ascending word from a *descending* one. Here, we append two columns to the right of an arbitrary word (the first descending and the second ascending); so (2.1) gives

$$(2.3) \quad x^2p^5s \frac{-p^{1+2k}s^k B(p^{-2}) + p^{2k}s^k B(p^{-1}) + pB(s) - B(ps)}{(1-p)(1-p^2s)} + x^2p^4s \frac{-B(s) + psB(s) + B(ps) + s(1-p)B(1) - sB(ps)}{(1-p)(1-s)(1-ps)}.$$

(5) In the last subcase, we construct an ascending word from a word that is either *steady* or *ascending*. Let $R(s) := R(x, p, s)$ be the generating function of words that are steady or ascending.

The generating function for our fifth and last class of words is then

$$\frac{sxp^3}{1-sp^2}R(x, p, s).$$

Since $R(x, p, s)$ counts all words, except the descending ones, case 3 of the current construction gives

$$R(s) = B(s) - \frac{px}{1-s}(sB(1) - B(s))$$

and hence constructing an ascending word from a non-descending word gives:

$$\frac{sxp^3}{1-sp^2} \left(B(s) - \frac{px}{1-s}(sB(1) - B(s)) \right).$$

Since the maximum height of the last column is k , we have for the last column

$$x \sum_{c=b+1}^k s^c p^3 (p^2)^{c-b-1} = \frac{xp^3(s^{b+1} - p^{2k-2b}s^{k+1})}{1-p^2s}.$$

Thus, generalizing from this single last column to any class of non-descending words, we obtain

$$\begin{aligned} & \frac{xp^3s}{1-p^2s} \left[R(s) - s^k p^{2k} R(p^{-2}) \right] \\ (2.4) \quad & = \frac{xp^3s}{1-p^2s} \left[B(s) - \frac{px}{1-s}(sB(1) - B(s)) - s^k p^{2k} \left(B(p^{-2}) - \frac{px}{1-p^{-2}}(p^{-2}B(1) - B(p^{-2})) \right) \right]. \end{aligned}$$

Adding the five cases completes the proof. □

In order to solve the functional equation in (2.2), we need the following notation.

Definition 2.2. Let $|p|, |x| < 1$ and

$$\begin{aligned} F(s) &= \frac{(1 + px - p^2x)(1 - p^2s + p^2s^2) + (p^5x^2 - p^3x + p^2x - 1)s}{(1 - s)(1 - sp^2)}, \\ G(s) &= -\frac{x^2p^4s}{(1 - ps)(1 - sp^2)}, \\ H_0(s) &= -\frac{sxp^4(1 - (sp^2)^k)}{1 - sp^2}, \\ H_1(s) &= -\frac{pxs}{1 - s} - \frac{x^2p^4s^2}{(1 - s)(1 - ps)} - \frac{x^2p^4s^2(p^2 - 1 - s^{k-1}(1 - s)p^{2k})}{(1 - s)(1 - p^2)(1 - sp^2)}, \\ H_2(s) &= -\frac{x^2p^{2k+5}s^{k+1}}{(1 - p)(1 - sp^2)}, \\ H_3(s) &= \frac{xp^{2k+3}s^{k+1}(1 - p^2 + xp^4)}{(1 - p^2)(1 - sp^2)}. \end{aligned}$$

Moreover, let $A_m(s) = H_m(s) + \sum_{j \geq 1} (-1)^j \frac{H_m(p^j s) \prod_{i=0}^{j-1} G(p^i s)}{\prod_{i=1}^{j-1} F(p^i s)}$.

Now, we express the generating function $B(s)$ in terms of $B(1)$, $B(1/p)$ and $B(1/p^2)$.

Lemma 2.3. Under the notation of Definition 2.2, we have

$$F(s)B(x, p, s) + A_1(s)B(x, p, 1) + A_2(s)B(x, p, p^{-1}) + A_3(s)B(x, p, p^{-2}) + A_0(s) = 0.$$

Proof. Define $\beta_j = B(x, p, p^j s)$ for all $j \geq 0$. By (2.2),

$$\beta_j + \frac{G(p^j s)}{F(p^j s)}\beta_{j+1} + \frac{H_1(p^j s)}{F(p^j s)}B(x, p, 1) + \frac{H_2(p^j s)}{F(p^j s)}B(x, p, p^{-1}) + \frac{H_3(p^j s)}{F(p^j s)}B(x, p, p^{-2}) + \frac{H_0(p^j s)}{F(p^j s)} = 0.$$

Thus, by iterating the above equation for $j \geq 0$ and using the fact that $\lim_{j \rightarrow \infty} B(x, p, p^j s) = B(x, p, 0) = 0$, we obtain an explicit formula for $\beta_0 = B(x, p, s)$:

$$F(s)\beta_0 + A_1(s)B(x, p, 1) + A_2(s)B(x, p, p^{-1}) + A_3(s)B(x, p, p^{-2}) + A_0(s) = 0,$$

as claimed. □

Theorem 2.4. *Under the notation of Definition 2.2, we have*

$$B(x, p, 1) = -\frac{A_0(r_+)A_3(r_-) - A_0(r_-)A_3(r_+)}{A_1(r_+)A_3(r_-) - A_1(r_-)A_3(r_+)},$$

where

$$r_+ = \frac{g + \sqrt{g^2 - 4p^2(xp^2 - px - 1)^2}}{2p^2(xp^2 - px - 1)} = 1 + px + \frac{p^3x^2}{1-p} + \dots,$$

$$r_- = \frac{g - \sqrt{g^2 - 4p^2(xp^2 - px - 1)^2}}{2p^2(xp^2 - px - 1)} = \frac{1}{p^2} - \frac{x}{p} + \frac{(2p-1)x^2}{p-1} + \dots$$

with $g = x^2p^5 + xp^4 - 2xp^3 - p^2 + xp^2 - 1$.

Proof. Note that $H_2(s)/H_3(s) = \frac{-(p+1)p^2x}{p^4x-p^2+1}$, so $A_2(s)/A_3(s) = \frac{-(p+1)p^2x}{p^4x-p^2+1}$. Thus, Lemma 2.3 gives

$$\frac{F(s)}{A_3(s)}B(x, p, s) = \frac{A_1(s)}{A_3(s)}B(x, p, 1) + \frac{A_2(s)}{A_3(s)}B(x, p, p^{-1}) + B(x, p, p^{-2}) + \frac{A_0(s)}{A_3(s)},$$

which is equivalent to

$$(2.5) \quad \frac{F(s)}{A_3(s)}B(x, p, s) = \frac{A_1(s)}{A_3(s)}B(x, p, 1) + K(x, p) + \frac{A_0(s)}{A_3(s)},$$

where

$$K(x, p) = \frac{-(p+1)p^2x}{p^4x-p^2+1}B(x, p, p^{-1}) + B(x, p, p^{-2}).$$

The functional equation in (2.5) can be solved by the kernel method as follows. Note that the generating functions r_+ and r_- satisfy $F(r_+) = F(r_-) = 0$ and $r_+r_- = \frac{1}{p^2}$. Moreover, as power series, we have

$$A_3(r_+) = \frac{2p^{2k+3}}{2(1-p^2)}x + \dots \neq 0, \quad A_3(r_-) = 1 + \frac{p^4 - p^3 - p^2 - kp(1-p^2)}{1-p^2}x + \dots \neq 0.$$

Hence, by substituting $s = r_+$ or $s = r_-$ into (2.5), we obtain

$$\begin{cases} \frac{A_1(r_+)}{A_3(r_+)}B(x, p, 1) + K(x, p) + \frac{A_0(r_+)}{A_3(r_+)} = 0, \\ \frac{A_1(r_-)}{A_3(r_-)}B(x, p, 1) + K(x, p) + \frac{A_0(r_-)}{A_3(r_-)} = 0, \end{cases}$$

which implies

$$\left(\frac{A_1(r_+)}{A_3(r_+)} - \frac{A_1(r_-)}{A_3(r_-)} \right) B(x, p, 1) + \frac{A_0(r_+)}{A_3(r_+)} - \frac{A_0(r_-)}{A_3(r_-)} = 0.$$

Since $\frac{A_1(r_+)}{A_3(r_+)} \neq \frac{A_1(r_-)}{A_3(r_-)}$, the result follows. □

Next, to find the mean value, it is sufficient to differentiate (2.2) with respect to p which leads to a more manageable equation.

Recall that $R(s)$ and $B(s)$ are functions of s , x and p ; we shall denote them as $R(p, s)$ and $B(p, s)$.

Also, we use the notation $B_p(p, r)$ and $B_s(p, r)$ for the partial derivatives $\frac{\partial B(p,s)}{\partial p}|_{s=r}$ and $\frac{\partial B(p,s)}{\partial s}|_{s=r}$ respectively. Now, we differentiate (2.2) with respect to p , and obtain an expression of the form

$$\begin{aligned}
 B_p(p, s) &= a(p, s) + b(p, s)B(p, 1) + c(p, s)B(p, s) + d(p, s)B(p, ps) + e(p, s)B(p, 1/p) \\
 &+ f(p, s)B(p, 1/p^2) + g(p, s)B_p(p, 1) + h(p, s)B_p(p, s) + k(p, s)B_p(p, 1/p) + l(p, s)B_p(p, ps) \\
 &+ m(p, s)B_p(p, 1/p^2) + n(p, s)B_s(p, 1/p) + r(p, s)B_s(p, ps) + B_s(p, 1/p^2).
 \end{aligned}$$

After substituting $p = 1$ we obtain

$$\begin{aligned}
 B_p(1, s) &= a(1, s) + b(1, s)B(1, 1) + c(1, s)B(1, s) + d(1, s)B(1, s) + e(1, s)B(1, 1) \\
 &+ f(1, s)B(1, 1) + g(1, s)B_p(1, 1) + h(1, s)B_p(1, s) + k(1, s)B_p(1, 1) + l(1, s)B_p(1, s) \\
 &+ m(1, s)B_p(1, 1) + n(1, s)B_s(1, 1) + r(1, s)B_s(1, s) + B_s(1, 1),
 \end{aligned}$$

for certain functions $a(p, s)$ up to $r(p, s)$.

The constant term $a(p, s)$ and the coefficients of the terms $B(1, s)$, $B(1, 1)$, $B_s(1, 1)$, $B_s(1, s)$, $B_p(1, s)$ and $B_p(1, 1)$ in the above expression are shown in the table below:

Constant term	$\frac{2sx(2 - s - (2 + k)s^k + (1 + k)s^{1+k})}{(1 - s)^2}$
$B(1, s)$	$\frac{x(1 - s(1 + x) + s^2(1 + 2x) - s^3)}{(1 - s)^3}$
$B(1, 1)$	$\frac{sx(1 - 2s + s^{1+k}(4 + 4k - 2x) + s^2(1 - x) - s^k(3 + 2k - x) - s^{2+k}(1 + 2k - x))}{(1 - s)^3}$
$B_s(1, 1)$	$\frac{s^{1+k}(4 - x)x}{2(1 - s)}$
$B_s(1, s)$	$\frac{s^2x^2}{(1 - s)^2}B_s(1, s)$
$B_p(1, s)$	-1
$B_p(1, 1)$	$\frac{s(1 - s^k)x}{1 - s}$

However, $B(1, 1)$ is the well-known generating function for words over the alphabet $[k]$, i.e., $B(1, 1) = \frac{kx}{1 - kx}$. Also, the generating function for words where s marks the last column is $B(1, s) = \frac{sx(1 - s^k)}{(1 - s)(1 - kx)}$.

Therefore $B_s(1, s) = \frac{x(1 - (1+k)s^k + ks^{1+k})}{(1 - s)^2(1 - kx)}$, and $B_s(1, 1) = \frac{kx(1+k)}{2(1 - kx)}$.

Using the above expressions in $B_p(1, s)$ we obtain

$$\begin{aligned}
 B_p(1, s) &= \frac{2sx(2 - s - (2 + k)s^k + (1 + k)s^{1+k})}{(1 - s)^2} + \frac{k(1 + k)s^{1+k}(4 - x)x^2}{4(1 - s)(1 - kx)} \\
 &+ \frac{s^2(1 - (1 + k)s^k + ks^{1+k})x^3}{(1 - s)^4(1 - kx)}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{k s x^2(1-2 s+s^{1+k}(4+4 k-2 x)+s^2(1-x)+s^k(-3-2 k+x)+s^{2+k}(-1-2 k+x))}{(1-s)^3(1-k x)} \\
 & - \frac{s(1-s^k) x^2(-1+s(1+x)-s^2(1+2 x)+s^3)}{(1-s)^4(1-k x)} \\
 (2.6) \quad & + \frac{s x(1-s^k)}{1-s} B_p(1,1)+\frac{s^{1+k}(-5 k-3 k^2+2 k^3) x^3}{12(1-s)(1-k x)} x^3.
 \end{aligned}$$

Taking the limit as $s \rightarrow 1$, we finally obtain an equation of the form

$$(2.7) \quad B_p(1, s)=k x B_p(1, s)+\frac{k x\left(2\left(18+4 x+x^2\right)+k\left(12-24 x-5 x^2\right)-4 k^2 x(2-x)-k^3 x^2\right)}{12(1-k x)}.$$

Solving for $B_p(1, s)$ yields Theorem 2.5.

Theorem 2.5. *The generating function for the sum of the site-perimeter of words over the alphabet $[k]$, where x marks the size of the word, is*

$$\frac{k x\left(36+12 k+8 x-24 k x-8 k^2 x+2 x^2-5 k x^2+4 k^2 x^2-k^3 x^2\right)}{12(1-k x)^2}.$$

In the table below, we list the values for total site perimeter for the alphabet $[k]$, where $k = 1$ to 7, and the size of the word $n = 1$ to 7:

$k \backslash n$	1	2	3	4	5	6	7
1	4	6	8	10	12	14	16
2	10	28	72	176	416	960	2176
3	18	74	281	1020	3591	12366	41877
4	28	152	762	3664	17120	78336	352768
5	40	270	1680	10050	58500	333750	1875000
6	54	436	3238	23160	161352	1102464	7420896
7	70	658	5677	47236	383131	3049270	23916361

We illustrate the bold case above (i.e., $k = 2$ and $n = 3$):



FIGURE 4. The eight words where $k = 2$ and $n = 3$, with total site perimeter 72

Now extract the exact coefficients of x^n in the generating function in Theorem 1. These are for $n \geq 2$

$$(2.8) \quad C(n)=\frac{k^{n-2}}{12}\left(n\left(3 k^3+16 k^2+3 k+2\right)+10 k^3+16 k^2+2 k-4\right),$$

and for $n = 1$ we have $C(1) = k^2 + 3k$. Since there are k^n partitions over the alphabet $[k]$ we have our next result

Theorem 2.6. For $n \geq 2$, the average site perimeter per word of length n over the alphabet $[k]$ is

$$\frac{2 + 3k + 16k^2 + 3k^3}{12k^2}n + \frac{-2 + k + 8k^2 + 5k^3}{6k^2}.$$

3. Direct count of the total site-perimeter

As an alternative method, we evaluate the total site-perimeter of words of length n over the alphabet $[k]$ by a direct counting argument. Let $C(n)$ be the total site-perimeter of all words of length n over $[k]$ and let $C(n|ab)$ be the total site-perimeter of all words of length n over $[k]$ starting with the letters a then b . Considering all possible cases, we add a letter of size j in front of a .

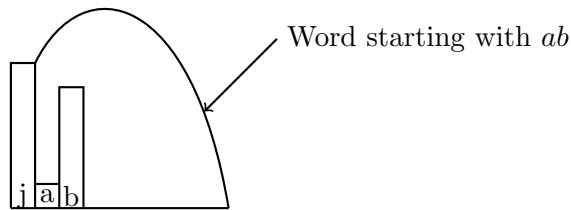


FIGURE 5. Adding j in front of a word that starts with ab

We have nine disjoint cases covering all possibilities; they are listed in the table below, together with the additional contributions to the site-perimeter as a result of adding an extra j . We will explain how the additional contribution of the cases $j < a \leq b$ and $a < b < j$ in bold are obtained. The other cases are similar.

A factor of k^{n-2} for all the possibilities of the remaining $n - 2$ columns in each case is included later.

Case	Additional contribution
$j < a \leq b$	$\sum_{b=2}^k \sum_{j=1}^{b-1} \sum_{a=j+1}^b 1 = \frac{1}{6}(k-1)(k+1)k$
$j \leq b < a$	$\sum_{a=2}^k \sum_{j=1}^{a-1} \sum_{b=j}^{a-1} 1 = \frac{1}{6}(k-1)(k+1)k$
$a < j \leq b$	$\sum_{b=2}^k \sum_{a=1}^{b-1} \sum_{j=a+1}^b (j-a+2) = \frac{1}{24}(k-1)(k+1)(k+10)k$
$a < b < j$	$\sum_{j=3}^k \sum_{a=1}^{j-2} \sum_{b=a+1}^{j-1} (2j-a-b+2) = \frac{1}{24}(3k^4 + 2k^3 - 27k^2 + 22k)$
$b < j < a$	$\sum_{a=3}^k \sum_{b=1}^{a-2} \sum_{j=b+1}^{a-1} 1 = \frac{1}{6}(k^3 - 3k^2 + 2k)$
$b < a < j$	$\sum_{j=3}^k \sum_{b=1}^{j-2} \sum_{a=b+1}^{j-1} (2j-2a+1) = \frac{1}{12}(k^4 - 7k^2 + 6k)$
$a = b < j$	$\sum_{j=2}^k \sum_{a=1}^{j-1} (2j-2a+1) = \frac{1}{6}(2k^3 + 3k^2 - 5k)$
$b < j = a$	$\sum_{j=2}^k \sum_{b=1}^{j-1} 2 = (k-1)k$
$j = a \leq b$	$\sum_{b=1}^k \sum_{j=1}^b 2 = (k+1)k$

Explanation for the two bold cases in the table: Words of size $n + 1$ are constructed from words of size n by the addition of the letter j at the start of the word. First, consider the case $j < a \leq b$: We shall only focus on the extra contribution due to the new column j . Under these circumstances there is only one extra contribution to the site-perimeter which is represented by the circle in the diagram below. Hence the summand in this case is 1.

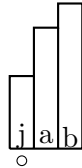


FIGURE 6. Extra contribution: $j < a \leq b$ case

For the case, $a < b < j$ the extra contributions $(j - b) + (j - a) + 2$ are shown in the diagram below. So the summand in this case is $2j - a - b + 2$.

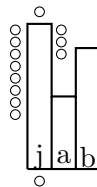


FIGURE 7. Extra contribution: $a < b < j$ case

The total contribution from all nine cases is $\frac{1}{12}(3k^3 + 16k^2 + 3k + 2)k$. We therefore obtain the following recursion

$$(3.1) \quad C(n + 1) = kC(n) + k^{n-2} \frac{1}{12}(3k^3 + 16k^2 + 3k + 2)k,$$

with initial condition $C(2) = \frac{1}{12}(16k^3 + 48k^2 + 8k)$. We solve this to obtain

Theorem 3.1. For $n \geq 2$, the total site-perimeter of all words of length n over $[k]$ is

$$C(n) = \frac{k^{n-2}}{12} \left(n(3k^3 + 16k^2 + 3k + 2) + 10k^3 + 16k^2 + 2k - 4 \right),$$

where $C(1) = k^2 + 3k$.

This corresponds to the result in equation (2.8) in Section 2.

4. Minimum and maximum site-perimeter for words over $[k]$

In this section, we consider the maximal and minimal values of the site-perimeter. Clearly the minimum site-perimeter occurs for the words

$$\underbrace{11111 \cdots 1}_n$$

with site-perimeter equal to $2n + 2$.

For the maximum site-perimeter, we separately consider three cases $n = 3s + 1$, $n = 3s$ and $n = 3s + 2$ where $s \in \mathbb{N}$.

Theorem 4.1. *The maximum site-perimeter is equal to*

$$\begin{cases} 2ks + 2k + 2s + 2, & \text{if } n = 3s + 1, \\ 2ks + k + 2s + 2, & \text{if } n = 3s, \\ 2ks + 2k + 2s + 4, & \text{if } n = 3s + 2. \end{cases}$$

A canonical form for the maximum consists of repeated copies of the three columns $k11$ with an ending that varies depending on $n \pmod 3$. In Figure 8, we have illustrated the latter part of the canonical forms (with only one or two copies of $k11$) for the three cases.

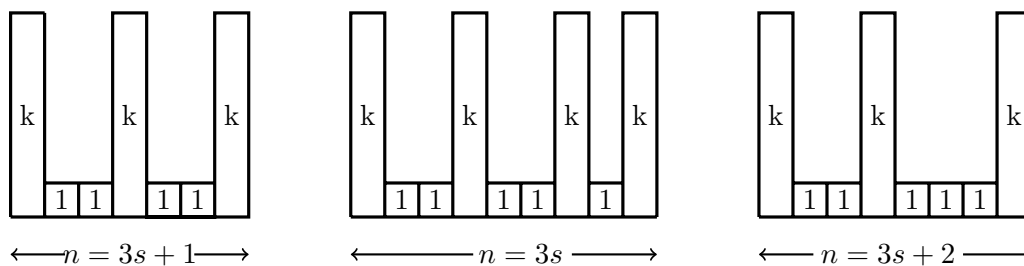


FIGURE 8. Latter end of three canonical forms for the maximal site-perimeter of words over $[k]$

The maximal values are obtained as follows: For $n = 3s + 1$ the combined horizontal site-perimeter above and below the bargraph is $2(3s + 1)$ and the vertical site-perimeter is $(s + 1)2k - 4s$.

For $n = 3s$, the canonical form has only one 1 between the latter two k 's, whereas for $n = 3s + 2$ the canonical form has three 1's between the latter two k 's.

To show that these configurations are maximal for each of the cases $n = 3s + 1$, $n = 3s$ and $n = 3s + 2$ one uses induction where the base cases are $n = 3, 4$ and 5 respectively.

5. Conclusion

Besides the use of generating functions to extend previous results on site perimeter of bargraphs to the case of words, we have demonstrated here a parallel direct method which we previously used in the case of permutations (see [2]), and have shown that it may also be used for other combinatorial objects such as words.

REFERENCES

- [1] A. Bacher, Average site perimeter of directed animals on the two-dimensional lattices, *Discrete Math.*, **312** (2012) 1038–1058.
- [2] A. Blecher, C. Brennan and A. Knopfmacher, The site-perimeter in permutations, *Utilitas Math.*, in press.
- [3] M. Bousquet-Mélou, A method for the enumeration of various classes of column-convex polygons, *Discrete Math.*, **154** (1996) 1–25.
- [4] M. Bousquet-Mélou and A. Rechnitzer, The site-perimeter of bargraphs, *Adv. in Appl. Math.*, **31** (2003) 86–112.
- [5] M. Delest, D. Gouyou-Beauchamps and B. Vauquelin, Enumeration of parallelogram polyominoes with given bond and site perimeter, *Graphs Combin.*, **3** (1987) 325–339.
- [6] G. Fülep and N. Sieben, Polyiamonds and polyhexes with minimum site-perimeter and achievement games, *Elect. J. Combin.*, **17** (2010) #R65.
- [7] G. Grimmett, *Percolation*, Springer-Verlag, New York, 1989.
- [8] S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, Discrete Mathematics and its applications, CRC press, Taylor and Francis group, 2010.
- [9] N. Sieben, Polyominoes with minimum site-perimeter and full set achievement games, *Europ. J. Combin.*, **29** (2008) 108–117.
- [10] D. Stauffer and A. Aharony, *An Introduction to Percolation Theory*, 2nd edition, Taylor and Francis, London, 1992.

Aubrey Blecher

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

Email: Aubrey.Blecher@wits.ac.za

Charlotte Brennan

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, niversity of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

Email: Charlotte.Brennan@wits.ac.za

Arnold Knopfmacher

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, niversity of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

Email: Arnold.Knopfmacher@wits.ac.za

Toufik Mansour